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Some Properties Associated with Adequate Transversals

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Abstract. In this paper, another relationship between the quasi-ideal adequate transversals of an abundant semigroup is given. We introduce the concept of a weakly multiplicative adequate transversal and the classic result that an adequate transversal is multiplicative if and only if it is weakly multiplicative and a quasi-ideal is obtained. Also, we give two equivalent conditions for an adequate transversal to be weakly multiplicative. We then consider the case when I and Λ (defined below) are bands. This is analogous to the inverse transversal if the regularity condition is adjoined.

1 Introduction

The concept of an inverse transversal of a regular semigroup was first introduced by Blyth and McFadden in 1982 [3]. Afterwards, this class of regular semigroups attracted several authors' attention, and a series of important results were obtained [1, 3, 11–14]. If S is a regular semigroup, then an inverse transversal of S is an inverse subsemigroup S^o such that S^o meets $V(a)$ precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{x \in S : axa = a \text{ and } xax = x\}$, denotes the set of inverses of a. The intersection of $V(a)$ and S^o is denoted by $V_{S^o}(a)$ and the unique element of $V_{S^o}(a)$ is denoted by a^o. It is well known that the sets $I = \{e \in S : ee^o = e\}$ and $\Lambda = \{f \in S : f^o f = f\}$ are left regular and right regular bands respectively, and play an important role in the study of regular semigroups with inverse transversals. An analogue of an inverse transversal, which is termed an adequate transversal, was introduced for abundant semigroups by El-Qallali [5]. Luo [10] has shown that if an abundant semigroup S has a quasi-ideal adequate transversal and if both I and Λ are sub-bands of S, then all quasi-ideal adequate transversals of S are isomorphic. This generalized a result of Saito [12], which showed that the quasi-ideal inverse transversals of a regular semigroups are mutually isomorphic, and a more general result that all inverse transversals of a regular semigroup are isomorphic [2]. In [12], Saito also showed that if a regular semigroup S has a multiplicative inverse transversal, then any quasi-ideal inverse transversal of S is multiplicative. The purpose of Section 2 is to show that if an abundant semigroup has a multiplicative adequate transversal, then any quasi-ideal adequate transversal is multiplicative. The relationship between the quasi-ideal adequate transversals of an abundant semigroup has been generalized and enriched. The weakly multiplicative inverse transversal was first introduced by Saito [11], and followed by Blyth and Almeida Santos [1]. As shown by Saito [11], the

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inverse transversal S^o is weakly multiplicative if and only if $I\Lambda = \{ef : e \in I, f \in \Lambda\}$ is the idempotent-generated subsemigroup of S with an inverse transversal $E(S^o)$. In [1], Blyth and Almeida Santos showed that an inverse transversal is multiplicative if and only if it is weakly multiplicative and a quasi-ideal. In Section 3, the concept of a weakly multiplicative adequate transversal is introduced and the classic connection between the properties of being multiplicative, weakly multiplicative, and a quasi-ideal is obtained. Also, we give two equivalent conditions for an adequate transversal to be weakly multiplicative. It is well known that if S^o is a quasi-ideal inverse transversal of a regular semigroup S, then I is a left normal band and Λ is a right normal band. In the general case, if S^o is an inverse transversal of a regular semigroup, Tang [14] has proved that both I and Λ are subsemigroups of S. But the corresponding result in the abundant case fails to be true [4]. Chen [4] gave an example illustrating that I and Λ need not be subsemigroups even if S^o is a quasi-ideal adequate transversal of S. In Section 4, we consider the case when I and Λ are bands and obtain an equivalent condition for I to be a band. If S satisfies the regularity condition, then I is a left regular band and Λ is a right regular band, analogous to the result of a regular semigroup with an inverse transversal. This further demonstrates that the adequate transversal is the natural generalization of the inverse transversal in the abundant case.

2 Preliminaries

We begin by recalling some results about abundant semigroups (see [7]). Let S be a semigroup and $a, b \in S$. By $a\mathcal{R}^*b$ we mean that $xa = ya$ if and only if $xb = yb$ for all $x, y \in S^1$. The relation $\mathcal{L}^*$ can be defined dually. $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence on S. If $a, b \in RegS$, the set of regular elements of S, then $a\mathcal{R}^*b$ ($a\mathcal{L}^*b$) if and only if $a\mathcal{R}b$ ($a\mathcal{L}b$). What is more, if S is a regular semigroup, then $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{L}^* = \mathcal{L}$. A semigroup S is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. An abundant semigroup S is called *quasi-adequate* if its idempotents form a band. Quasi-adequate semigroups are analogues of orthodox semigroups in the range of abundant semigroups. An *adequate semigroup* is a quasi-adequate semigroup in which the idempotents commute. Each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of an adequate semigroup contains a unique idempotent. For an element a of an adequate semigroup, the idempotent in the $\mathcal{L}^*$-class containing a will be denoted by a^*, the idempotent in the $\mathcal{R}^*$-class by a^+. Let S and S^o be semigroups. Throughout this paper we denote the set of idempotents of S and S^o by E and E^o, respectively. We list some basic results, which are frequently used in this paper.

