MAXIMAL SUBSETS OF PAIRWISE NONCOMMUTING ELEMENTS OF SOME $p$-GROUPS OF MAXIMAL CLASS

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Abstract

Let $G$ be a group. A subset $X$ of $G$ is a set of pairwise noncommuting elements if $xy \neq yx$ for any two distinct elements $x$ and $y$ in $X$. If $|X| \geq |Y|$ for any other set of pairwise noncommuting elements $Y$ in $G$, then $X$ is said to be a maximal subset of pairwise noncommuting elements. In this paper we determine the cardinality of a maximal subset of pairwise noncommuting elements for some $p$-groups of maximal class. Specifically, we determine this cardinality for all 2-groups and 3-groups of maximal class.

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1. Introduction

Let $G$ be a nonabelian group and let $X$ be a maximal subset of pairwise noncommuting elements of $G$. The cardinality of such a subset is denoted by $\omega(G)$. Also $\omega(G)$ is the maximal clique size in the noncommuting graph of a group $G$. Let $Z(G)$ be the center of $G$. The noncommuting graph of a group $G$ is a graph with $G \setminus Z(G)$ as the vertices and join all distinct vertices $x$ and $y$ such that $xy \neq yx$. By a famous result of Neumann [8], answering a question of Erdős, the finiteness of $\omega(G)$ in $G$ is equivalent to the finiteness of the factor group $G/Z(G)$. Chin [4] has obtained upper and lower bounds for $\omega(G)$ for an extra-special $p$-group $G$, where $p$ is an odd prime number. For $p = 2$, it has been shown by Isaacs (see [3, p. 40]) that $\omega(G) = 2n + 1$ for any extra-special group $G$ of order $2^{2n+1}$. In [1, 2], $\omega(GL(n, q))$ is given for $n \in \{2, 3\}$.

Let $G$ be a $p$-group of maximal class and order $p^n$, where $p$ is a prime number. In this paper we find $\omega(G)$ when $G$ satisfies some extra conditions. On the other hand, for these groups of maximal class we show that $\omega(G) = p^{n-2} + 1$ or $\omega(G) = p^{n-2} + p + 1$ (Theorems 3.4 and 3.7). Then by the above observation we determine $\omega(G)$ for all 2-groups and 3-groups $G$ of maximal class. In particular, we show that $\omega(G) = 2^{n-2} + 1$ for any 2-group $G$ of maximal class and order $2^n$ (Corollary 3.10). Also for a 3-group $G$ of maximal class and order $3^n$ we show that $\omega(G) = 3^{n-2} + 1$.
when $G$ possesses an abelian maximal subgroup, and $\omega(G) = 3^{n-2} + 4$ when $G$
possesses no abelian maximal subgroups (Corollary 3.11).

Throughout this paper we use the following notation. The letter $p$ denotes a prime
number. The terms of the lower central series of $G$ are denoted by $\gamma_i = \gamma_i(G)$. A
group $G$ is called an $AC$-group if the centralizer of every noncentral element of $G$ is
abelian and $C_G(x)$ is the centralizer of an element $x$ in a group $G$.

**2. Some basic results**

In this section we give some basic results that are needed for the main results of the
paper.

**Lemma 2.1.** The following conditions on a group $G$ are equivalent:

(i) $G$ is an $AC$-group;

(ii) if $[x, y] = 1$ then $C_G(x) = C_G(y)$, where $x, y \in G \setminus Z(G)$.

**Proof.** This is straightforward. See also [9, Lemma 3.2].

**Lemma 2.2.** Let $G$ be an $AC$-group.

(i) If $a, b \in G \setminus Z(G)$ with distinct centralizers, then $C_G(a) \cap C_G(b) = Z(G)$.

(ii) If $G = \bigcup_{i=1}^k C_G(a_i)$, where $C_G(a_i)$ and $C_G(a_j)$ are distinct for $1 \leq i < j \leq k$,
then $\{a_1 \cdots a_k\}$ is a maximal set of pairwise noncommuting elements in $G$.

**Proof.** (i) We see that $Z(G) \leq C_G(a) \cap C_G(b)$. If $Z(G) < C_G(a) \cap C_G(b)$, then
there exists an element $x$ in $C_G(a) \cap C_G(b)$ such that $x \not\in Z(G)$. This means that
$C_G(a) = C_G(x)$ and $C_G(b) = C_G(x)$ by Lemma 2.1(ii), which is impossible.

