NUMBERS OF CONJUGACY CLASSES
IN SOME FINITE CLASSICAL GROUPS

I.G. MACDONALD

In this paper we calculate the number of conjugacy classes in the following finite classical groups: $GL_n(F_q)$; $PGL_n(F_q)$, $SL_n(F_q)$, and more generally $G(F_q)$, where $G$ is any algebraic group isogenous to $SL_n$; $PSL_n(F_q)$; $U_n(F_{q^2})$; $PU_n(F_{q^2})$, $SU_n(F_{q^2})$ and more generally $G(F_{q^2})$ where $G$ is any group isogenous to $SU_n$ over $F_q$; and $PSU_n(F_{q^2})$.

Introduction

Let $G$ be a semisimple algebraic group isogenous to $SL_n$ and let $k$ be a finite field. In this paper we calculate the number of conjugacy classes in the finite group $G(k)$ of $k$-rational points of $G$, for all choices of $G$ and $k$. The result is as follows. If $G$ is the image of $SL_n$ under a central isogeny of degree $f$, and $e = n/f$, then the number of conjugacy classes in $G(k)$ is

$$(q-1)^{-1} \sum_{d_1, d_2} \phi_1(d_1) \phi_2(d_2) \sigma_n/d_1d_2$$

summed over all pairs of positive integers $d_1, d_2$ such that $d_1$ (respectively $d_2$) divides $f$ and $q - 1$ (respectively $e$ and $q - 1$).

Received 25 July 1980.
and the notation is as follows: - \( q \) is the number of elements in \( k \); \( c_n \) is the number of conjugacy classes in \( \text{GL}_n(k) \); and for any positive integers \( n \) and \( r \),

\[
\phi_r(n) = n^r \prod_{p|n} \left( 1 - \frac{1}{p^r} \right)
\]

(product over the primes dividing \( n \)), so that \( \phi_1 \) is Euler’s \( \phi \)-function.

In particular, the expression

\[
(q-1)^{-1} \sum_{d|(n,q-1)} \phi_r(d)c_{n/d}
\]

gives the number of conjugacy classes in \( \text{PGL}_n(k) \) (respectively \( \text{SL}_n(k) \)) when \( r = 1 \) (respectively \( r = 2 \)). These two formulas were also found by Wall [7].

For convenience of exposition, we deal with these two cases first, in §2 and §3 respectively, and the general case in §4. We also calculate in §5 the number of conjugacy classes in the simple group \( \text{PSL}_n(k) \). Finally, in §6, we establish analogous formulas for unitary groups. For example, if \( k_2 \) is the quadratic extension of \( k \) and if \( \gamma_n \) is the number of conjugacy classes in the unitary group \( U_n(k_2) \), the expression

\[
(q+1)^{-1} \sum_{d|(n,q+1)} \phi_r(d)\gamma_{n/d}
\]

gives the number of conjugacy classes in \( \text{PU}_n(k_2) \) (respectively \( \text{SU}_n(k_2) \)) when \( r = 1 \) (respectively \( r = 2 \)).

We conclude this introduction with a few remarks on notation and terminology. A partition is any finite or infinite sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of integers such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \) and \( |\lambda| = \sum \lambda_i < \infty \). The non-zero \( \lambda_i \) are called the parts of \( \lambda \), and we denote by \( m_i(\lambda) \), for each \( i \geq 1 \), the number of parts \( \lambda_j \) equal to \( i \).

If \( \lambda, \mu \) are partitions, \( \lambda \cup \mu \) denotes the partition for which
Conjugacy classes in finite groups

\[ m_\ell(\lambda \cup \mu) = m_\ell(\lambda) + m_\ell(\mu), \text{ and similarly for any set of partitions.} \]

Finally, for any finite group \( G \), we denote by \( c(G) \) the number of conjugacy classes in \( G \).

1. Conjugacy classes in \( \text{GL}_n(k) \)

Let \( g \in \text{GL}_n(k) \). Then \( g \) acts on the vector space \( k^n \), and hence defines on \( k^n \) a \( k[t] \)-module structure (where \( t \) is an indeterminate) such that \( t \cdot v = gv \) for \( v \in k^n \). Let \( V_g \) denote this \( k[t] \)-module. It is clear that two elements \( g, h \in \text{GL}_n(k) \) are conjugate in \( \text{GL}_n(k) \) if and only if the \( k[t] \)-modules \( V_g \) and \( V_h \) are isomorphic. The conjugacy classes in \( \text{GL}_n(k) \) are therefore in one-one correspondence with the isomorphism classes of \( k[t] \)-modules \( V \) such that

(i) \( \dim_k V = n \),

(ii) \( tv = 0 \) implies \( v = 0 \).

Now \( k[t] \) is a principal ideal domain, and therefore \( V_g \) is a direct sum of cyclic modules of the form \( k[t]/(f)^m \), where \( m \geq 1 \) and \( f \) is an irreducible monic polynomial in \( k[t] \), the polynomial \( t \) being excluded, and \( (f) \) is the ideal generated by \( f \) in \( k[t] \). Let \( \Phi \) denote the set of these polynomials. Then we may write

\[ V_g \cong \bigoplus_{f \in \Phi} \bigoplus_{i \geq 1} (k[t]/(f))^{\lambda_i(f)} \]

where the direct sum is over all \( f \in \Phi \) and integers \( i \geq 1 \), and the exponents satisfy \( \lambda_1(f) \geq \lambda_2(f) \geq \ldots \geq 0 \), so that \( \lambda(f) = (\lambda_1(f), \lambda_2(f), \ldots) \) is a partition for each \( f \in \Phi \). Since \( \dim_k (k[t]/(f)^m) = m \deg(f) \), the partition-valued function \( \lambda \) on \( \Phi \) must satisfy

\[ \sum_{f \in \Phi} |\lambda(f)| \deg(f) = n. \]
The conjugacy classes of $\text{GL}_n(k)$ are thus in one-one correspondence with the partition-valued functions $\lambda$ on $\Phi$ which satisfy (1.2).

For later use it is convenient to modify this parametrization. Let $\overline{k}$ be an algebraic closure of $k$, and let $M = \overline{k}^*$ be its multiplicative group. The Frobenius automorphism $F : x \mapsto x^q$ (where $q = \text{Card}(k)$) acts on $M$, and the roots in $\overline{k}$ of a polynomial $f \in \Phi$ form a single $F$-orbit in $M$, and conversely. We may therefore replace $\lambda$ by the partition-valued function $\mu$ on $M$ defined by $\mu(x) = \lambda(f)$, where $f$ is the minimal polynomial of $x$ over $k$. The condition (1.2) becomes

$$\sum_{x \in M} |\mu(x)| = n$$

and moreover $\mu$ must satisfy

$$\mu(Fx) = \mu(x)$$

for all $x \in M$. The conjugacy classes in $\text{GL}_n(k)$ are now parametrized by partition-valued functions $\mu$ on $M$ satisfying (1.3) and (1.4).