The first result, Hall [9, Result 2], establishes a useful link in a semigroup between the regularity of a product of arbitrary regular elements and that of idempotents. Note that we denote the set of regular elements of a semigroup S by $Reg(S)$.

Lemma 2.1 ([9]) *Let S be an arbitrary semigroup. Then the following are equivalent:*

(i) *for all idempotents e and f of S the element ef is regular;*
(ii) *$\langle E \rangle$ is a regular subsemigroup;*
(iii) *$Reg(S)$ is a regular subsemigroup.*

Any semigroup satisfying this lemma will be said to satisfy the regularity condition.

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Lemma 2.2 ([7]) *Let a be an element of a semigroup S and e an idempotent of S. Then the following statements are equivalent:*

(i) $e\mathcal{R}^*a$;
(ii) $ea = a$ *and for all* $x, y \in S^1$, $xa = ya$ *implies* $xe = ye$.

Lemma 2.3 ([6]) *Let S be a semigroup. Then the following conditions are equivalent:*

(i) *S is adequate;*
(ii) *each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of S contains a unique idempotent and the subsemigroup generated by E is regular.*

Lemma 2.4 ([6]) *If S is an adequate semigroup, then*

(i) *for all* $a, b \in S, (ab)^* = (a^*b)^*$ *and* $(ab)^+ = (ab^+)^+$, *and*
(ii) *for all* $a, b \in S, a\mathcal{R}^*b$ *if and only if* $a^+ = b^+$; $a\mathcal{L}^*b$ *if and only if* $a^* = b^*$.

Let S be an abundant semigroup and U an abundant subsemigroup of S, U is called a $*$-subsemigroup of S if for any $a \in U$, there exists an idempotent $e \in L_a^*(S) \cap U$ and an idempotent $f \in R_a^*(S) \cap U$. As pointed out in [5], an abundant subsemigroup U of an abundant semigroup S is a $*$-subsemigroup of S if and only if $\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)$ and $\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$.

Definition 2.5 ([5]) Let S be an abundant semigroup, and let S^o be a $*$-adequate subsemigroup of S. S^o is called *an adequate transversal* of S if for each $x \in S$, there are a unique element $\overline{x} \in S^o$ and idempotents $e, f \in E$ such that $x = e\overline{x}f$, where $e\mathcal{L}\overline{x}^+$ and $f\mathcal{R}\overline{x}^*$. It can easily be shown that e and f are uniquely determined by x and S^o (see [5]). Hence we denote e by e_x and f by f_x.

Let S^o be an adequate transversal of an abundant semigroup S, then for any $x \in S, x = e_x\overline{x}f_x$, moreover, $e_x\mathcal{R}^*x\mathcal{L}^*f_x$. If $a \in S^o$, we can easily check that $e_a = a^+$, $f_a = a^*$ and $\overline{a} = a$. Furthermore, if $a \in E^o$, then $\overline{a} = e_a = f_a = a$. We say that the adequate transversal S^o is *multiplicative* if for any $x, y \in S$, $f_xe_y \in E^o$. A subsemigroup S^o of S is called *a quasi-ideal* of S if $S^oSS^o \subseteq S^o$.

Let S^o be an adequate transversal of an abundant semigroup S. We write

$$I = \{e_x : x \in S\}; \quad \Lambda = \{f_x : x \in S\}.$$

Lemma 2.6 ([4]) *Let S be an abundant semigroup with an adequate transversal S^o, then*

(i) $I \cap \Lambda = E^o$;
(ii) $I = \{e \in E : (\exists l \in E^o)\, l\mathcal{L}e\}, \Lambda = \{f \in E : (\exists h \in E^o)\, h\mathcal{R}f\}$;
(iii) $IE^o \subseteq I, E^o\Lambda \subseteq \Lambda$.