(ii) By Lemma 2.1(ii), $\{a_1, a_2, \ldots, a_k\}$ is a set of pairwise noncommuting
elements. Suppose to the contrary that $\{b_1, b_2, \ldots, b_t\}$ is another set of noncom-
muting elements of $G$ with $t > k$. Then we see that there exist positive integers $r, s$ and
$i$ with $1 \leq r < s \leq t$ and $1 \leq i \leq k$ such that $b_r, b_s \in C_G(a_i)$. This yields
$C_G(b_r) = C_G(b_s)$ by Lemma 2.1(ii), or equivalently $b_rb_s = b_sb_r$, which is a contradiction.

**Lemma 2.3.** Let $G$ be a finite group of order $p^n$ with the central quotient of order $p^2$,
where $p$ is a prime number. Then $\omega(G) = p + 1$.

**Proof.** First we show that $G$ is an $AC$-group. Suppose that $a$ is a noncentral element
of $G$. So $Z(G) < C_G(a)$. Therefore $|C_G(a)| = p^{n-1}$. Since $C_G(a) = \langle Z(G), a \rangle$, we
see that $C_G(a)$ is abelian and so $G$ is an $AC$-group. Now since $G$ is finite we may write
$G = \bigcup_{i=1}^k C_G(a_i)$, where $C_G(a_i)$ and $C_G(a_j)$ are distinct for $1 \leq i < j \leq k$. Therefore
$X = \{a_1, a_2, \ldots, a_k\}$ is a maximal subset of pairwise noncommuting elements of $G$
by Lemma 2.2(ii). Thus by Lemma 2.2(i),

$$|G| = \sum_{i=1}^k (|C_G(a_i)| - |Z(G)|) + |Z(G)|.$$

This yields $p^n = k \times (p^{n-1} - p^{n-2}) + p^{n-2}$ and so $k = p + 1$. 


3. Main results

Let \( G \) be a \( p \)-group of maximal class and order \( p^n \) (\( n \geq 4 \)), where \( p \) is a prime. Following [7], we define the 2-step centralizer \( K_i \) in \( G \) to be the centralizer in \( G \) of \( \gamma_i(G)/\gamma_{i+2}(G) \) for \( 2 \leq i \leq n-2 \) and define \( P_1 = P_1(G) \) by \( P_0 = G, \ P_1 = K_2, \ P_i = \gamma_i(G) \) for \( 2 \leq i \leq n \). The degree of commutativity \( l = l(G) \) of \( G \) is defined to be the maximum integer such that \([P_1, P_j] \leq P_{i+j+1}\) for all \( i, j \geq 1 \) if \( P_1 \) is not abelian and \( l = n - 3 \) if \( P_1 \) is abelian.

In this section we determine \( \omega(G) \) for any \( p \)-group \( G \) of maximal class and order \( p^n \), with positive degree of commutativity when \([P_1, P_3] = 1\). Then we deduce \( \omega(G) \) for all 2-groups and 3-groups of maximal class.

**Lemma 3.1.** Let \( G \) be a \( p \)-group of maximal class which possesses an abelian maximal subgroup. Then \( P_1 \) is abelian.

**Proof.** Let \( M \) be an abelian maximal subgroup of \( G \). Then \([M, \gamma_2(G)] = 1\). This implies that \( P_1 = M \) by using the definition of \( P_1 \).

**Lemma 3.2** [7, Corollary 3.2.7]. Let \( G \) be a \( p \)-group of maximal class. The degree of commutativity of \( G \) is positive if and only if the 2-step centralizers of \( G \) are all equal.

