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The Universal Enveloping Algebra of the Schrödinger Algebra and its Prime Spectrum

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Abstract. The prime, completely prime, maximal, and primitive spectra are classified for the universal enveloping algebra of the Schrödinger algebra. The explicit generators are given for all of these ideals. A counterexample is constructed to the conjecture of Cheng and Zhang about non-existence of simple singular Whittaker modules for the Schrödinger algebra (and all such modules are classified). It is proved that the conjecture holds 'generically'.

1 Introduction

In this paper, module means left module, $\mathbb{K}$ is a field of characteristic zero, $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$, $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, and $\mathbb{N}_+ := \{1, 2, \dots\}$. The *Schrödinger algebra* $\mathfrak{s}$ is a 6-dimensional Lie algebra that admits a $\mathbb{K}$-basis $\{F, H, E, Y, X, Z\}$, where the Lie bracket on $\mathfrak{s}$ is as follows:

$$
\begin{array}{llll}
[H, E] = 2E, & [H, F] = -2F, & [E, F] = H, & [H, X] = X, \\
[H, Y] = -Y, & [E, Y] = X, & [E, X] = 0, & [F, X] = Y, \\
[F, Y] = 0, & [X, Y] = Z, & [Z, \mathfrak{s}] = 0. &
\end{array}
$$

The Lie algebra $\mathfrak{s}$ is a semidirect product $\mathfrak{s} = \mathrm{sl}_2 \ltimes \mathcal{H}$ of Lie algebras where $\mathrm{sl}_2 = \mathbb{K}F \oplus \mathbb{K}H \oplus \mathbb{K}E$ and $\mathcal{H} = \mathbb{K}X \oplus \mathbb{K}Y \oplus \mathbb{K}Z$ is the three dimensional *Heisenberg Lie algebra*.

The Schrödinger algebra plays an important role in mathematical physics. A classification of simple lowest weight modules of the Schrödinger algebra is given in [13]. The fact that all the weight spaces of a simple weight module have the same dimension is proved in [17]. By using Mathieu's twisting functor, a classification of simple weight modules with finite dimensional weight spaces over the Schrödinger algebra is given in [11]. In [16], the author studied the finite dimensional indecomposable modules of the Schrödinger algebra. The BGG category $\mathcal{O}$ of the Schrödinger algebra was considered in [12], and the simple non-singular Whittaker modules and quasi-Whittaker modules were classified in [9, 18]. In [7], a classification of simple weight $U(\mathfrak{s})$-modules is given.

The primitive spectrum of $U(\mathfrak{s})$ Let $\mathcal{S} := U(\mathfrak{s})$ be the universal enveloping algebra of the Schrödinger Lie algebra $\mathfrak{s}$. The primitive ideals of $U(\mathfrak{s})$ with *non-zero* central

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charge were described in [12]: Each such primitive ideal is the annihilator of a simple highest weight $U(\mathfrak{s})$-module with nonzero central charge, [12, Corollary 30]. In [12], the author mentioned that "the problem of classification of primitive ideals in $U(\mathfrak{s})$ for *zero* central charge might be very difficult". In [5], using the classification of prime ideals of the algebra $A := U(\mathfrak{s})/(Z)$ ([5, Theorem 2.8]), a classification of primitive ideals of $U(\mathfrak{s})$ with zero central charge is given. In this paper, using a different (ring theoretic) approach, all the primitive ideals (including with zero central charge) are classified and their generators are given explicitly; see Theorem 4.4.

The prime, completely prime, and maximal spectra of $U(\mathfrak{s})$ The prime, completely prime, and maximal spectra are classified for the universal enveloping algebra of the Schrödinger algebra (Theorem 3.3 and Corollary 3.4). An explicit set of generators is given for all of these ideals.

Existence of simple singular Whittaker A-modules It is conjectured that there is no simple *singular* Whittaker module for the algebra A [18, Conjecture 4.2]. We construct a family of such A-modules (Proposition 4.6). Furthermore, we classify all such modules (Theorem 4.7 and Theorem 4.8). We also proved that the conjecture holds 'generically', *i.e.,* for any non-zero central charge (Proposition 4.5).

2 The Centre of $\mathcal{S}$ and Some Related Algebras

In this section, we show that the localization $\mathcal{S}_Z$ of the algebra $\mathcal{S}$ at the powers of the central element Z is isomorphic to the tensor product of algebras $\mathbb{K}[Z^{\pm 1}] \otimes U(\mathrm{sl}_2) \otimes A_1$; see (2.8). Using this fact, a short proof is given of the fact that the centre of the algebra $\mathcal{S}$ is a polynomial algebra in two explicit generators (Proposition 2.6). The central element C was found in [15]; see also [1, 12]. The fact that the centre $Z(\mathcal{S})$ of $\mathcal{S}$ is a polynomial algebra was proved in [12] by using the Harish-Chandra homomorphism. In the above papers, it was not clear how this element was found. In this paper, we clarify the 'origin' of C which is *the Casimir element of the 'hidden'* $U(\mathrm{sl}_2)$, which is a tensor component in the decomposition (2.8).

The n-th *Weyl algebra* $A_n = A_n(\mathbb{K})$ is an associative algebra that is generated by elements $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the defining relation: $[x_i, x_j] = 0$, $[y_i, y_j] = 0$ and $[y_i, x_j] = \delta_{ij}$, where $[a, b] := ab - ba$ and δ_{ij} is the Kronecker delta function. The Weyl algebra A_n is a simple Noetherian domain of Gelfand–Kirillov dimension $2n$. Let $U := U(\mathrm{sl}_2)$ be the enveloping algebra of the Lie algebra sl_2. Then the centre of the algebra U is a polynomial algebra, $Z(U) = \mathbb{K}[\Delta]$, where $\Delta := 4FE + H^2 + 2H$ is called the *Casimir* element of U. For an algebra R, we denote its centre by $Z(R)$. For an element $r \in R$, we denote by (r) the ideal of R generated by the element r.