Lemma 2.7 ([4]) *Let S^o be an adequate transversal of an abundant semigroup S, then*

(i) *S^o is a quasi-ideal of S if and only if $\Lambda I \subseteq S^o$*
(ii) *if S^o is multiplicative, then S^o is a quasi-ideal of S.*

Lemma 2.8 ([4]) *Let S be an abundant semigroup with an adequate transversal S^o, $RegS$ denote the set of regular elements of S. Then*

(i) $(\forall x \in RegS) \quad |V(x) \cap S^o| \leq 1$, *and*

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(ii) *if S^o is a quasi-ideal of S, then $(\forall x \in RegS)$ $|V(x) \cap S^o| = 1$.*

Lemma 2.9 ([4]) *Let S^o be an adequate transversal of an abundant semigroup S, then*

(i) *$x\mathcal{R}^* y$ if and only if $e_x = e_y$;*
(ii) *$x\mathcal{L}^* y$ if and only if $f_x = f_y$.*

Lemma 2.10 ([10]) *Let S be an abundant semigroup with a quasi-ideal adequate transversal S^o, for any $x, y \in S$, then*

(i) *$\overline{xy} = \overline{x} f_x e_y \overline{y}$;*
(ii) *$e_{xy} = e_x(\overline{x} f_x e_y)^+$;*
(iii) *$f_{xy} = (f_x e_y \overline{y})^* f_y$.*

3 Another Relationship

Theorem 3.1 *Let S be an abundant semigroup with a multiplicative adequate transversal S^o, and let $\langle E \rangle$ be the idempotent-generated subsemigroup of S. Then $\langle E \rangle = \{x \in S : \overline{x} \in E^o\}$. Consequently, E^o is the semilattice transversal of $\langle E \rangle$.*

Proof It is clear that if $\overline{x} \in E^o$, then $x = e_x \overline{x} f_x \in \langle E \rangle$. For the converse part, it suffices to prove that $\overline{xy} \in E^o$ for all $x, y \in E$. By [8, Proposition 3.1], $\overline{x}$, $\overline{y} \in E^o$, so that $\overline{xy} = \overline{x} f_x e_y \overline{y} \in E^o$. ■

Theorem 3.2 *Let S^o be an adequate transversal of an abundant semigroup S. Let $a, b \in S^o$ and $c \in S$ be such that $a\mathcal{R}^* c\mathcal{L}^* b$, then $c \in S^o$. (In particular, $\mathcal{D}^{*S^o} = \mathcal{D}^{*S} \cap (S^o \times S^o)$.)*

Proof Since $a, b \in S^o$ and S^o is adequate, there exist $a^+, b^* \in E^o$, such that

$$a^+\mathcal{R}^* a\mathcal{R}^* c\mathcal{L}^* b\mathcal{L}^* b^*.$$

From the definition of an adequate transversal, we have $c = e_c \overline{c} f_c$, where $e_c \mathcal{L} \overline{c}^+$ and $f_c \mathcal{R} \overline{c}^*$. Moreover, $e_c \mathcal{R}^* c\mathcal{L}^* f_c$. We deduce that $a^+\mathcal{R}^* c\mathcal{R}^* e_c \mathcal{L} \overline{c}^+$, consequently $e_c \in \Lambda \cap I = E^o$. Similarly, $\overline{c}^* \mathcal{R} f_c \mathcal{L}^* c\mathcal{L}^* b^*$, $f_c \in E^o$. That is, $c = e_c \cdot \overline{c} \cdot f_c \in E^o \cdot S^o \cdot E^o \subseteq S^o$. ■

Theorem 3.3 *Let S^o be an adequate transversal of an abundant semigroup S, then for every regular element x of S, $|V(x) \cap S^o| = 1$.*

Proof For every regular element x, since x, e_x, and f_x are all regular, from $e_x \mathcal{R}^* x\mathcal{L}^* f_x$ we deduce that $e_x \mathcal{R} x \mathcal{L} f_x$, so by the Miller–Clifford Theorem x has an inverse $x' \in R_{f_x} \cap L_{e_x}$. Thus $\overline{x}^* \mathcal{R} f_x \mathcal{R} x' \mathcal{L} e_x \mathcal{L} \overline{x}^+$, so by Theorem 3.2 $x' \in S^o$. Now let $(x')^{-1}$ denote the unique inverse of x' in the adequate semigroup S^o. Then from $x = xx' \cdot (x')^{-1} \cdot x'x$ and the definition of an adequate transversal, the uniqueness of $x' \in S^o$ is obvious. That is, $|V(x) \cap S^o| = 1$. ■

Remark Theorem 3.3 shows that, in the regular case, the definition of abundant semigroups with adequate tranversals coincides with the definition of regular semigroups with inverse transversals. Consequently, let x^o denote the unique inverse of the regular element x in S^o.