**Theorem 3.3.** Let \( G \) be a \( p \)-group of maximal class and order \( p^n (n \geq 4) \) with positive degree of commutativity and let \( s \in G \setminus P_1 \), \( s_1 \in P_1 \setminus P_2 \) and \( s_i = [s_{i-1}, s] \) for \( 1 \leq i \leq n - 1 \). Then:

(i) \( G = \langle s, s_1 \rangle, P_1 = \langle s_i, \ldots, s_{n-1} \rangle, |P_1| = p^{n-i} \) for \( 1 \leq i \leq n - 1 \) and \( P_{n-1}(G) = Z(G) \) is of order \( p \);

(ii) \( C_G(s) = \langle s \rangle P_{n-1}, s^p \in P_{n-1} \) and \( |C_G(s)| = p^2 \). So

\[
C_G(s) = \{s^i s_j^{s_{n-1}} : 0 \leq i, j \leq p - 1\};
\]

(iii) \( C_G(s) \cap (G \setminus P_1) = \{s^i s_j^{s_{n-1}} : 1 \leq i \leq p - 1, 0 \leq j \leq p - 1\};\)

(iv) \( C_G(s) \cap P_1 = Z(G)\);

(v) if \( s, s' \in G \setminus P_1 \) and \( [s, s'] \neq 1 \), then \( C_G(s) \cap C_G(s') = Z(G) \).

**Proof.** (i) This is obvious by [7, Lemma 3.2.4].

(ii) This follows from Lemma 3.2 and [6, Hilfssatz III 14.13].

(iii) and (iv) are evident.

(v) We have \( Z(G) \leq C_G(s) \cap C_G(s') < C_G(s) \). Also by (i) and (ii), we see that \( |Z(G)| = p \) and \( |C_G(s)| = p^2 \) which completes the proof.

**Theorem 3.4.** Let \( G \) be a \( p \)-group of maximal class and order \( p^n (n \geq 4) \) with positive degree of commutativity which possesses an abelian maximal subgroup. Then:

(i) \( G \) is an AC-group;

(ii) \( \omega(G) = p^{n-2} + 1 \).

**Proof.** (i) By Lemma 3.1, \( P_1 \) is abelian and so \( C_G(x) = P_1 \) for any \( x \in P_1 \setminus Z(G) \). Moreover, if \( x \in G \setminus P_1 \), then by Theorem 3.3(ii), \(|C_G(x)| = p^2\) as desired.
(ii) We may write \( G = \bigcup_{i=1}^{m} C_G(a_i) \), where \( C_G(a_i) \) and \( C_G(a_j) \) are distinct for \( 1 \leq i < j < m \) and \( a_i \notin Z(G) \). Therefore \( \{a_1, \ldots, a_m\} \) is a maximal subset of pairwise noncommuting elements of \( G \) by Lemma 2.2(ii). Now let \( s_1 \in P_1 \setminus P_2 \) as in Theorem 3.3. Then we may assume that \( s_1 \in C_G(a_1) \) and so \( C_G(s_1) = C_G(a_1) \) by Lemma 2.1. Moreover, \( C_G(s_1) = P_1 \) by considering the proof of (i). Therefore \( |C_G(a_1)| = |P_1| = p^{n-1} \). So \( a_i \notin P_1 \) for \( 2 \leq i \leq m \). Therefore \( |C_G(a_i)| = p^2 \) for \( 2 \leq i \leq m \) by Theorem 3.3(ii). On the other hand,

\[
|G| = |C_G(a_1)| + \sum_{i=2}^{m} (|C_G(a_i)| - |Z(G)|)
\]

by Lemma 2.2(i). This means that \( p^n = p^{n-1} + (m-1)(p^2-p) \) which yields \( \omega(G) = p^{n-2} + 1 \).

**Lemma 3.5.** Let \( G \) be a \( p \)-group of maximal class and order \( p^n (n \geq 4) \) with positive degree of commutativity and \( |P_1, P_3| = 1 \) such that \( G \) possesses no abelian maximal subgroup. Then \( P_1 \) is not abelian and \( \omega(P_1) = p+1 \).

**Proof.** Note that \( P_1 \) is not abelian since \( P_1 \) is a maximal subgroup of \( G \). Also, \( P_3 \leq Z(P_1) \) by the fact that \( |P_1, P_3| = 1 \). Therefore \( |P_1/Z(P_1)| = p^2 \) by Theorem 3.3(i). Now the result follows from Lemma 2.3.

**Lemma 3.6.** By the assumption of Lemma 3.5, if \( Y = \{y_1, \ldots, y_l\} \) is a maximal subset of pairwise noncommuting elements in \( G \setminus P_1 \), then \( t = p^{n-2} \).