An automorphism γ of $\mathcal{S}$ The algebra $\mathcal{S}$ admits an automorphism γ defined by

$$\begin{aligned} &\gamma(F) = E, && \gamma(H) = -H, && \gamma(E) = F, \\ &\gamma(Y) = -X, && \gamma(X) = -Y, && \gamma(Z) = -Z. \end{aligned} \tag{2.1}$$

Clearly, $\gamma^2 = \mathrm{id}_{\mathcal{S}}$.

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The subalgebra $\mathbb{H}$ of $\mathcal{S}$ Let $\mathbb{H}$ be the subalgebra of $\mathcal{S}$ generated by the elements X, Y, and Z. Then the generators of the algebra $\mathbb{H}$ satisfy the defining relations

$$XY - YX = Z, \quad ZX = XZ, \quad \text{and} \quad ZY = YZ.$$

So $\mathbb{H} = U(\mathcal{H})$ is the universal enveloping algebra of the three dimensional Heisenberg algebra $\mathcal{H}$. In particular, $\mathbb{H}$ is a Noetherian domain of Gelfand–Kirillov dimension 3. Let $\mathbb{H}_Z$ be the localization of $\mathbb{H}$ at the powers of the element Z, and let $\mathscr{X} := Z^{-1}X \in \mathbb{H}_Z$. Then the algebra $\mathbb{H}_Z$ is a tensor product of algebras

$$\mathbb{H}_Z = \mathbb{K}[Z^{\pm 1}] \otimes A_1, \tag{2.2}$$

where $A_1 := \mathbb{K}\langle \mathscr{X}, Y\rangle$ is the (first) Weyl algebra, since $[\mathscr{X}, Y] = 1$.

Lemma 2.1 ([14, Lemma 14.6.5]) *Let B be a $\mathbb{K}$-algebra, let $S = B \otimes A_n$ be the tensor product of the algebra B and the Weyl algebra A_n, and let δ be a $\mathbb{K}$-derivation of S and $T = S[t;\delta]$. Then there exists an element $s \in S$ such that the algebra $T = B[t';\delta'] \otimes A_n$ is a tensor product of algebras where $t' = t + s$ and $\delta' = \delta + \mathrm{ad}_s$.*

The subalgebra $\mathbb{E}$ of $\mathcal{S}$ Let $\mathbb{E}$ be the subalgebra of $\mathcal{S}$ generated by the elements X, Y, Z, and E. Then

$$\mathbb{E} = \mathbb{H}[E;\delta] \tag{2.3}$$

is an Ore extension where δ is the $\mathbb{K}$-derivation of the algebra $\mathbb{H}$ defined by $\delta(Y) = X, \delta(X) = 0$ and $\delta(Z) = 0$. Let $\mathbb{E}_Z$ be the localization of $\mathbb{E}$ at the powers of the element Z. Then

$$\mathbb{E}_Z = \mathbb{H}_Z[E;\delta] = \big(\mathbb{K}[Z^{\pm 1}] \otimes A_1\big)[E;\delta],$$

where δ is defined as in (2.3), in particular, $\delta(\mathscr{X}) = 0$. Now, the element $s = -\frac{1}{2}Z\mathscr{X}^2$ satisfies the conditions of Lemma 2.1. In particular, the element $E' := E + s = E - \frac{1}{2}Z^{-1}X^2$ commutes with the elements of A_1. Hence, $\mathbb{E}_Z$ is a tensor product of algebras

$$\mathbb{E}_Z = \mathbb{K}[E', Z^{\pm 1}] \otimes A_1 = \mathbb{K}[E'] \otimes \mathbb{H}_Z. \tag{2.4}$$

In particular, $\mathbb{E}$ and $\mathbb{E}_Z$ are Noetherian domains of Gelfand–Kirillov dimension 4.

The subalgebra $\mathbb{F}$ of $\mathcal{S}$ Let $\mathbb{F} := \gamma(\mathbb{E})$. Then $\mathbb{F}$ is the subalgebra of $\mathcal{S}$ generated by the elements X, Y, Z and F. Notice that the automorphism γ (see (2.1)) can be naturally extended to an automorphism of $\mathcal{S}_Z$ by setting $\gamma(Z^{-1}) = -Z^{-1}$ where $\mathcal{S}_Z$ is the localization of the algebra $\mathcal{S}$ at the powers of the element Z. Let $\mathbb{F}_Z$ be the localization of $\mathbb{F}$ at the powers of the central element Z and $F' := \gamma(E') = F + \frac{1}{2}Z^{-1}Y^2 \in \mathbb{F}_Z$. Then $\mathbb{F}_Z$ is a tensor product of algebras

$$\mathbb{F}_Z = \mathbb{K}[F', Z^{\pm 1}] \otimes A_1 = \mathbb{K}[F'] \otimes \mathbb{H}_Z, \tag{2.5}$$

where A_1 is as above; see (2.2).