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Theorem 3.4 *Let S^o be an adequate transversal of an abundant semigroup S, then the relation $\mathcal{H}^*$ on S saturates S^o. That is, S^o is the union of some $\mathcal{H}^*$-classes on S.*

Proof Let H^* be an $\mathcal{H}^*$-class of S, $H^* \cap S^o \neq \varnothing$. We will prove that $H^* \subseteq S^o$. Take $s^o \in H^* \cap S^o$, since S^o is adequate, then there exist $e^o, f^o \in E^o$ such that $e^o \mathcal{R}^*(S^o) s^o \mathcal{L}^*(S^o) f^o$ and since S^o is $*$-adequate subsemigroup, then we can assume $e^o \mathcal{R}^*(S) s^o \mathcal{L}^*(S) f^o$. Now take $x \in H^*, x = e_x \overline{x} f_x$, and $e_x \in I$, $f_x \in \Lambda$. It follows from $e_x \mathcal{R}^* x$ that $e_x \mathcal{R}^* e^o$, hence $e_x \in \Lambda$. Consequently $e_x \in I \cap \Lambda = E^o$. Similarly $f_x \in E^o$. Therefore $x = e_x \overline{x} f_x \in E^o \cdot S^o \cdot E^o \subseteq S^o$. ∎

In 2003, Luo [10] showed that if an abundant semigroup S has a quasi-ideal adequate transversal and if both I and Λ are sub-bands of S then all quasi-ideal adequate transversals of S are isomorphic. But she did not give an isomorphism map. In what follows we will give one such map, but we only give the outline of the proof; for more details, see [10]. Using this map, we shall show another relationship between the adequate transversals of an abundant semigroup.

Theorem 3.5 *Let S be an abundant semigroup that satisfies the regularity condition. For any quasi-ideal adequate transversals S^o and $S^\square$ of S there is an isomorphism $\varphi\colon S^o \to S^\square$ given by $\varphi(\overline{x}) = (\overline{x} f_x)^*$, where*

$$x \in Q = \{x \in S : e_x \in E(S^o)\} \cap \{x \in S : r_x \in E(S^\square)\}.$$

Proof In fact, the map φ is well defined. For $\overline{x}, \overline{y} \in S^o$, if $\overline{x} = \overline{y}$, then $f_x = f_y$ for $x, y \in Q$. Thus $\overline{x} f_x = \overline{y} f_y$, and whence $(\overline{x} f_x)^* = (\overline{y} f_y)^*$.

For any $\overline{x},\ \overline{y} \in S^o$, we have

$$\begin{aligned} \varphi(\overline{x} \cdot \overline{y}) &= \varphi(\overline{xy}) = (\overline{xy} f_{xy})^* = (\overline{x} f_x e_y \overline{y} \cdot (f_x e_y \overline{y})^* f_y)^* \\ &= (\overline{x} f_x e_y \overline{y} f_y)^* = (\overline{x} f_x \cdot y)^* = (xy)^* \\ &= x^* y^* = (\overline{x} f_x)^* (\overline{y} f_y)^* = \varphi(\overline{x}) \varphi(\overline{y}) \end{aligned}$$

and so φ is a homomorphism.

To show that ρ is surjective, for any $z \in S^\square$, we have $z = z^{*+} z z^{**} = e_z \overline{z} \cdot f_z$ and $\varphi(\overline{z}) = (\overline{z} f_z)^* = z^* = z$, since $z \in S^\square$.

Suppose that $\varphi(\overline{x}) = \varphi(\overline{y})$ for $\overline{x}, \overline{y} \in S^o$. Then $(\overline{x} f_x)^* = (\overline{y} f_y)^*$, that is, $x^* = y^*$. Similarly, $r_x = r_y$, where $x, y \in Q$. Thus $\overline{x^* r_x} = \overline{y^* r_y}$, that is , $\overline{x} = \overline{y}$. ∎

Theorem 3.6 *Let S be an abundant semigroup and suppose that S has a multiplicative adequate transversal. Then any quasi-ideal adequate transversal of S is multiplicative.*

Proof Let $S^\square$ be a multiplicative adequate transversal of S and S^o a quasi-ideal adequate transversal of S. Then by [8, Proposition 3.4], S satisfies the regularity condition, and so by Theorem 3.5, S^o is isomorphic to $S^\square$ under the mapping $\varphi(\overline{x}) = (\overline{x} f_x)^* = x^*$ for each $\overline{x} \in S^o$. Let $x, y \in S$. Then, since $f_x e_y \in \langle E \rangle$ and $f_x e_y \in S^o$, let $f_x e_y = \overline{z}$ for some $z \in S$. Then $\varphi(f_x e_y) = \varphi(\overline{z}) = (\overline{z} f_z)^*$, and since $\overline{z} f_z = f_x e_y f_z \in \langle E \rangle$, from Theorem 3.1 we deduce that $(\overline{z} f_z)^* \in E(S^\square)$. Therefore $f_x e_y$ is an idempotent in S^o, since φ is an isomorphism. Consequently, S^o is multiplicative. ∎