The function $\mu$ rather than $\lambda$ arises when we decompose the $\overline{k}[t]$-module $V_g = V_g \otimes_k \overline{k}$, for we have

$$\overline{V}_g \cong \bigoplus_{x, i} \overline{k}[t]/(t-x)^{u_i(x)}$$

(direct sum over all $x \in M$ and $i \geq 1$).

Now define polynomials $u_i$ by

$$u_i = \prod_{x \in M} \frac{1}{(1-\lambda_i x)}^{m_i(\mu(x))}$$

where $m_i(\mu(x))$ is the number of parts equal to $i$ in the partition $\mu(x)$, for each $i \geq 1$. Clearly $u_i(0) = 1$, and $u_i \in k[t]$ by virtue of (1.4). Moreover, if $g \in \text{GL}_n(k)$ is in the conjugacy class parametrized by $\mu$, we have

$$\det(1-tg) = \prod_{i \geq 1} u_i(t)^i.$$
For the characteristic polynomial of $g$ is
\[
\det(t-g) = \prod_{f \in \Phi} f(t) |\lambda(f)| = \prod_{x \in \mathcal{M}} (t-x) |\mu(x)|
\]
and therefore
\[
\det(1-tg) = \prod_{x \in \mathcal{M}} (1-tx) |\mu(x)| = \prod_{i \geq 1} u_i(t)^i.
\]

The sequence of polynomials $u = (u_1, u_2, \ldots)$ determines the function $\mu$ by (1.6), and hence the conjugacy class. The $u_i$ must satisfy
\[
(1.8) \quad u_i \in k[t], \quad u_i(0) = 1 \quad (i \geq 1)
\]
and
\[
(1.9) \quad \sum_{i \geq 1} i \deg u_i = n
\]
(by (1.3) or (1.7)). It follows that the conjugacy classes of $GL_n(k)$ are in one-one correspondence with the set of sequences $u = (u_1, u_2, \ldots)$ which satisfy (1.8) and (1.9).

If $\deg(u_i) = n_i$, so that by (1.7),
\[
v = (1^{n_1} 2^{n_2} \ldots)
\]
is a partition of $n$, we call $v$ the type of the conjugacy class corresponding to $u$. In terms of $\mu$, we have
\[
(1.10) \quad v = \mathop{\sum_{x \in \mathcal{M}}} u_i(x).
\]

The number of polynomials $u_i$ of degree $n_i$ which satisfy (1.8) is
\[
q^n - q^{-1}_{n_i} \quad \text{if} \quad n_i > 0 \quad (\text{and is 1 if} \quad n_i = 0). \quad \text{Hence the number of conjugacy classes of type} \quad v \quad \text{in} \quad GL_n(k) \quad \text{is}
\]
\[
(1.11) \quad \alpha_v = \prod_{n_i > 0} \left( q^{n_i} - q^{-1}_{n_i} \right)
\]
and the total number of conjugacy classes in $GL_n(k)$ is

$$(1.12) \quad c_n = c(GL_n(k)) = \sum_{|\nu| = n} a_{\nu}.$$ 

Since $q^n - q^{n-1}$ is the coefficient of $t^n$ in $(1-t)(1-qt)^{-1}$, if $n > 0$, it follows from (1.11) and (1.12) that the generating function for the numbers $c_n$ is

$$(1.13) \quad C(t) = \sum_{n=0}^{\infty} c_n t^n = \sum_{r=1}^{\infty} \frac{1-t^r}{1-qt^r}.$$ 

This formula is due to Feit and Fine [1].

If we combine (1.13) with the well-known identity

$$\prod_{r=1}^{\infty} (1-qt^r)^{-1} = \sum_{r=0}^{\infty} q^r t^r \prod_{k=1}^{r} (1-t^k)^{-1},$$

we obtain

$$(1.14) \quad C(t) = \sum_{r=0}^{\infty} q^r t^r \prod_{k>r} (1-t^k).$$

We can transform this further, as follows. From the identity

$$\prod_{k=1}^{\infty} (1+at^k)^{-1} = \sum_{s=0}^{\infty} a^s t^s(s+1)/2 \prod_{j=1}^{s} (1-t^j)^{-1},$$

with $a = -t^r$, we obtain

$$\prod_{k>r} (1-t^k)^{-1} = \sum_{s=0}^{\infty} (-1)^s t^{rs+s(s+1)/2} \prod_{j=1}^{s} (1-t^j)^{-1}$$

and hence (1.14) becomes

$$C(t) = \sum_{r=0}^{\infty} q^r \sum_{s=0}^{\infty} (-1)^s t^{r(s+1)(r+s/2)} \prod_{j=1}^{s} (1-t^j)^{-1} \left[ q^r t^r - \frac{q^r t^{2r+1}}{1-t} + \frac{q^r t^{3r+3}}{(1-t)(1-t^2)} - \cdots \right]$$

from which it follows that $c_n$ is a polynomial in $q$ of the form...
Conjugacy classes in finite groups

\[ c_n = q^n - \left( q^a + q^{a-1} + \ldots + q^{b+1} + q^b \right) + \ldots \]

where \( a = [\frac{1}{3}(n-1)] \), \( b = \left[ \frac{1}{3}n \right] \), and the terms not written at the end have degree less than \( b \). Also, from Euler's pentagonal number theorem, the constant term in \( c_n \) is 0 unless \( n \) is of the form \( \frac{1}{3}m(3m+1) \) for some \( m \in \mathbb{Z} \), in which case \( c_n = (-1)^m \).

Finally, all the \( c_n \) (\( n \geq 1 \)) are divisible by \( q - 1 \), as one sees by putting \( q = 1 \) in (1.13).

In Table 1 below we list the polynomials \( c_n \) for \( 0 \leq n \leq 12 \), and in Table 2 the polynomials \( c'_n = c_n / (q-1) \) for \( 1 \leq n \leq 12 \).