The algebra $\mathcal{A}$ Let $\mathcal{A}$ be the subalgebra of $\mathcal{S}$ generated by the elements H, E, Y, X, and Z. The algebra $\mathcal{A}$ is the enveloping algebra $U(\mathfrak{a})$ of the solvable Lie subalgebra $\mathfrak{a}$ of $\mathfrak{s}$ with basis elements H, E, Y, X and Z. The algebra $\mathcal{A}$ is an Ore extension

$$\mathcal{A} = \mathbb{E}[H;\delta] \tag{2.6}$$

where δ is a $\mathbb{K}$-derivation of the algebra $\mathbb{E}$ defined by $\delta(E) = 2E, \delta(Y) = -Y, \delta(X) = X$ and $\delta(Z) = 0$. Let $\mathcal{A}_Z$ be the localization of the algebra $\mathcal{A}$ at the powers of the central element Z. Then

$$\mathcal{A}_Z = \mathbb{E}_Z[H; \delta] = \big(\mathbb{K}[E', Z^{\pm 1}] \otimes A_1\big)[H; \delta],$$

where δ is defined as in (2.6), in particular, $\delta(\mathscr{X}) = \mathscr{X}$. The element $s = \mathscr{X} Y - \frac{1}{2} = Z^{-1}XY - \frac{1}{2}$ satisfies the conditions of Lemma 2.1. In particular, the element $H' := H + s = H + Z^{-1}XY - \frac{1}{2}$ commutes with the elements of A_1 and $[H', E'] = 2E'$. Hence, $\mathcal{A}_Z$ is a tensor product of algebras

$$\mathcal{A}_Z = \mathbb{K}[Z^{\pm 1}] \otimes \mathbb{K}[H'][E'; \sigma] \otimes A_1, \tag{2.7}$$

where σ is the automorphism of the algebra $\mathbb{K}[H']$ such that $\sigma(H') = H' - 2$. In particular, $\mathcal{A}_Z$ is a Noetherian domain of Gelfand–Kirillov dimension 5.

Lemma 2.2 (i) *Let $E' := E - \frac{1}{2}Z^{-1}X^2$, $F' := F + \frac{1}{2}Z^{-1}Y^2$, and $H' := H + Z^{-1}XY - \frac{1}{2}$. Then the following commutation relations hold in the algebra $\mathcal{S}_Z$:*

$$[H', E'] = 2E', \quad [H', F'] = -2F', \quad [E', F'] = H',$$

i.e., *the Lie algebra $\mathbb{K}F' \oplus \mathbb{K}H' \oplus \mathbb{K}E'$ is isomorphic to* sl_2. *Moreover, the subalgebra U' of $\mathcal{S}_Z$ generated by H', E', and F' is isomorphic to the enveloping algebra $U(\mathrm{sl}_2)$. Furthermore, the elements E', F', and H' commute with X and Y.*

(ii) *The localization $\mathcal{S}_Z$ of the algebra $\mathcal{S}$ at the powers of Z is $\mathcal{S}_Z = \mathbb{K}[Z^{\pm 1}] \otimes U' \otimes A_1$.*

Proof (i) It is straightforward to verify that the commutation relations in the lemma hold. The fact that the elements E', F', and H' commute with the elements X and Y follows from (2.4), (2.5), and (2.7), respectively. Let U be the universal enveloping algebra of the Lie algebra $\mathrm{sl}_2 = \langle F', H', E'\rangle$. The algebra U' is an epimorphic image of the algebra U under a natural epimorphism $f: U \to U'$. The kernel of f, say $\mathfrak{p}$, is a (completely) prime ideal of U, since U' is a domain. Suppose that $\mathfrak{p} \neq 0$; we seek a contradiction. Then $\mathfrak{p} \cap \mathbb{K}[\Delta'] \neq 0$ (it is known fact) where Δ' is the Casimir element of U; see (2.9). In particular, there is a non-scalar monic polynomial $P(t) = t^n + \lambda_{n-1}t^{n-1} + \cdots + \lambda_0 \in \mathbb{K}[t]$ such that $P(\Delta') = 0$ in $\mathcal{S}_Z$. Then necessarily $Z^n P(\Delta') \equiv 0 \mod \mathcal{S}Z$, *i.e.*, $(EY^2 + HXY - FX^2)^n \equiv 0 \mod \mathcal{S}Z$, a contradiction, since $\mathcal{S}/\mathcal{S}Z \simeq U(\mathrm{sl}_2 \ltimes V_2)$, where $\mathrm{sl}_2 \ltimes V_2$ is a semidirect product of Lie algebras and $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ is an abelian 2-dimensional Lie algebra.

(ii) Using the defining relations of the algebra $\mathcal{S}$, we see that the algebra $\mathcal{S}$ is a skew polynomial algebra $\mathcal{S} = \mathcal{A}[F; \sigma, \delta]$ where σ is the automorphism of the algebra $\mathcal{A}$ defined by $\sigma(H) = H + 2, \sigma(E) = E, \sigma(Y) = Y, \sigma(X) = X$, and $\sigma(Z) = Z$; and δ is the σ-derivation of $\mathcal{A}$ given by the rule: $\delta(H) = \delta(Y) = \delta(Z) = 0$, $\delta(E) = -H$ and $\delta(X) = Y$. Then, by (2.7) and statement 1,

$$\begin{aligned} \mathcal{S}_Z = \mathcal{A}_Z[F'; \sigma', \delta'] &= \big(\mathbb{K}[Z^{\pm 1}] \otimes \mathbb{K}[H'][E'; \sigma] \otimes A_1\big)[F'; \sigma', \delta'] \\ &= \mathbb{K}[Z^{\pm 1}] \otimes U' \otimes A_1 \end{aligned} \tag{2.8}$$

is a tensor product of algebras where σ' is an automorphism of $\mathcal{A}_Z$ such that $\sigma'(Z) = Z, \sigma'(H') = H' + 2, \sigma'(E') = E', \sigma'(X) = X$, and $\sigma'(Y) = Y$; and δ' is a σ'-derivation

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of the algebra $\mathcal{A}_Z$ such that $\delta'(Z) = \delta'(H') = \delta'(X) = \delta'(Y) = 0$ and $\delta'(E') = -H'$. In particular, $\mathcal{S}_Z$ is a Noetherian domain of Gelfand–Kirillov dimension 6. ∎

Definition 2.3 The algebra U' is called the *hidden* $U(\mathrm{sl}_2)$.