Theorem 3.7 *Let S be an abundant semigroup with an adequate transversal S^o. Then S^o is multiplicative if and only if S^o is a quasi-ideal of S that satisfies the regularity condition and $\overline{e} \in E^o$ for every $e \in E$.*

Proof If S^o is multiplicative, it is clear that S^o is a quasi-ideal. By [8, Proposition 3.1], $\overline{e} \in E^o$ for every $e \in E$, and by [8, Proposition 3.4], S satisfies the regularity condition.

Conversely, let $f \in \Lambda$, and $e \in I$, then by the regularity condition fe is a regular element of S. Let x denotes the unique inverse of fe in S^o. Then exf is idempotent and $fe \in V(exf) \cap S^o$, so $fe = (exf)^o$ and from the assumption we have that fe is idempotent. Consequently, $\Lambda I \subseteq S^o \cap E = E^o$, so S^o is multiplicative.

In the regular case, if S^o is the inverse transversal, then S^o is multiplicative if and only if S^o is a quasi-ideal and $e^o \in E^o$ for every idempotent e of the regular semigroup S. In the abundant case, the regularity condition cannot be removed. An example was given by Chen [4] in 2000.

Let $S = \{e, g, h, w, f\}$ with the following multiplication table.

	e	g	h	w	f
e	e	g	e	g	g
g	g	g	g	g	g
h	h	g	h	g	g
w	w	g	w	g	g
f	g	g	w	w	f

Notice that g is the zero element of S; it is easy to check the associativity. From the multiplication table we obtain

(i) $e\mathcal{L}^*h\mathcal{L}^*w\mathcal{R}^*f; E = \{e, g, h, f\}$;
(ii) $S^o = \{w, e, f, g\}$ is a subsemigroup of S and $S^oS \subseteq S^o$;
(iii) $L_g^*(S^o) = L_g^*(S) = R_g^*(S^o) = L_g^*(S) = \{g\}$;
(iv) $L_w^*(S^o) = \{w, e\}; L_w^*(S) = \{w, e, h\}$;
(v) $R_w^*(S^o) = R_w^*(S) = \{w, f\}$.

Therefore S^o is a $*$-adequate subsemigroup with the semilattice of idempotents $E^o = \{e, f, g\}$. Each element of S can be uniquely written in the form

$$e = e \cdot e \cdot e; \quad w = f \cdot w \cdot e; \quad g = g \cdot g \cdot g; \quad f = f \cdot f \cdot f; \quad h = h \cdot e \cdot e.$$

$S^oS \subseteq S^o$ implies $S^oSS^o \subseteq S^o$. Hence S^o is a quasi-ideal adequate transversal of S.

This example also illustrates situations where an abundant semigroup S with a quasi-ideal adequate transversal S^o and $E = I = \{e, h, f, g\}$, where $\overline{e} = \overline{h} = e \in E^o$, and $f, g \in E^o$, that is, $\overline{e} \in E^o$ for all $e \in E$, does not satisfy the regularity condition and is not multiplicative, since $fh = w \notin E^o$. ■

4 Weakly Multiplicative Adequate Transversals

Lemma 4.1 *Let S be an abundant semigroup with an adequate transversal S^o. If $\overline{g} \in E^o$ for $g \in S$, then g is regular and $\overline{g}$ is the unique inverse in S^o of g. In this case, $g^o = \overline{g}$ and $e_g = g\overline{g}$, $f_g = \overline{g}g$.*

Proof From the assumption $\overline{g} \in E^o$ we deduce that $(\overline{g})^+ = (\overline{g})^* = \overline{g}$, hence $g = e_g \cdot \overline{g} f_g = e_g f_g$. Consequently,

$$g\overline{g}g = e_g f_g \cdot \overline{g} \cdot e_g f_g = e_g \overline{g} f_g = g \text{ and } \overline{g}g\overline{g} = \overline{g} e_g f_g \overline{g} = \overline{g},$$

that is, $\overline{g} \in V(g) \cap S^o$. We can therefore write g in the form $g = g\overline{g} \cdot \overline{g} \cdot \overline{g}g$, where $g\overline{g}\mathcal{L}\overline{g}\mathcal{R}\overline{g}g$. Consequently, $e_g = g\overline{g}$ and $f_g = \overline{g}g$. ∎

Definition 4.2 Let S be an abundant semigroup with an adequate transversal S^o. Then we say that the adequate transversal S^o is weakly multiplicative if S satisfies the regularity condition and for any $x, y \in S$, $\overline{f_x e_y} \in E^o$.