<table>
<thead>
<tr>
<th>TABLE 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 = 1 )</td>
</tr>
<tr>
<td>( c_1 = q - 1 )</td>
</tr>
<tr>
<td>( c_2 = q^2 - 1 )</td>
</tr>
<tr>
<td>( c_3 = q^3 - q )</td>
</tr>
<tr>
<td>( c_4 = q^4 - q )</td>
</tr>
<tr>
<td>( c_5 = q^5 - q^2 - q + 1 )</td>
</tr>
<tr>
<td>( c_6 = q^6 - q^2 )</td>
</tr>
<tr>
<td>( c_7 = q^7 - q^3 - q^2 + 1 )</td>
</tr>
<tr>
<td>( c_8 = q^8 - q^3 - q^2 + q )</td>
</tr>
<tr>
<td>( c_9 = q^9 - q^4 - q^3 + q )</td>
</tr>
<tr>
<td>( c_{10} = q^{10} - q^4 - q^3 + q )</td>
</tr>
<tr>
<td>( c_{11} = q^{11} - q^5 - q^4 - q^3 + q^2 + q )</td>
</tr>
<tr>
<td>( c_{12} = q^{12} - q^5 - q^4 + q^2 + q - 1 )</td>
</tr>
</tbody>
</table>
TABLE 2

<table>
<thead>
<tr>
<th>$c_1'$</th>
<th>$c_2'$</th>
<th>$c_3'$</th>
<th>$c_4'$</th>
<th>$c_5'$</th>
<th>$c_6'$</th>
<th>$c_7'$</th>
<th>$c_8'$</th>
<th>$c_{10}'$</th>
<th>$c_{11}'$</th>
<th>$c_{12}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$q+1$</td>
<td>$q^2+q$</td>
<td>$q^3+q^2+q$</td>
<td>$q^4+q^3+q^2-1$</td>
<td>$q^5+q^4+q^3+q^2$</td>
<td>$q^6+q^5+q^4+q^3-q-q$</td>
<td>$q^7+q^6+q^5+q^4+q^3-q$</td>
<td>$q^8+q^7+q^6+q^5+q^4-q^2-q$</td>
<td>$q^9+q^8+q^7+q^6+q^5+q^4-q^2-q$</td>
<td>$q^{11}+q^{10}+q^9+q^8+q^7+q^6+q^5-q^3-q^2+1$</td>
</tr>
</tbody>
</table>

2. $\text{PGL}_n(k)$

The group $k^*$ acts on $\text{GL}_n(k)$ by multiplication, and hence on the set of conjugacy classes in $\text{GL}_n(k)$. The conjugacy classes of $\text{PGL}_n(k)$ may be identified with the orbits of this action. If $\xi \in k^*$ is such that $\xi g$ is conjugate to $g$ in $\text{GL}_n(k)$, then $\det(g) = \det(\xi g) = \xi^n \det(g)$, so that $\xi^n = 1$.

If the conjugacy class of $g$ in $\text{GL}_n(k)$ is represented by a sequence $u = (u_1(t), u_2(t), \ldots)$ of polynomials as in §1, then the conjugacy class of $\xi g$ is represented by $\xi u = (u_1(\xi t), u_2(\xi t), \ldots)$.

For each $d$ dividing $n$, let $\mu_d(k)$ denote the group of $d$th roots of unity in $k$. The order of $\mu_d(k)$ is
Conjugacy classes in finite groups

\[ h_d = (d, q-1) \]

(highest common factor). A sequence \( u = (u_1(t), u_2(t), \ldots) \) is fixed by \( \mu_d(k) \) if and only if \( \xi u = u \) for all \( \xi \in k^* \) such that \( \xi^d = 1 \), that is to say if and only if \( u_\xi(t) \in k[t^d] \) for all \( i \). It follows that the number of conjugacy classes in \( GL_n(k) \) fixed by \( \mu_d(k) \) is \( \sigma_{n/d} \), and therefore the number for which the isotropy group is exactly \( \mu_d(k) \) is

\[ \sum_{d'} \mu(d') \sigma_{n/dd'} \]

where \( \mu \) is the Möbius function and the sum is over all divisors \( d' \) of \( n/d \). But if the isotropy group is \( \mu_d(k) \), the number of elements in the orbit is \((q-1)/h_d\). Hence the total number of orbits, that is, the number of conjugacy classes in \( PGL_n(k) \), is

\[ \overline{c}_n = \sigma(PGL_n(k)) = \sum_{d} \frac{h_d}{q-1} \sum_{d'} \mu(d') \sigma_{n/dd'}, \]

where the sum is over all pairs of positive integers \( d, d' \) such that \( dd' \) divides \( n \).

An alternative formula for \( \overline{c}_n \) is

\[ \overline{c}_n = \sum_{d} \phi(d) \sigma'_{n/d} \]

where \( \phi \) is Euler's function and the sum is over the divisors \( d \) of \((q-1, n) = h_n\).

This is a consequence of the following lemma. For each positive integer \( k \) and positive integer \( n \) we define

\[ \phi_p(n) = n^p \prod_{p|n} (1-p^{-r}) \]

where the product on the right is over the prime factors of \( n \); \( \phi_p(n) \)
is the number of elements of order \( n \) in the abelian group \((\mathbb{Z}/n\mathbb{Z})^r\).

When \( r = 1 \), \( \phi_1 = \phi \) is Euler’s function; when \( r \geq 2 \), \( \phi_r \) is Jordan’s generalization of Euler’s function.

(2.4) Let \( N \) be a positive integer. Then

\[
\sum_{d|N} (N, d)^r \mu(n/d) = \phi_r(n) \quad \text{if} \quad n|N,
\]

\[
= 0 \quad \text{otherwise}.
\]

Proof. The function \( d \to (N, d)^r \) is multiplicative, that is,

\[
(N, dd')^r = (N, d)^r(N, d')^r \quad \text{if} \quad (d, d') = 1.
\]

Hence if \( n = \prod_i p_i^{v_i} \) is the prime factorization of \( n \) we have

\[
\sum_{d|n} (N, d)^r \mu(n/d) = \prod_i \left[ \left( N, p_i^{v_i} \right)^r - \left( N, p_i^{v_i-1} \right)^r \right].
\]

If \( n \) does not divide \( N \), at least one of the factors on the right is zero. If \( n \) divides \( N \), the product on the right is just \( \phi_r(n) \). //

If we take \( r = 1 \) and \( N = q - 1 \) in (2.4), we have

\[
\sum_{d|N} \mu(n/d) = \phi(n) \quad \text{if} \quad n|q-1,
\]

\[
= 0 \quad \text{otherwise}.
\]

Substitution of this result in (2.1) gives (2.2).

From (2.2), the generating function for the numbers \( \overline{\sigma}_n \) is (with \( \overline{\sigma}_0 = 1 \))

(2.5)

\[
\overline{C}(t) = \sum_{n=0}^{\infty} \overline{\sigma}_n t^n = \frac{1}{q-1} \sum_{d|q-1} \phi(d)C(t^d)
\]

where \( C(t) \) is the generating function (1.13).