Corollary 2.4 $\mathrm{Fract}(\mathcal{S}) \simeq \mathrm{Fract}(A_2(Q_2))$ *where* $A_2(Q_2)$ *is the second Weyl algebra over the field* $Q_2 = K(x_1, x_2)$ *of rational functions in two variables.*

Proof The statement follows from Lemma 2.2(ii). ∎

The centre of the algebra $\mathcal{S}_Z$ Let $\Delta' := 4F'E' + H'^2 + 2H'$ be the Casimir element of U'; then the centre $Z(U') = \mathbb{K}[\Delta']$ is a polynomial algebra. Using the explicit expressions of the elements F', E', and H' (see Lemma 2.2(i)), the element Δ' can be written as

$$\Delta' = (4FE + H^2 + H) + 2Z^{-1}(EY^2 + HXY - FX^2) - \frac{3}{4}. \tag{2.9}$$

Let

$$\begin{aligned} C &:= Z\Delta' + \frac{3}{4}Z = Z(4FE + H^2 + H) + 2(EY^2 + HXY - FX^2),\\ C' &:= Z\Delta' = C - \frac{3}{4}Z. \end{aligned} \tag{2.10}$$

Lemma 2.5 $Z(\mathcal{S}_Z) = \mathbb{K}[Z^{\pm1}, C]$.

Proof By (2.8),

$$\begin{aligned} Z(\mathcal{S}_Z) &= Z\big(\mathbb{K}[Z^{\pm1}]\big) \otimes Z(U') \otimes Z(A_1) = \mathbb{K}[Z^{\pm1}] \otimes \mathbb{K}[\Delta'] \\ &= \mathbb{K}[Z^{\pm1}, \Delta'] = \mathbb{K}[Z^{\pm1}, C]. \end{aligned}$$ ∎

Let $\mathcal{U}$ be the subalgebra of $\mathcal{S}$ generated by the elements Z, $e := ZE'$, $f := ZF'$, and $h := ZH'$. Since $e = ZE - \frac{1}{2}X^2$, $f = ZF + \frac{1}{2}Y^2$ and $h = Z(H - \frac{1}{2}) + XY$, the elements Z, e, f, h belong to the ideal $(X) = (X, Y, Z)$ of $\mathcal{S}$. By the very definition of $\mathcal{U}$, $\mathcal{U}_Z = \mathbb{K}[Z^{\pm1}] \otimes U'$, *i.e.*, the algebra $\mathcal{U}$ is a 'Z-homogenized' version of $U' \simeq U(\mathrm{sl}_2)$. Since the algebra U' is the generalized Weyl algebra (GWA) $\mathbb{K}[H', \Delta'][E', F'; \sigma', a' = \frac{1}{4}(\Delta' - H'^2 - 2H')]$, where $\sigma(H') = H' - 2$ and $\sigma(\Delta') = \Delta'$; see [2], the algebras $\mathcal{U} = \mathbb{K}[Z, h, C'][e, f; \sigma, \alpha = \frac{1}{4}(ZC' - h^2 - 2Zh)]$ and $\mathcal{U}_Z = \mathbb{K}[Z^{\pm1}, h, C'][e, f; \sigma, \alpha]$ are GWAs, where $\sigma(Z) = Z$, $\sigma(h) = h - 2Z$ and $\sigma(C') = C'$.

Using the $\mathbb{Z}$-grading of the GWA $\mathcal{U}_Z = \oplus_{i\in\mathbb{Z}}\mathcal{U}_{Z,i}$ (where $\mathcal{U}_{Z,i} = \mathbb{K}[Z^{\pm1}, h, C']v_i$ and $v_j = e^j$ and $v_{-j} = f^j$ for $j \geqslant 0$) and the fact that modulo (Z) the elements e, f, and h of $\mathcal{U}$ are equal to $-\frac{1}{2}X^2$, $\frac{1}{2}Y^2$ and XY, respectively, we have $\mathcal{U}_Z \cap \mathcal{S} = \mathcal{U}$.

The factor algebra $S/(Z)$ The set $\mathbb{K}Z$ is an ideal of the Lie algebra $\mathfrak{s}$ and $\mathfrak{s}/\mathbb{K}Z \simeq \mathrm{sl}_2 \ltimes V_2$ is a semidirect product of Lie algebras where $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ is a 2-dimensional abelian Lie algebra. So,

$$\mathcal{S}/(Z) \simeq U(\mathfrak{s}/\mathbb{K}Z) \simeq U(\mathrm{sl}_2 \ltimes V_2).$$

The element $\mathsf{c} := FX^2 - HXY - EY^2$ is in the centre of the algebra $A := U(\mathrm{sl}_2 \ltimes V_2)$. In fact, $Z(A) = \mathbb{K}[\mathsf{c}]$; see [5].

The centre of the algebra $\mathcal{S}$ The next proposition shows that the centre of $\mathcal{S}$ is a polynomial algebra in two variables.

Proposition 2.6 $Z(\mathcal{S}) = \mathbb{K}[Z, C] = \mathbb{K}[Z, C']$.