If S^o is weakly multiplicative, then by Lemma 4.1, $\overline{f_x e_y}$ is just the unique inverse of $f_x e_y$ in S^o. If S^o is the multiplicative adequate transversal of S, then S satisfies the regularity condition and for any $x, y \in S$, $f_x e_y \in E^o$, consequently, $\overline{f_x e_y} = f_x e_y \in E^o$, that is, S^o is weakly multiplicative.

Theorem 4.3 *Let S be an abundant semigroup with a weakly multiplicative adequate transversal S^o. Then $\overline{e} \in E^o$ for every $e \in E$.*

Proof For any $g \in E$, let g^o be the unique inverse of g in S^o. Then $g^o = g^o g g^o = g^o g \cdot g g^o = f_g e_g$. Thus $g^o = \overline{g^o} = \overline{f_g e_g} \in E^o$, since $g^o \in S^o$ and S^o is weakly multiplicative. From $g = gg^o \cdot g^o \cdot g^o g$ we deduce that $\overline{g} = g^o \in E^o$.

In fact, we have the following connection between the properties of being multiplicative, weakly multiplicative and a quasi-ideal. ∎

Theorem 4.4 *Let S be an abundant semigroup with an adequate transversal S^o. Then S^o is multiplicative if and only if S^o is weakly multiplicative and a quasi-ideal.*

Proof The conditions are clearly necessary.

Conversely, if S^o is weakly multiplicative and a quasi-ideal, then S satisfies the regularity condition and by Theorem 4.3, $\overline{e} \in E^o$ for every $e \in E$. Consequently, by Theorem 3.7, S^o is multiplicative. ∎

Theorem 4.5 *Let S be an abundant semigroup with an adequate transversal S^o. Then S^o is weakly multiplicative if and only if $\langle E \rangle = \{x \in S : \overline{x} \in E^o\}$.*

Proof Suppose that S^o is weakly multiplicative, then by Definition 4.2, S satisfies the regularity condition. Take $x \in S$. If $\overline{x} \in E^o$, from the definition of an adequate transversal,

$$x = e_x \cdot \overline{x} \cdot f_x \in \langle E \rangle.$$

For the other inclusion, it suffices to prove that $\overline{xy} \in E^o$ for all $x, y \in E$. By the regularity condition, for $x, y \in E$, xy is regular and $(xy)^o$ is the unique inverse of xy in S^o. Similar to the inverse transversal, it is a routine matter to prove that $(xy)^o = y^o (f_x e_y)^o x^o$, where x^o, y^o denote the unique inverse in S^o of x, y, respectively. From Theorem 4.3, we have $x^o, y^o \in E^o$, and from Definition 4.2 we have $\overline{f_x e_y} \in E^o$, hence $(f_x e_y)^o = \overline{f_x e_y} \in E^o$ by Lemma 4.1. Consequently, $(xy)^o = y^o (f_x e_y)^o x^o \in E^o$, and thus $\overline{xy} = (xy)^o \in E^o$.

Conversely, if $\langle E\rangle = \{x \in S : \overline{x} \in E^o\}$, then for any $x, y \in S$, $\overline{f_x e_y} \in E^o$, since $f_x e_y \in \Lambda I$. From Lemma 4.1 we deduce that every element in $\langle E\rangle$ is regular, that is, E generates a regular semiband. By Definition 4.2, S^o is weakly multiplicative.

In this case, E^o is the semilattice transversal of the regular semigroup $\langle E\rangle$. ■

Theorem 4.6 *Let S be an abundant semigroup with an adequate transversal S^o. Then S^o is weakly multiplicative if and only if $I\Lambda = \{ef : e \in I, f \in \Lambda\}$ is the idempotent-generated subsemigroup of S.*

Proof Suppose that S^o is weakly multiplicative. Take $x \in \langle E\rangle$, then by Theorem 4.5, we have $\overline{x} \in E^o$. In this case, $\overline{x}$ is the unique inverse of x in S^o. Thus $x = x\overline{x}x = x\overline{x} \cdot \overline{x}x = e_x f_x \in I\Lambda$, and $I\Lambda \subseteq \langle E\rangle$ is obvious.