In Table 3 we list the values of \( \overline{\sigma}_n \) for \( 1 \leq n \leq 10 \). They are easily computed from (2.1) and Table 2.
TABLE 3

\[ \begin{array}{c}
\sigma_1 = 1 \\
\sigma_2 = q + h_2 \\
\sigma_3 = q^2 + q + h_3 - 1 \\
\sigma_4 = q^3 + q^2 + h_2 q + h_4 - 1 \\
\sigma_5 = q^4 + q^3 + q^2 + h_5 - 2 \\
\sigma_6 = q^5 + q^4 + q^3 + h_2 q^2 + (h_2 + h_3 - 2)q + h_6 - h_2 \\
\sigma_7 = q^6 + q^5 + q^4 + q^3 - q + h_7 - 2 \\
\sigma_8 = q^7 + q^6 + q^5 + h_2 q^3 + (h_2 - 1)q^2 + (h_2 - 2)q + h_8 - h_2 \\
\sigma_9 = q^8 + q^7 + q^6 + q^5 + h_2 - 2(q^2 + q) + h_9 - h_3 \\
\sigma_{10} = q^9 + q^8 + q^7 + q^6 + q^5 + h_2 q^4 + (h_2 - 1)q^3 + (h_2 - 2)q^2 + (h_2 - 3)q + (h_10 - 2h_2 + 1)
\end{array} \]

3. \(SL_n(k)\)

For each partition \(\nu = (\nu_1, ..., \nu_r)\) of \(n\), let \(h_\nu\) denote the highest common factor of \(q - 1\) and \(\nu_1, \nu_2, ..., \nu_r\).

(3.1) The number of \(GL_n(k)\)-conjugacy classes of type \(\nu\) contained in \(SL_n(k)\) is \(h_\nu c_\nu = h_\nu c_\nu / (q - 1)\).

Proof. Consider a conjugacy class \(c\) of type \(\nu\) represented as in §1 by a sequence of polynomials \(u = (u_1(t), u_2(t), ...)\), where

\[ u_i = a_i t^{n_i} + ... + 1 \quad (a_i \in k^*, n_i = m_i(\nu)) \]

From (1.7), the class \(c\) is contained in \(SL_n(k)\) (that is, its elements have determinant 1) if and only if \(\prod_{i=1}^r a_i = (-1)^n\).

Let \(S\) be the set of positive integers \(i\) such that \(n_i > 0\), so
that \( s = |S| \) is the number of different parts of \( \nu \). Consider the homomorphism \( \phi : (k^*)^S \to k^* \) defined by

\[
(a_i)_{i \in S} \mapsto (-1)^n \prod a_i^i.
\]

If \( C \) and \( K \) are the cokernel and kernel of \( \phi \), the exact sequence

\[
1 \to K \to (k^*)^S \to k^* \to C \to 1
\]

shows that \( |K| = (q-1)^{s-1} |C| \). Also \( C \) is the quotient of \( k^* \) by the subgroup generated by the \( i \)-th powers for all \( i \in S \), that is to say by the \( j_i \)-th powers \( (1 \leq j_i \leq n) \). Hence \( C \) is a cyclic group of order \( h_\nu \), and so \( |K| = (q-1)^{s-1} \cdot h_\nu \). It follows that the number of \( \text{GL}_n(k) \)-conjugacy classes of type \( \nu \) contained in \( \text{SL}_n(k) \) is

\[
(q-1)^{s-1} \cdot h_\nu \prod_{i \in S} q^{i-1} = h_\nu c_\nu / (q-1). \]

Next we need to know how many \( \text{SL}_n(k) \)-conjugacy classes are contained in a \( \text{GL}_n(k) \)-class. For this purpose we shall use the following lemma:

(3.2) Let \( G, H \) be finite groups, \( \delta : G \to H \) a surjective homomorphism, \( K \) the kernel of \( \delta \). Let \( X \) be a set on which \( G \) acts. Then for each \( x \in X \), the \( G \)-orbit of \( x \) in \( X \) splits up into

\[
n_x = |\text{Coker}(\delta | G_x)|
\]

\( K \)-orbits, where \( G_x \) is the subgroup of \( G \) which fixes \( x \).

Proof. Since \( K \) is a normal subgroup of \( G \), the \( K \)-orbits contained in \( G \cdot x \) are permuted transitively by \( G \), and therefore the number of them is

\[
n_x = \frac{|G \cdot x|}{|K \cdot x|} = \frac{|G|}{|K|} \frac{|G_x|}{|K_x|} = \frac{|G|}{|K|} \frac{|G_x|}{|G_x/K_x|} = \frac{|H|}{|\delta(G_x)|} = |\text{Coker}(\delta | G_x)|,
\]

since \( K_x = K \cap G_x \) is the kernel of \( \delta | G_x \). //

We shall apply (3.2) with \( G = \text{GL}_n(k) \), \( H = k^* \) and \( \delta \) the determinant homomorphism (so that \( K = \text{SL}_n(k) \) ) and \( X = \text{SL}_n(k) \) with \( G \)
Conjugacy classes in finite groups

acting by inner automorphisms.

(3.3) Let \( g \in \text{SL}_n(k) \) and let \( Z(g) \) be the centralizer of \( g \) in \( \text{GL}_n(k) \). If \( \nu = (\nu_1, \ldots, \nu_r) \) is the type of \( g \), then \( \det(Z(g)) \) is the subgroup \( P_\nu \) of \( k^* \) generated by the \( \nu_1 \)th, \ldots, \( \nu_r \)th powers.

Proof. As in §1, let \( \overline{V} = \overline{V}_g \) be the \( k[t] \)-module defined by \( g \). We have

\[
\overline{V} = \bigoplus_{x \in M} \overline{V}(x),
\]

where

\[
(1) \quad \overline{V}(x) \cong \bigoplus_{t \geq 1} k[t]/(t-x)^{\mu_t(x)}
\]

is the characteristic submodule of \( \overline{V} \) consisting of the elements of \( \overline{V} \) killed by some power of \( t - x \).

If \( h \in Z(g) \), each submodule \( \overline{V}(x) \) is stable under \( h \) and therefore decomposes relative to the action of \( h \) : say

\[
(2) \quad \overline{V}(x) = \bigoplus_{y \in M} \overline{V}(x, y)
\]

where \( \overline{V}(x, y) \) is the subspace of elements of \( \overline{V}(x) \) killed by some power of \( h - y \), and is a submodule of \( \overline{V}(x) \) because \( h \) is a module automorphism of \( \overline{V} \). Let \( \pi(x, y) \) be the type of \( \overline{V}(x, y) \), so that

\[
(3) \quad \overline{V}(x, y) \cong \bigoplus_{t \geq 1} k[t]/(t-x)^{\pi_t(x,y)}
\]

From (1), (2) and (3) it follows that

\[
\mu(x) = \bigcup_{y \in M} \pi(x, y)
\]

and therefore

\[
(4) \quad \nu = \bigcup_{x \in M} \mu(x) = \bigcup_{x, y} \pi(x, y).
\]

On the other hand, the determinant of \( h|\overline{V}(x, y) \) is

\[
y \dim \overline{V}(x, y) = y \left| \pi(x, y) \right|
\]
and therefore

\[ \det(h) = \prod_{x,y} \left| \pi(x,y) \right| . \]  

From (4) and (5) it is clear that \( \det(h) \in \mathbb{P} \), for all \( h \in \mathcal{Z}(g) \).