Proof The second equality is obvious. By Lemma 2.5, $Z(\mathcal{S}) = \mathcal{S} \cap Z(\mathcal{S}_Z) = \mathcal{S} \cap \mathbb{K}[Z^{\pm 1}, C] \supseteq \mathbb{K}[Z, C]$. It remains to show that $Z(\mathcal{S}) = \mathbb{K}[Z, C]$. Suppose that this is not the case; we seek a contradiction. Then $Z^{-1} f(C) \in Z(\mathcal{S})$ for some non-scalar polynomial $f(C) \in \mathbb{K}[C]$ (since $Z^{-1} \notin \mathcal{S}$). Hence, by (2.10),

$$0 \equiv f(C) \equiv f(\mathrm{c}) \mod \mathcal{S}Z;$$

i.e., the element c is algebraic in $U(\mathrm{sl}_2 \ltimes V_2)$, a contradiction. ∎

3 Prime, Completely Prime, and Maximal Ideals of $\mathcal{S}$

The aim of this section is to classify the prime, completely prime, and maximal ideals of the algebra $\mathcal{S}$ and to give their explicit generators (Theorem 3.3 and Corollary 3.4).

For an algebra R, let $\mathrm{Spec}(R)$ be the set of its prime ideals. The set $(\mathrm{Spec}(R), \subseteq)$ is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$ where $x \in R$. An element $a \in R$ is called a *normal element* if $Ra = aR$.

Proposition 3.1 ([6]) *Let R be a Noetherian ring and let s be an element of R such that $S_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that $\ker(\cdot s_R) \subseteq \ker(s_R \cdot)$). Then $\mathrm{Spec}(R) = \mathrm{Spec}(R, s) \sqcup \mathrm{Spec}_s(R)$ where $\mathrm{Spec}(R, s) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid s \notin \mathfrak{q}\}$ and*

(i) *the map $\mathrm{Spec}(R, s) \to \mathrm{Spec}(R/(s))$, $\mathfrak{p} \mapsto \mathfrak{p}/(s)$, is a bijection with the inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ where $\pi : R \to R/(s), r \mapsto r + (s)$;*
(ii) *the map $\mathrm{Spec}_s(R) \to \mathrm{Spec}(R_s)$, $\mathfrak{p} \mapsto S_s^{-1}\mathfrak{p}$, is a bijection with the inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$ where $\sigma : R \to R_s := S_s^{-1}R$, $r \mapsto \frac{r}{1}$;*
(iii) *for all $\mathfrak{p} \in \mathrm{Spec}(R, s)$ and $\mathfrak{q} \in \mathrm{Spec}_s(R)$, $\mathfrak{p} \notin \mathfrak{q}$.*

The prime ideals $(\mathfrak{q}, I'_n)$ of $\mathcal{S}$ where $\mathfrak{q} \in \mathcal{Q} := \mathrm{Max}(\mathbb{K}[Z]) \smallsetminus \{(Z)\}$ and $n \in \mathbb{N}_+$

If $\mathfrak{q} \in \mathcal{Q}$, then the element Z is a *unit* in the factor algebras $\mathbb{K}[Z]/\mathfrak{q}$, and so $L_\mathfrak{q} := \mathbb{K}[Z]/\mathfrak{q} \simeq \mathbb{K}[Z^{\pm 1}]/\mathfrak{q}_Z$. Then, by (2.8), the algebra

$$\mathcal{S}/\mathfrak{q}\mathcal{S} \simeq \mathcal{S}_Z/\mathfrak{q}\mathcal{S}_Z \simeq L_\mathfrak{q} \otimes U' \otimes A_1 \tag{3.1}$$

is a domain; *i.e.,* the ideal $\mathfrak{q}\mathcal{S}$ is completely prime. For each natural number $n \geqslant 1$, there exists a unique simple n-dimensional U'-module V'_n. Its annihilator $I'_n := \mathrm{ann}_{U'}(V'_n)$ is a prime ideal of U' such that $U'/I'_n \simeq M_n(\mathbb{K})$, the ring of $n \times n$ matrices over $\mathbb{K}$. The Casimir element $\Delta' = 4F'E' + H'^2 + 2H'$ acts on V'_n as the scalar $\lambda_n := n^2 - 1$, *i.e.,* $\Delta' - \lambda_n \in I'_n$.

Lemma 3.2 *Let $\mathfrak{q} \in \mathrm{Max}(\mathbb{K}[Z]) \smallsetminus \{(Z)\}$, $L_\mathfrak{q} = \mathbb{K}[Z]/\mathfrak{q}$ and $n \in \mathbb{N}_+$.*

(i) *The ideal $(\mathfrak{q}, I'_n)$ of the algebra $\mathcal{S}$ (see, (3.1)) is a maximal (hence, prime) ideal of $\mathcal{S}$ and $\mathcal{S}/(\mathfrak{q}, I'_n) \simeq L_\mathfrak{q} \otimes U'/I'_n \otimes A_1 \simeq L_\mathfrak{q} \otimes M_n(\mathbb{K}) \otimes A_1$.*

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(ii) *The ideal* $(\mathfrak{q}, I'_n)$ *is completely prime if and only if* $n = 1$.
(iii) $(\mathfrak{q}, I'_n) \cap \mathbb{K}[Z, C'] = (\mathfrak{q}, C' - \lambda_n Z)$, $I'_n \mathcal{S}_Z \cap \mathbb{K}[Z, C'] = (C' - \lambda_n Z)$, *where* $\lambda_n = n^2 - 1$ *and* $I'_n \mathcal{S}_Z = \mathbb{K}[Z^{\pm 1}] \otimes I'_n \otimes A_1$.