Conversely, take $x \in \langle E\rangle$, then $x = e_a f_b$ for some $a, b \in S$. Consequently, we have

$$
\begin{aligned}
x \cdot \overline{b}^* \overline{a}^+ \cdot x &= e_a f_b \cdot \overline{b}^* \overline{a}^+ \cdot e_a f_b \\
&= e_a \overline{b}^* \overline{a}^+ f_b \qquad (e_a \mathcal{L} \overline{a}^+,\ f_b \mathcal{R} \overline{b}^*) \\
&= e_a \cdot \overline{a}^+ \cdot \overline{b}^* f_b \quad (\overline{a}^+, \overline{b}^* \in E^o) \\
&= e_a f_b \qquad (e_a \mathcal{L} \overline{a}^+,\ f_b \mathcal{R} \overline{b}^*) \\
&= x
\end{aligned}
$$

and

$$
\overline{b}^* \overline{a}^+ \cdot x \cdot \overline{b}^* \overline{a}^+ = \overline{b}^* \overline{a}^+ \cdot e_a f_b \cdot \overline{b}^* \overline{a}^+ \cdot = \overline{b}^* \overline{a}^+ \cdot \overline{b}^* \overline{a}^+ = \overline{b}^* \overline{a}^+.
$$

Thus $\overline{b}^* \overline{a}^+ \in V(x) \cap E^o$, that is, $\overline{x} = \overline{b}^* \overline{a}^+ \in E^o$. From the proof of Theorem 4.5 we deduce that S^o is weakly multiplicative.

As for the inverse transversal, if the regular semigroup S is orthodox, then all the inverse transversals are weakly multiplicative. For the adequate transversal, we have a similar result. ■

Theorem 4.7 *Let S be an abundant semigroup with an adequate transversal S^o. If S is quasi-adequate, then S^o is weakly multiplicative.*

Proof If S is quasi-adequate, then certainly S satisfies the regularity condition, since the set of idempotents of S form a band. For any $g \in E$, by hypothesis, $g = e_g \overline{g} f_g$. Since $e_g \mathcal{L} \overline{g}^+$ and $f_g \mathcal{R} \overline{g}^*$, we have that $\overline{g}^+ = \overline{g}^+ e_g$ and $\overline{g}^* = f_g \overline{g}^*$. So that

$$
\overline{g} = \overline{g}^+ \overline{g}\, \overline{g}^* = \overline{g}^+ e_g \overline{g} f_g \overline{g}^* = \overline{g}^+ g \overline{g}^* \in E \cap S^o = E^o.
$$

Consequently, for any $x, y \in S$, $f_x e_y \in \Lambda I \subseteq E$, hence $\overline{f_x e_y} \in E^o$. By the definition, S^o is weakly multiplicative. ■

Corollary 4.8 *Let S be a quasi-adequate semigroup with an adequate transversal S^o. Then S^o is multiplicative if and only if S^o is a quasi-ideal.*

Theorem 4.9 *Let S be an abundant semigroup with an adequate transversal S^o. Then the following conditions are equivalent:*

(i) *S is quasi-adequate;*

(ii) *for all $x, y \in RegS$ ($RegS$ denotes the set of regular elements of S), $(xy)^o = y^o x^o$;*

(iii) *for all $x, y \in E$, $(xy)^o = y^o x^o$.*

Proof (i) $\Longrightarrow$ (ii). If S is quasi-adequate, for all $x, y \in RegS$, let x^o, y^o denote the unique inverses of x and y in S^o respectively. Computing,

$$\begin{aligned} xy \cdot y^o x^o \cdot xy &= x \cdot x^o xyy^o \cdot x^o xyy^o \cdot y \\ &= x \cdot x^o xyy^o \cdot y \qquad (x^o xyy^o \in E \cdot E \subseteq E) \\ &= xy, \\ y^o x^o \cdot xy \cdot y^o x^o &= y^o \cdot yy^o\, x^o x \cdot yy^o\, x^o x \cdot x^o \\ &= y^o \cdot yy^o\, x^o x \cdot x^o \\ &= y^o x^o, \end{aligned}$$

hence $y^o x^o \in V(xy) \cap S^o$, that is, $(xy)^o = y^o x^o$.

(ii) $\Longrightarrow$ (iii). This is trivial.