To show that \( \det(\mathcal{Z}(g)) \) is the whole of \( \mathbb{P} \), it is enough to show that we can choose \( h \in \mathcal{Z}(g) \) so that \( \det(h) = \xi^v \cdot \eta^j \) for any \( \xi \in k^* \) and any \( j \geq 1 \). Suppose that \( v_j = \lambda_{\xi}(f) \) in the notation of §1. Then \( v_g \) has a direct summand isomorphic to \( k[t]/(f)^{v_j} \), and it is enough to produce an automorphism of this cyclic module with determinant \( \xi^v \cdot \eta^j \). In the field \( k_f = k[t]/(f) \), choose an element \( \zeta \) whose norm (from \( k_f \) to \( k \)) is \( \xi \); if \( \zeta \) is the image in \( k_f \) of a polynomial \( z(t) \in k[t] \), then multiplication by \( z(t) \) will induce an automorphism of \( k[t]/(f)^{v_j} \) with the required determinant. //

From (3.2) and (3.3) it follows that each \( \text{GL}_n(k) \)-conjugacy class of type \( v \) contained in \( \text{SL}_n(k) \) is the union of

\[ [k^* : \mathbb{P}_v] = h_v \]

\( \text{SL}_n(k) \)-conjugacy classes. Hence from (3.1) the total number of conjugacy classes in \( \text{SL}_n(k) \) is

\[ \bar{\sigma}_n = \sigma(\text{SL}_n(k)) = \sum_{|v|=n} h_v \sigma'_v . \]

Now it is clear from (1.11) that if \( k \) divides each part \( v_i \) of \( v \) we have \( \sigma'_v = \sigma'_v / k \), where \( v / k \) is the partition \( \{ v_1 / k, v_2 / k, \ldots \} \).

Hence if we define

\[ \sigma''_n = \sum_v \sigma'_v \]

summed over all partitions \( v \) of \( n \) such that the highest common factor
Conjugacy classes in finite groups

of \( \nu_1, \nu_2, \ldots \) is 1, we have

\[
\sigma'_n = \sum_{|\nu|=n} \sigma'_\nu = \sum_{d|n} \sigma''_d,
\]

and therefore by Möbius inversion

\[
\sigma''_n = \sum_{d|n} \mu(d)\sigma'_{n/d}.
\]

Now in (3.1*) the possible values of \( h_\nu \) are \( h_d \), for \( d \) dividing \( n \), and hence (3.4) takes the form

\[
\tilde{\sigma}_n = \sum_{d|n} h_d^2 \sigma''_{n/d};
\]

that is,

\[
(3.5)
\tilde{\sigma}_n = \sum_{d|d'} h_d^2 \mu(d')\sigma'_{n/dd'}.
\]

Comparison of (3.5) with (2.1) shows that \( \tilde{\sigma}_n \) is derived from \( \sigma_n \) by replacing each coefficient \( h_d \) by \( h_d^2 \). Hence \( \tilde{\sigma}_n \) for \( 1 \leq n \leq 10 \) can be read off from Table 3.

From (3.5) and (2.4) we deduce an alternative formula for \( \tilde{\sigma}_n \):

\[
(3.6)
\tilde{\sigma}_n = \sum_d \phi_2(d)\sigma'_n/d
\]

summed over all divisors \( d \) of \( (q-1, n) = h_n \). Hence the generating function for the \( \tilde{\sigma}_n \) (with \( \tilde{\sigma}_0 = 1 \)) is

\[
(3.7)
\tilde{c}(t) = \sum_{n=0}^{\infty} \tilde{\sigma}_n t^n = \frac{1}{q-1} \sum_{d|q-1} \phi_2(d)c(t^d) - q + 2
\]

(since \( \sum_{d|q-1} \phi_2(d) = (q-1)^2 \)).

REMARKS. 1. From (3.1) it follows that the number of \( \text{GL}_n(k) \)-classes contained in \( \text{SL}_n(k) \) is
\[ \sum_{|\nu|=n} h_{\nu}^{\sigma'} \]

which by the same argument that led from (3.4) to (3.5) is equal to

\[ \sum_{d',d} h_{d}^{\mu(d')} \sigma_{n/\Delta}\]

and hence is equal to \( \overline{a}_{n} \): in other words, the number of \( GL_{n}(k) \)-classes contained in \( SL_{n}(k) \) is equal to the number of conjugacy classes in \( PGL_{n}(k) \). This fact was first observed by Lehrer [2].

2. Instead of counting conjugacy classes we could instead have counted the irreducible representations of \( SL_{n}(k) \), using the parametrization of [3], §5. The details are rather similar to those of this section and lead (fortunately) to the same result (3.5) or (3.6).

4. Other groups isogenous to \( SL_{n} \)

Let \( e, f \) be positive integers such that \( ef = n \). Let \( A_{e} \) be the kernel of the homomorphism \( \delta : GL_{n} \times GL_{1} \rightarrow GL_{1} \) defined by

\[ \delta(g, x) = x^{-e} \det(g), \]

and let \( B_{e} \) be the image of the homomorphism \( \varepsilon : GL_{1} \rightarrow GL_{n} \times GL_{1} \) defined by

\[ \varepsilon(x) = \left( x^{1/n}, x^{f} \right). \]

Then \( B_{e} \) is isomorphic to \( GL_{1} \), and is a closed normal subgroup of \( A_{e} \). Let \( G_{e} = A_{e} / B_{e} \).

The mapping \( g \mapsto (g, 1) \) embeds \( SL_{n} \) in \( A_{e} \), hence defines a homomorphism \( SL_{n} \rightarrow A_{e} \rightarrow G_{e} \), which is easily seen to be surjective, with kernel \( \mu_{f}^{1/n} \), where \( \mu_{f} \) is the group of \( f \)th roots of unity. Hence \( G_{e} \) is a connected algebraic group isogenous to \( SL_{n} \). In particular, \( G_{1} = PGL_{n} \) and \( G_{n} = SL_{n} \).
Now let $k$ be a finite field, and consider the group $A_e(k)$ of $k$-rational points of the algebraic group $A_e$. We define the type of an element $(g, x)$ in $GL_n(k) \times k^*$ (or of its conjugacy class) to be the type of $g$ (hence a partition of $n$).