Proof (i) The claim about the isomorphisms is obvious. Then statement (i) follows.
(ii) Statement (ii) follows from statement (i).
(iii) By (2.8),

$$\begin{aligned}(\mathfrak{q}, I'_n)_Z \cap \mathbb{K}[Z^{\pm 1}, C'] &= (\mathfrak{q}, I'_n)_Z \cap \mathbb{K}[Z^{\pm 1}, \Delta'] = (\mathfrak{q}, \Delta' - \lambda_n)_Z = (\mathfrak{q}, Z\Delta' - \lambda_n Z)_Z \\ &= (\mathfrak{q}, C' - \lambda_n Z)_Z.\end{aligned}$$

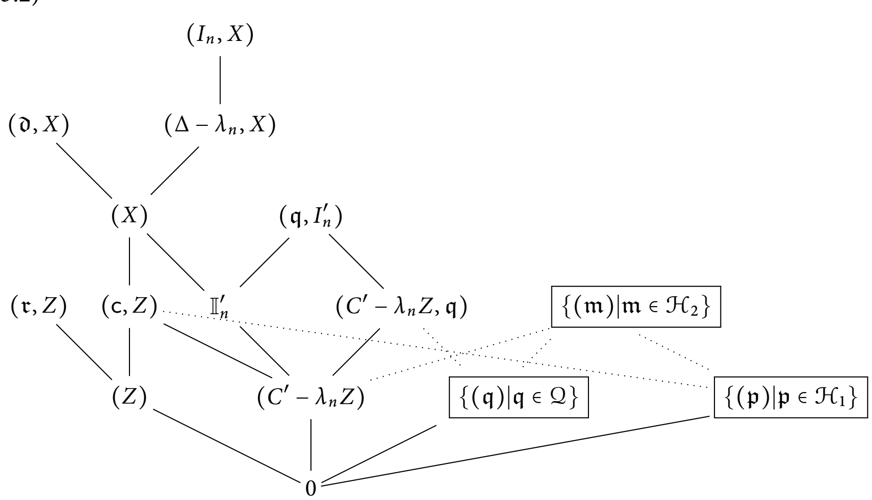
Hence, $(\mathfrak{q}, I'_n) \cap \mathbb{K}[Z, C'] = (\mathfrak{q}, C' - \lambda_n Z)$, since $\frac{\mathbb{K}[Z,C']}{(\mathfrak{q},C'-\lambda_n Z)} \simeq \frac{\mathbb{K}[Z^{\pm 1},C']}{(\mathfrak{q},C'-\lambda_n Z)_Z}$.
Now,

$$\begin{aligned}(C' - \lambda_n Z) \subseteq I'_n \mathcal{S}_Z \cap \mathbb{K}[Z, C'] &= \bigcap_{\mathfrak{q} \in \mathcal{Q}} (\mathfrak{q}, I'_n) \cap \mathbb{K}[Z, C'] = \bigcap_{\mathfrak{q} \in \mathcal{Q}} (\mathfrak{q}, C' - \lambda_n Z) \\ &= (C' - \lambda_n Z),\end{aligned}$$

and so $(C' - \lambda_n Z) = I'_n \mathcal{S}_Z \cap \mathbb{K}[Z, C']$. ∎

Theorem 3.3 *The set of prime ideals of the algebra* $\mathcal{S}$ *and inclusions between primes are given in the following diagram (where dotted line means all the inclusions between height* 1 *and* 2 *primes):*

(3.2)



where $n \in \mathbb{N}_+$; $\lambda_n = n^2 - 1$; $\mathfrak{q} \in \mathcal{Q} := \mathrm{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}$; $\mathfrak{r} \in \mathcal{R} := \mathrm{Max}(\mathbb{K}[\mathfrak{c}] \setminus \{(\mathfrak{c})\})$; $\mathfrak{d} \in \mathcal{D} := \mathrm{Max}(\mathbb{K}[\Delta]) \setminus \{(\Delta - \lambda_n), n \in \mathbb{N}_+\}$; $\mathcal{H}_1 := \{\mathfrak{p} \in \mathrm{Spec}(\mathbb{K}[C', Z]) | \mathrm{ht}(\mathfrak{p}) = 1\} \setminus \{\mathcal{E}_1 \cup \mathcal{Q} \cup (Z)\}$ *where* $\mathcal{E}_1 := \{(C' - \lambda_n Z)\mathbb{K}[C', Z] | n \in \mathbb{N}_+\}$; $\mathcal{H}_2 := \{\mathfrak{m} \in \mathrm{Max}(\mathbb{K}[C', Z]) | Z \notin \mathfrak{m}\} \setminus \{(C' - \lambda_n Z, \mathfrak{q}) | n \in \mathbb{N}_+, \mathfrak{q} \in \mathcal{Q}\}$ *and* $\mathbb{I}'_n := \mathcal{S} \cap I'_n \mathcal{S}_Z$.

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Remark The inclusions represented by the dotted line are easy to describe, as they are precisely the inclusions between the primes of the polynomial algebra, *e.g.*, $(\mathfrak{p})\cdots\cdots(\mathfrak{m})$ if and only if $(\mathfrak{p}) \subseteq (\mathfrak{m})$ in $\mathcal{S}$ if and only if $\mathfrak{p} \subseteq \mathfrak{m}$ in $\mathbb{K}[C', Z]$.