(iii) $\Longrightarrow$ (i). If $(xy)^o = y^o x^o$ for all $x, y \in E$, then $g \in E$ if and only if $g^o \in E^o$. For this, we first notice that $g \in E$ implies $g^o = (g \cdot g)^o = g^o g^o$, hence $g^o \in E^o$. Secondly, $g^o \in E^o$ implies $g^o = (g^o)^2 = (g^2)^o \in E^o$, hence g^2 is regular and $(g^2)^o$ is the unique inverse in S^o of g^2 . Consequently,

$$g = gg^o g = g(g^o)^2 g = g(g^2)^o g = g(g^2)^o \cdot g^2 \cdot (g^2)^o \cdot g = gg^o g \cdot gg^o g = gg,$$

that is, $g \in E$. Let $g, h \in E$. Then $(gh)^o = h^o g^o \in E^o$, so that $gh \in E$. Thus S is quasi-adequate.

If one of the conditions in Theorem 4.9 is satisfied, then by Theorem 4.7, the adequate transversal S^o is weakly multiplicative. ■

5 The Case when I is a Band

Lemma 5.1 *Let S be an abundant semigroup with an adequate transversal S^o. If S satisfies the regularity condition, then for any $x, y \in RegS$, we have*

$$(xy)^o = (x^o xy)^o x^o = y^o (xyy^o)^o,$$

where a^o denotes the unique inverse in S^o of a .

Proof Notice that

$$xy \cdot (x^o xy)^o x^o \cdot xy = x \cdot x^o xy \cdot (x^o xy)^o \cdot x^o xy = x \cdot x^o xy = xy,$$

$$(x^o xy)^o x^o \cdot xy \cdot (x^o xy)^o x^o = (x^o xy)^o x^o.$$

Therefore $(x^o xy)^o x^o = (xy)^o$ and similarly $y^o (xyy^o)^o = (xy)^o$. ■

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Theorem 5.2 *Let S be an abundant semigroup with an adequate transversal S^o. If S satisfies the regularity condition, then I and Λ are bands. Consequently, I is a left regular band, and Λ is a right regular band.*

Proof Suppose that $e, f \in I$ and consider the element $g = f(ef)^o e$, where $(ef)^o$ denotes the unique inverse in S^o of ef. Clearly, $g \in E$, $ge = g = fg$, and $egf = ef$. By Lemma 5.1, $(xy)^o = (x^o xy)^o x^o = y^o(xyy^o)^o$ for any $x, y \in RegS$, and if $e \in I$, then $e^o \in E^o$ and $e\mathcal{L}e^o$. Thus we have

$$g = f(e^o ef)^o e^o e = f(e^o ef)^o e^o = f(ef)^o.$$

Consequently, $eg = ef(ef)^o \in I$. We then have $(eg)^o \in E^o$ and $g\mathcal{L}^* eg\mathcal{L}^*(eg)^o$, so that $g \in I$ and $g^o = (eg)^o$. Also,

$$g^o = [f(ef)^o]^o = [f^o f(ef)^o]^o f^o = [f^o f(ef)^o]^o f^o f = g^o f.$$

Consequently, $g = gg^o = gg^o f = gf$. Therefore $ef = egf = eg$ and $eg \in I$. Hence I is a subsemigroup; and similarly so is Λ. By Lemma 2.9, I is left regular, and Λ is right regular. ■

Theorem 5.3 *Let S be an abundant semigroup with an adequate transversal S^o. Then I is a band if and only if $E^o I \subseteq I$.*

Proof The condition is clearly necessary.

Conversely, suppose that $E^o I \subseteq I$. Take $e, f \in I$, then $e^o, f^o \in E^o$, and $e^o\mathcal{L}e$, $f^o\mathcal{L}f$. By assumption $e^o f \in I$, thus $(e^o f)^o \in E^o$. Computing,

$$(e^o f)^o e^o \cdot e \cdot e^o f \cdot (e^o f)^o e^o = (e^o f)^o \cdot e^o f \cdot (e^o f)^o e^o = (e^o f)^o e^o,$$
$$e \cdot e^o f \cdot (e^o f)^o e^o \cdot e \cdot e^o f = e \cdot e^o f \cdot (e^o f)^o \cdot e^o f = e \cdot e^o f.$$

Consequently, $(e^o f)^o e^o$ is the unique inverse in S^o of $ee^o f = ef$. Consider $g = f(e^o f)^o e^o$, then $g \in E$, $ge = g = fg$, and $egf = ef$. Since

$$g = f(e^o f)^o e^o \mathcal{L}^* f^o (e^o f)^o e^o = (e^o f)^o e^o f^o \in E^o,$$

thus $g \in I$ and $g^o = (e^o f)^o e^o f^o$, consequently $g^o f = g^o$. Hence $ef = egf = egg^o f = egg^o = eg$ and $eg = ef(e^o f)^o e^o = ef \cdot (ef)^o \in I$. Therefore, $ef \in I$ and I is a band. ■

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