(4.1) For each partition $\nu = (\nu_1, \ldots, \nu_r)$ of $n$, the number of conjugacy classes of type $\nu$ in $A_e(k)$ is

$$h_{e, \nu}^2,$$

where $h_{e, \nu}$ is the highest common factor of $q - 1$, $e$, and $\nu_1, \ldots, \nu_r$.

Proof. The same argument as in (3.1) shows that the number of $GL_n(k) \times k^*$-conjugacy classes of type $\nu$ contained in $A_e(k)$ is $h_{e, \nu}$. Next, by applying the lemma (3.2) to the homomorphism $\delta : GL_n(k) \times k^* \to k^*$, we see that the $GL_n(k) \times k^*$-conjugacy class of an element $(g, e) \in A_e(k)$ of type $\nu$ splits up into $|Coker(\delta|_{Z(g) \times k^*})|$ $A_e(k)$-conjugacy classes. By (3.3), the group $\delta(Z(g) \times k^*)$ is the subgroup $P_{e, \nu}$ of $k^*$ generated by the $\nu_1$-th, ..., $\nu_r$-th and $e$th powers, and therefore

$$|Coker(\delta|_{Z(g) \times k^*})| = (k^* : P_{e, \nu}) = h_{e, \nu}.$$ //

We have now to pass from $A_e(k)$ to the group $G_e(k)$. Since $B_e$ is connected, it follows from Lang's theorem ([5], §10) that $G_e(k) \cong A_e(k)/B_e(k) \cong A_e(k)/k^*$. The conjugacy classes of $G_e(k)$ may therefore be identified with the orbits of $k^*$ acting on the set of conjugacy classes of $A_e(k)$, the action of $k^*$ on $A_e(k)$ being given by $\xi.(g, x) = (\xi g, \xi^e x)$.

If $\xi \in k^*$ fixes the class of $(g, x)$ in $A_e(k)$, we must have $x = \xi^e x$, so that $\xi \in \mu_f(k)$. As in §2, if the conjugacy class of $g$ in $GL_n(k)$ is represented by a sequence $u = (u_1(t), u_2(t), \ldots)$ of polynomials $u_i \in k[t]$, then the conjugacy class of $\xi g$ is represented by
\[ \xi_k = (u_1(\xi t), u_2(\xi t), \ldots) \]; also \( \deg u_{\xi} = m_{\xi}(\nu) \) where \( \nu \) is the type of \( g \).

Suppose that \( \mu_d(k) \), where \( d \) is a divisor of \( f \), fixes the conjugacy class of \( (g, x) \) in \( A_\epsilon(k) \). Then each \( u_{\xi} \) must be a polynomial in \( t^d \), and therefore \( d \) divides each \( m_{\xi}(\nu) \). Let \( d\backslash\nu \) denote the partition of \( n/d \) such that \( m_{\xi}(d\backslash\nu) = m_{\xi}(\nu)/d \). Since \( \nu \) and \( d\backslash\nu \) have the same parts (with different multiplicities), we have \( h_{e,\nu} = h_{e,d\backslash\nu} \). Hence the number of \( A_\epsilon(k) \)-conjugacy classes of type \( \nu \) for which the isotropy group is exactly \( \mu_d(k) \) is, by (4.1),

\[
\sum_{d|f} h_{e,\nu}^2 \sum_{d'|\nu} \mu(d') \sigma_{d\backslash\nu} \]

summed over all divisors \( d' \) of \( f/d \) such that \( dd' \) divides each \( m_{\xi}(\nu) \) (so that the partition \( dd'\backslash\nu \) is defined). Since \( |\mu_d(k)| = h_d \), there are \( (q-1)/h_d \) elements in each orbit, and hence the total number of conjugacy classes in the group \( G_\epsilon(k) \) is

\[
\sigma(G_\epsilon(k)) = \sum_{|\nu|=n} h_{e,\nu}^2 \sum_{d,d'|\nu} h_{d\nu}(d') \sigma_{dd'\backslash\nu},
\]

in which the first sum is over all partitions \( \nu \) of \( n \), and the second is over all pairs of positive integers \( d, d' \) such that \( dd' \) divides \( f \) and each \( m_{\xi}(\nu) \). By (2.4), the expression (4.2) is equal to

\[
\sum_{|\nu|=n} h_{e,\nu}^2 \sum_{d_1|\nu} \phi(d_1) \sigma_{d_1\nu},
\]

where \( d_1 \) runs through the common divisors of \( f, q-1 \) and the \( m_{\xi}(\nu) \).

Since \( h_{e,\nu} = h_{e,d_1\nu} \), (4.3) can be rewritten in the form

\[
\sum_{d_1} \phi(d_1) \sum_{\nu} h_{e,\nu}^2 \sigma_{\nu},
\]

where \( d_1 \) runs through the divisors of \( h_f \) and \( \nu \) through the partitions
of $n/d_1$. As in §3, the inner sum is equal to

$$\sum_{d_2} \phi(d_2) c'_{n/d_1d_2}$$

where $d_2$ runs through the divisors of $h_e$. Hence

(4.4) The number of conjugacy classes in the group $G_e(k)$ is

$$c(G_e(k)) = \sum_{d_1,d_2} \phi(d_1) \phi(d_2) c'_{n/d_1d_2}$$

summed over pairs of positive integers $d_1, d_2$ such that $d_1$ divides $h_f$ and $d_2$ divides $h_e$.

REMARKS. 1. When $f = 1$ and $e = n$ (respectively $e = 1$ and $f = n$), the group $G_e(k)$ is $SL_n(k)$ (respectively $PGL_n(k)$), and the expression (4.4) reduces to (3.6) (respectively (2.2)).

2. From (4.4) and (2.4) we deduce an alternative formula for

$$c(G_e(k)) :$$

summed over all quadruples $a_1, a_2, b_1, b_2$ of positive integers such that $a_1a_2b_1b_2$ divides $n$, where $h_{b_1,f}$ (respectively $h_{b_2,e}$) is the highest common factor of $b_1, f$ and $q - 1$ (respectively $b_2, e$ and $q - 1$). When $e = n$ and $f = 1$ (respectively $e = 1$ and $f = n$), the formula (4.5) reduces to (3.5) (respectively (2.1)).

3. The group $GL_n(k)$ acts by inner automorphisms on $A_e(k)$:

$$h.(g, x) = (hgh^{-1}, x).$$

This action fixes $B_e(k)$ pointwise, and hence defines an action of $GL_n(k)$ on the group $G_e(k)$. The method used to prove (4.4) shows that the number of $GL_n(k)$-orbits in $G_e(k)$ is

(4.6)

$$\sum_{d_1,d_2} \phi(d_1) \phi(d_2) c'_{n/d_1d_2}$$
summed over pairs of positive integers $d_1, d_2$ such that $d_1$ divides $h_f$ and $d_2$ divides $h_e$. Since (4.6) is unaltered by interchange of $e$ and $f$, it follows that the number of $G_e(k)$-orbits is the same in $G_e(k)$ and $G_f(k)$. This result generalizes Remark 1 of §3.