Proof Notice that $A = \mathcal{S}/(Z)$. Then by Proposition 3.1,

$$\operatorname{Spec}(\mathcal{S}) = \operatorname{Spec}(A) \sqcup \operatorname{Spec}(\mathcal{S}_Z),$$

where the sets $\operatorname{Spec}(A)$ and $\operatorname{Spec}(\mathcal{S}_Z)$ are identified with subsets of $\operatorname{Spec}(\mathcal{S})$ via the following injections

$$\operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(\mathcal{S}), \mathfrak{p} \longmapsto \pi^{-1}(\mathfrak{p}) \quad \text{and} \quad \operatorname{Spec}(\mathcal{S}_Z) \longrightarrow \operatorname{Spec}(\mathcal{S}), \mathfrak{q} \longmapsto \mathfrak{q} \cap \mathcal{S},$$

where $\pi\colon \mathcal{S} \to A, a \mapsto a + (Z)$. So, $\operatorname{Spec}(A) = \{P \in \operatorname{Spec}(\mathcal{S}) | Z \in P\}$ and $\operatorname{Spec}(\mathcal{S}_Z) = \{Q \in \operatorname{Spec}(\mathcal{S}) | Z \notin Q\}$. By (2.8), $\mathcal{S}_Z = \mathbb{U} \otimes A_1$ where $\mathbb{U} := \mathbb{K}[Z^{\pm 1}] \otimes U'$ and $U' = U(\mathrm{sl}_2)'$. Since the Weyl algebra A_1 is a central simple algebra, the map

$$\operatorname{Spec}(\mathbb{U}) \longrightarrow \operatorname{Spec}(\mathcal{S}_Z), \quad \mathfrak{p} \mapsto \mathfrak{p} \otimes A_1,$$

is a bijection, and so we can write

$$\operatorname{Spec}(\mathcal{S}_Z) = \operatorname{Spec}(\mathbb{U}). \tag{3.3}$$

The prime spectrum of the algebra A was described in [5]. It comprises precisely the prime ideals over (Z) in (3.2).

(i) *For all* $\mathfrak{m} \in \operatorname{Max}(\mathbb{K}[C', Z])$ *such that* $Z \notin \mathfrak{m}$, *the ideals* $(\mathfrak{m})$ *of* $\mathcal{S}$ *are completely prime*:

By the choice of the maximal ideal $\mathfrak{m}$ of $\mathbb{K}[C', Z]$, the element Z of the field $F_\mathfrak{m} := \mathbb{K}[C', Z]/\mathfrak{m}$ is not equal to zero. In particular, Z is a unit of $F_\mathfrak{m}$. Then, by (2.8),

$$\mathcal{S}/(\mathfrak{m}) \simeq \mathcal{S}_Z/\mathfrak{m}\mathcal{S}_Z \simeq \mathbb{U}(\mathfrak{m}) \otimes A_1 \tag{3.4}$$

where $\mathbb{U}(\mathfrak{m}) := \mathbb{U}/\mathfrak{m}\mathbb{U} \simeq F_\mathfrak{m}[H'][E', F'; \sigma', a' \neq 0]$ is a generalized Weyl algebra (GWA), which is a domain; see [2]. Now statement (i) is obvious.

(ii) *Every nonzero prime ideal of* $\mathcal{S}$ *intersects non-trivially the centre of* $\mathcal{S}$:

Let $P \in \operatorname{Spec}(\mathcal{S}) \setminus \{0\}$. We can assume that $Z \notin P$. Then $P_Z \in \operatorname{Spec}(\mathcal{S}_Z) \setminus \{0\}$. By (3.3), we have to show that every nonzero prime ideal of the algebra $\mathbb{U}$ intersects non-trivially the centre $\mathbb{K}[Z^{\pm 1}, C']$ of $\mathbb{U}$. Let $T = \mathbb{K}[Z^{\pm 1}, C'] \setminus \{0\}$. Then the GWA

$$T^{-1}\mathbb{U} = \mathbb{K}(Z, C')[H'][E', F'; \sigma', a']$$

is *simple*, by [2] (since the group $\langle \sigma' \rangle$ acts freely on the maximal spectrum of the Dedekind domain $\mathbb{K}(Z, C')[H']$ and the element a' is irreducible in $\mathbb{K}(Z, C')[H']$). Now statement (ii) is obvious.

Let $V(Z) = \{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[Z, C']) | Z \in \mathfrak{p}\}$. The map $\operatorname{Spec}(\mathbb{K}[Z, C']) \setminus V(Z) \to \operatorname{Spec}(\mathbb{K}[Z^{\pm 1}, C'])$, $\mathfrak{p} \mapsto \mathfrak{p}_Z$, is a bijection with the inverse $\mathfrak{q} \mapsto \mathbb{K}[Z, C'] \cap \mathfrak{q}$. We identify the two sets above via the bijection $\mathfrak{p} \mapsto \mathfrak{p}_Z$.

(iii) *For every* $\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[Z, C']) \setminus V(Z)$, *the ideal* $(\mathfrak{p})$ *is a completely prime ideal of* $\mathcal{S}$:

By [12], the algebra $\mathcal{S}$ is a free module over its centre. Hence, $(\mathfrak{p}) = \mathcal{S} \cap (\mathfrak{p})_Z$, and so

$$\mathcal{S}/(\mathfrak{p}) = \frac{\mathcal{S}}{\mathcal{S} \cap (\mathfrak{p})_Z} \subseteq \frac{\mathcal{S}_Z}{(\mathfrak{p})_Z} \simeq \mathbb{U}/\mathfrak{p}\mathbb{U} \otimes A_1. \tag{3.5}$$

The GWA $\mathbb{U}/\mathfrak{p}\mathbb{U}$ is a domain, hence so are the algebras $\mathbb{U}/\mathfrak{p}\mathbb{U} \otimes A_1$ and $\mathcal{S}/(\mathfrak{p})$. Now statement (iii) follows.

Let $P \neq 0$ be a prime ideal of $\mathcal{S}$ such that $Z \notin P$. The nonzero prime ideal P_Z of $\mathcal{S}_Z$ is equal to $\mathbb{P} \otimes A_1$, where $\mathbb{P}$ is a nonzero prime ideal of $\mathbb{U}$. Then $P' := \mathbb{K}[Z, C'] \cap P$ is a nonzero prime ideal of the polynomial algebra $\mathbb{K}[Z, C']$ that does not contain the element Z.