**Proof.** On setting \( A_i = C_G(y_i) \cap (G \setminus P_1) \) for \( 1 \leq i \leq t \), we see that \( |A_i| = p(p-1) \) and \( A_i \cap A_j = \emptyset \) for \( 1 \leq i < j \leq t \) by Theorem 3.3(iii), (v). We claim that \( G \setminus P_1 = A_1 \cup \cdots \cup A_t \). For otherwise if \( y \in G \setminus P_1 \) and \( y \notin A_i \) for \( 1 \leq i \leq t \), then \( y \notin C_G(y_i) \). This implies that \( \{y, y_1, \ldots, y_l\} \) is a subset of pairwise noncommuting elements in \( G \setminus P_1 \), which is a contradiction. Therefore \( G \setminus P_1 = A_1 \cup \cdots \cup A_t \) is a partition for \( G \setminus P_1 \). Hence \( |G \setminus P_1| = \sum_{i=1}^{t} |A_i| \) and so \( p^n - p^{n-1} = p(p-1)t \), and consequently \( t = p^{n-2} \).

**Theorem 3.7.** Let \( G \) be a \( p \)-group of maximal class and order \( p^n (n \geq 4) \) with positive degree of commutativity and \( |P_1, P_3| = 1 \) such that \( G \) possesses no abelian maximal subgroup. Then \( \omega(G) = p^{n-2} + p + 1 \).

**Proof.** Let \( Y = \{y_1, \ldots, y_l\} \) and \( X = \{x_1, \ldots, x_{p+1}\} \) be maximal subsets of pairwise noncommuting elements in \( G \setminus P_1 \) and \( P_1 \), respectively. Obviously, by Lemma 3.6, \( t = p^{n-2} \). First we see that \( \{x_i, y_j\} \neq 1 \) for \( 1 \leq i \leq p+1 \) and \( 1 \leq j \leq t \). For otherwise \( x_i \in C_G(y_j) \) and so \( x_i \in C_G(y_j) \cap P_1 \) or, equivalently, \( x_i \in Z(G) \) by Theorem 3.3(iv), which is impossible. Therefore \( X \cup Y \) is a subset of pairwise noncommuting elements in \( G \) and so \( \omega(G) \geq t + p + 1 = p^{n-2} + p + 1 \). Now let \( \omega(G) = m \) and \( \{a_1, \ldots, a_m\} \) be a maximal subset of pairwise noncommuting elements in \( G \). We may assume that \( \{a_1, \ldots, a_k\} \subseteq G \setminus P_1 \) and \( \{a_{k+1}, \ldots, a_m\} \subseteq P_1 \). By Lemma 3.5, we see that \( m - k \geq \omega(P_1) = p + 1 \). Also, by Lemma 3.6, we have \( k \leq p^{n-2} \). Consequently \( \omega(G) = m - k + k \leq p + 1 + p^{n-2} \) as desired.
Now we determine $\omega(G)$ for all 2-groups and 3-groups $G$ of maximal class by using the following two theorems.

**Theorem 3.8** [7, Theorem 3.4.1]. Let $G$ be a 2-group of maximal class. Then $P_1$ is cyclic.

**Theorem 3.9** [7, Theorem 3.4.3]. Let $G$ be a 3-group of maximal class. Then $G$ has degree of commutativity $l \geq n - 4$.

**Corollary 3.10.** If $G$ is a 2-group of maximal class and order $2^n$, then $\omega(G) = 2^{n-2} + 1$.

**Proof.** This is evident for $n = 3$ by Lemma 2.3. Now since $P_1$ is abelian and $n \geq 4$, the degree of commutativity of $G$ is $n - 3$. Therefore we can complete the proof by Theorem 3.4.

**Corollary 3.11.** Let $G$ be a 3-group of maximal class and order $3^n$.

(i) If $G$ possesses an abelian maximal subgroup, then $\omega(G) = 3^{n-2} + 1$.

(ii) If $G$ possesses no abelian maximal subgroup, then $\omega(G) = 3^{n-2} + 4$.

**Proof.** This is obvious for $n = 3$ by Lemma 2.3. Also for $n = 4$, we see that $\omega(G) = 10$ by using GAP [5]. Now we may assume that $n \geq 5$. Moreover, $G$ has degree of commutativity $l \geq n - 4$ by Theorem 3.9 and so $[P_1, P_3] = 1$. Therefore we can complete the proof by using Theorems 3.4 and 3.7.

**References**


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