For a related result, see Lehrer [2].

5. $\text{PSL}_n(k)$

Let $\text{PSL}_n(k)$ denote the quotient of $\text{SL}_n(k)$ by its centre $\mu_n(k).1_n$. The conjugacy classes of $\text{PSL}_n(k)$ may be identified with the orbits of $\mu_n(k)$ acting on the set of conjugacy classes of $\text{SL}_n(k)$. From §3, the number of conjugacy classes of type $\nu$ in $\text{SL}_n(k)$ is $h^2_\nu c'_\nu$, for each partition $\nu$ of $n$. It follows as in §4 that the number of $\text{SL}_n(k)$-classes for which the isotropy group is $\mu_d(k)$, where $d$ is a divisor of $n$, is

$$h^2_\nu \sum_{d'} h(d') c'_{dd'}\nu$$

summed over all divisors $d'$ of $n/d$ such that $dd'$ divides each $m_\nu(\nu)$. The number of elements in each orbit is $(\mu_n(k) : \mu_d(k)) = h_n/h_d$, and hence we obtain the total number of conjugacy classes in $\text{PSL}_n(k)$:

$$\hat{c}_n = c(\text{PSL}_n(k)) = \frac{1}{h_n} \sum_{|\nu| = n} h^2_\nu \sum_{d,d'} h(d') c'_{dd'}\nu$$

in which the first sum is over all partitions $\nu$ of $n$, and the second is over all pairs of positive integers $d, d'$ such that $dd'$ divides $n$ and each $m_\nu(\nu)$. This formula can be turned around in the same way as in §4 to give

$$(5.1) \quad \hat{c}_n = \frac{1}{h_n} \sum_{d_1,d_2} \phi(d_1) \phi(d_2) c'_{n/d_1d_2}$$

summed over all pairs of divisors $d_1, d_2$ of $q-1$ such that $d_1d_2$ divides $n$; and equivalently
(5.2) \[ \hat{c}_n = \frac{1}{n} \sum_{a_1, a_2, b_1, b_2} u(a_1)u(a_2)h_1 h_2 b_1 b_2 c_n / a_1 a_2 b_1 b_2 , \]

summed over all quadruples \( a_1, a_2, b_1, b_2 \) of positive integers such that \( a_1 a_2 b_1 b_2 \) divides \( n \).

In Table 4 below we list the values of \( \hat{c}_n \) for \( 1 \leq n \leq 8 \); these are straightforward to calculate from (5.2) and Table 2.

<table>
<thead>
<tr>
<th>TABLE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{c}_1 = 1 )</td>
</tr>
<tr>
<td>( \hat{c}_2 = \frac{1}{h_2} \left( q^2 + qh_2 - 3 \right) )</td>
</tr>
<tr>
<td>( \hat{c}_3 = \frac{1}{h_3} \left( q^2 + q + qh_2 + qh_3 - 5 \right) )</td>
</tr>
<tr>
<td>( \hat{c}_4 = \frac{1}{h_4} \left( q^3 + q^2 + q(qh_2 - 3 + h_1 h_4 + 3h_2 - 5) \right) )</td>
</tr>
<tr>
<td>( \hat{c}_5 = \frac{1}{h_5} \left( q^4 + q^3 + q^2 + qh_5 - 8 \right) )</td>
</tr>
<tr>
<td>( \hat{c}_6 = \frac{1}{h_6} \left( q^5 + q^4 + q^3 + q^2 + q(qh_2 - 3 + 9h_3 - 9) + 5(qh_2 - 3)(h_3 - 1) \right) )</td>
</tr>
<tr>
<td>( \hat{c}_7 = \frac{1}{h_7} \left( q^6 + q^5 + q^4 + q^3 + 9h_7 - 10 \right) )</td>
</tr>
<tr>
<td>( \hat{c}_8 = \frac{1}{h_8} \left( q^7 + q^6 + q^5 + q^4 + q^3 (qh_2 - 3 + h_1 h_4 + 3h_2 - 6) \right) )</td>
</tr>
</tbody>
</table>

6. Unitary groups

Let \( k_2 \) be the unique quadratic extension of \( k \) contained in \( \overline{k} \).

The unitary group \( U_n(k_2) \) is the subgroup of all \( g \in \text{GL}_n(k_2) \) such that \( Fg = t g^{-1} \), where \( (Fg)_{i,j} = g_{i,j}^q \).

(6.1) Every conjugacy class of \( G = \text{GL}_n(k) \) which intersects \( U = U_n(k_2) \) does so in a single conjugacy class of \( U \).
Proof. Let $\sigma$ be the endomorphism of $G$ defined by $g^\sigma = (Fg)^{-1}$, so that $U$ is the group of fixed points of $\sigma$. Suppose that $u \in U$ and $g \in G$ are such that $gug^{-1} \in U$; we have to show that $u$ and $gug^{-1}$ are conjugate in $U$.

Since $gug^{-1} \in U$, we have $gug^{-1} = g^\sigma u g^{-\sigma}$, so that $g^{-1} g^\sigma \in C_u$, the centralizer of $u$ in $G$. Now centralizers in $G$ are connected ([4], III, 3.22), and $\sigma(C_u) \subset C_u$ because $u = u^\sigma$; since the kernel of $\sigma$ is finite it follows that $\sigma|_{C_u}$ is surjective, and hence by Lang's theorem ([5], §10) the mapping $h \mapsto h^{-1} h^\sigma$ from $C_u$ to $C_u$ is surjective. Hence we have $g^{-1} g^\sigma = h^{-1} h^\sigma$ for some $h \in C_u$, which shows that $v = gh^{-1} \in U$ and therefore $gug^{-1} = gh^{-1} v g^{-1} = uv^{-1}$. Hence $u$ and $gug^{-1}$ are conjugate in $U$. \hfill //

In particular, if two elements of $U_n(k_2)$ are conjugate in $GL_n(k_2)$, they are conjugate in $U_n(k_2)$. It follows that the conjugacy classes of $U_n(k_2)$ are parametrized bijectively by the partition-valued functions $\mu$ on $M = \overline{k}^*$ which satisfy (1.3) and $\mu(x^{-q}) = \mu(x)$ for all $x \in M$ in place of (1.4). Equivalently, as in §1, each conjugacy class of $U_n(k_2)$ may be represented by a sequence of polynomials $u = (u_1(t), u_2(t), \ldots)$ with coefficients in $k_2$ which satisfy (1.9) and are such that $u_i(0) = 1$ and the set of roots of each $u_i$ is stable under $x \rightarrow x^{-q}$ (in place of (1.8)). We define the type $\nu$ of a conjugacy class in $U_n(k_2)$ as in §1 to be the partition $(1^{n_1} 2^{n_2} \ldots)$ of $n$, where $n_i = \deg u_i$. It is easily seen that the number of possibilities for $u_i$ of degree $n_i$ is $q^{n_i} + q^{n_i-1}$ if $n_i > 0$ (and 1 if $n_i = 0$), so that the number of conjugacy classes of type $\nu$ in $U_n(k_2)$ is...
(6.2) \[ \gamma_v = \prod_{n_v > 0} \left( q_{n_v}^{\gamma_v} n_v^{-1} \right) \]

or equivalently

(6.2') \[ \gamma_v = q^{l(v)-p(v)(q+1)p(v)} \]

where \( l(v) \) is the number of parts of \( v \) and \( p(v) \) is the number of unequal parts. Hence the total number of conjugacy classes in \( U^n (k_2) \) is

(6.3) \[ \gamma_n = \sigma(U^n (k_2)) = \sum_{|\nu|=n} \gamma_v . \]

Since \( q^n + q^{n-1} \) is the coefficient of \( t^n \) in \( (1+t)/(1-qt)^{-1} \), if \( n > 0 \), it follows from (6.2) and (6.3) that the generating function for the numbers \( \gamma_n \) is

(6.4) \[ \Gamma(t) = \sum_{n=0}^{\infty} \gamma_n t^n = \prod_{r=1}^{\infty} \frac{1+t^r}{1-qt^r} . \]

This formula is due to Wall [6].

Each \( \gamma_n \) \( (n \geq 1) \) is a polynomial in \( q \), divisible by \( (q+1) \), as one sees for example by putting \( q = -1 \) in (6.4). Let \( \gamma'_n = \gamma_n/(q+1) \), so that from (6.2') and (6.3) we have

(6.5) \[ \gamma'_n = \sum_{|\nu|=n} q^{l(v)-p(v)(q+1)p(v)-1} . \]

In Table 5 (p. 46) we list the polynomials \( \gamma'_n \) for \( 1 \leq n \leq 10 \). They are easily calculated from the formula (6.5).

The formulas of §§2-5 all have their analogues for the unitary groups. Since the proofs are the same, we shall merely state the results.

For any positive integer \( d \), let \( \eta_d \) denote the highest common factor of \( d \) and \( q+1 \).

(i) \( \Gamma_{U^n (k_2)} \). Corresponding to (2.1) we have

(6.6) \[ \bar{\gamma}_n = \sigma(U^n (k_2)) = \sum_{d, d'} \eta_d \mu(d') \gamma'_{n/dd'} . \]
summed over all pairs of positive integers \( d, d' \) such that \( dd' \) divides \( n \), or alternatively

\[
(6.7) \quad \bar{\gamma}_n = \sum_{d} \phi(d) \gamma_{n/d}
\]

summed over the divisors \( d \) of \( \eta_n = (n, q+1) \). Hence the generating function for the \( \bar{\gamma}_n \) is (with \( \bar{\gamma}_0 = 1 \))

\[
(6.8) \quad \bar{\Gamma}(t) = \sum_{n=0}^{\infty} \bar{\gamma}_n t^n = \frac{1}{q+1} \sum_{d|q+1} \phi(d) \Gamma(t^d),
\]

where \( \Gamma(t) \) is the generating function (6.4).

TABLE 5

\[
\begin{align*}
\gamma_1' &= 1 \\
\gamma_2' &= q + 1 \\
\gamma_3' &= q^2 + q + 2 \\
\gamma_4' &= q^3 + q^2 + 3q + 2 \\
\gamma_5' &= q^4 + q^3 + 3q^2 + 4q + 3 \\
\gamma_6' &= q^5 + q^4 + 3q^3 + 5q^2 + 6q + 4 \\
\gamma_7' &= q^6 + q^5 + 3q^4 + 5q^3 + 8q^2 + 9q + 5 \\
\gamma_8' &= q^7 + q^6 + 3q^5 + 5q^4 + 9q^3 + 12q^2 + 13q + 6 \\
\gamma_9' &= q^8 + q^7 + 3q^6 + 5q^5 + 9q^4 + 14q^3 + 19q^2 + 17q + 8 \\
\gamma_{10}' &= q^9 + q^8 + 3q^7 + 5q^6 + 9q^5 + 15q^4 + 22q^3 + 27q^2 + 23q + 10
\end{align*}
\]

(ii) \( \text{SU}_n(k_2) \). In place of (3.5) and (3.6) we obtain

\[
(6.9) \quad \tilde{\gamma}_n = c(\text{SU}_n(k_2)) = \sum_{d,d'} \eta_{d}^2 \mu(d') \gamma_{n/dd'},
\]

so that \( \tilde{\gamma}_n \) is obtained from \( \bar{\gamma}_n \) by replacing each \( \eta_d \) by \( \eta_d^2 \), and equivalently
Conjugacy classes in finite groups

\begin{equation}
(6.10) \quad \hat{\gamma}_n = \sum_{d} \phi_2(d) \gamma'_n / d
\end{equation}

summed over the divisors $d$ of $\eta_n$. Consequently the generating function for the $\hat{\gamma}_n$ is (with $\hat{\gamma}_0 = 1$)

\begin{equation}
(6.11) \quad \hat{\Gamma}(t) = \sum_{n=0}^{\infty} \hat{\gamma}_n t^n = \frac{1}{q+1} \sum_{d \mid q+1} \phi_2(d) \Gamma^d(t^d) - q.
\end{equation}

(iii) Groups isogenous to $SU_n$. If $\Gamma_e$ is the quotient of $SU_n$ by its central subgroup isomorphic to $\mu_f$ (where $ef = n$ as in §4) then in place of (4.4) we find

\begin{equation}
(6.12) \quad d(\Gamma_e(k_2)) = \sum_{d_1, d_2} \phi(d_1) \phi_2(d_2) \gamma'_n / d_1 d_2
\end{equation}

summed over pairs of positive integers $d_1, d_2$ such that $d_1$ divides $\eta_f$ and $d_2$ divides $\eta_e$.

(iv) $PSU_n(k_2)$. This is the quotient of $SU_n(k_2)$ by its centre, and in place of (5.1) the number of its conjugacy classes is

\begin{equation}
(6.13) \quad \tilde{\gamma}_n = \sigma(PSU_n(k_2)) = \frac{1}{n^n} \sum_{d_1, d_2} \phi(d_1) \phi_2(d_2) \gamma'_n / d_1 d_2
\end{equation}

summed over all pairs of divisors $d_1, d_2$ of $q+1$ such that $d_1 d_2$ divides $n$.

References


Department of Pure Mathematics,  
Queen Mary College,  
University of London,  
Mile End Road,  
London E14NS,  
England.