(iv) *If P' is a maximal ideal of $\mathbb{K}[Z, C']$ such that $P' \in \operatorname{Spec}(\mathbb{K}[Z, C']) \smallsetminus V(Z)$, then either $P' \in \{(C' - \lambda_n Z, \mathfrak{q}) | n \in \mathbb{N}_+, \mathfrak{q} \in \mathcal{Q}\}$ and in this case $P = (C' - \lambda_n Z, \mathfrak{q})$ or $P = (\mathfrak{q}, I'_n) = (C' - \lambda_n Z, \mathfrak{q}, I'_n) \in \operatorname{Max}(\mathcal{S})$, or, otherwise, $P = (P')$ is a maximal ideal of $\mathcal{S}$:*

In the first case, the ideal P contains the prime ideal $(C' - \lambda_n Z, \mathfrak{q})$ of the algebra $\mathcal{S}$, by statement (i). By (3.4),

$$\mathcal{S}/(C' - \lambda_n Z, \mathfrak{q}) \simeq L_\mathfrak{q} \otimes U'/(\Delta'_n - \lambda_n) \otimes A_1.$$

Hence, either $P = (C' - \lambda_n Z, \mathfrak{q})$ or, otherwise, $P = (\mathfrak{q}, I'_n) = (C' - \lambda_n Z, \mathfrak{q}, I'_n) \in \operatorname{Max}(\mathcal{S})$.

In the second case, by (3.4), the factor algebra $\mathcal{S}/(P') \simeq \mathbb{U}(P') \otimes A_1$ is simple, since the GWA

$$\mathbb{U}(P') \simeq F_{P'}[H'][E', F'; \sigma', a']$$

is simple, by the description of ideals in [2]. (Since $a' = \frac{1}{4}(\Delta' - H'(H' + 2))$ is a polynomial of degree 2 in the variable H', the GWA $\mathbb{U}(P')$ is not simple if and only if

$$H'(H' + 2) - \Delta' = (H' + \mu)(H' + \mu - 2n) \text{ for some } \mu \in F_{P'} \text{ and } n \in \mathbb{N}_+.$$

Hence, $\mu = n + 1$ and $\Delta' = -\mu(\mu - 2n) = n^2 - 1 = \lambda_n$. Since $C' = Z\Delta'$, we have $C' - \lambda_n Z = Z^{-1}(\Delta' - \lambda_n) \in P'$, and so $P' = (C' - \lambda_n Z, \mathfrak{q})$ for some $\mathfrak{q} \in \mathcal{Q}$). Therefore, $P = (P')$ is a maximal ideal of $\mathcal{S}$.

(v) *For all $n \in \mathbb{N}_+$, $\mathbb{I}'_n \in \operatorname{Spec}(\mathcal{S})$:*

The statement follows from the fact that $(\mathbb{I}'_n)_Z = I'_n \mathcal{S}_Z$ is a prime ideal of the algebra $\mathcal{S}_Z$.

Clearly, $(C' - \lambda_n Z) \subseteq \mathbb{I}'_n \subseteq (\mathfrak{q}, I'_n)$ and $(C' - \lambda_n Z) \subseteq (C' - \lambda_n Z, \mathfrak{q}) \subseteq (\mathfrak{q}, I'_n)$ for all $n \in \mathbb{N}_+$ and $\mathfrak{q} \in \mathcal{Q}$.

It remains to consider the case when P' is not a maximal ideal of $\mathbb{K}[Z, C']$, *i.e.*, $\operatorname{ht}(P') = 1$ and $P' \neq (Z)$. We assume that these two conditions hold till the end of the proof.

(vi) *If $\mathfrak{q} := \mathbb{K}[Z] \cap P' \neq 0$, then $\mathfrak{q} \in \mathcal{Q}$ and $P = (\mathfrak{q})$:*

Clearly, $(\mathfrak{q}) \subseteq P$ and $(\mathfrak{q}) \in \operatorname{Spec}(\mathcal{S})$ by statement (iii). Now statement (vi) follows from (3.1) and the fact that every nonzero ideal of the algebra $L_\mathfrak{q} \otimes U'$ intersects its centre $L_\mathfrak{q}[\Delta'] = L_\mathfrak{q}[C']$ non-trivially.

(vii) *If $\mathbb{K}[Z] \cap P = 0$, then either $P \in \{(C' - \lambda_n Z), \mathbb{I}'_n | n \in \mathbb{N}_+\}$ or, otherwise, $P = (P')$ where $P' \in \mathcal{H}_1$:*

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Let Γ be the field of fractions of the commutative domain $\Lambda = \mathbb{K}[Z, C']/P'$. Clearly, $\Gamma = S^{-1}\Lambda$ where $S = \mathbb{K}[Z] \smallsetminus \{0\}$. By (3.5),

$$\mathcal{S}/P'\mathcal{S} \subseteq \mathcal{S}_Z/P'\mathcal{S}_Z = \Lambda[H'][E', F'; \sigma', a'] \otimes A_1 \subset \Gamma[H'][E', F'; \sigma', a'] \otimes A_1.$$

Using a similar argument as in the proof of statement (iv), we see that the GWA $\Gamma[H'][E', F'; \sigma', a']$ is not simple if and only if $C' - \lambda_n Z \in P'$ for some $n \in \mathbb{N}_+$, *i.e.*, $P' = (C' - \lambda_n Z)$. If $P' = (C' - \lambda_n Z)$, then

$$\begin{aligned}\Gamma[H'][E', F'; \sigma', a'] &= \mathbb{K}(Z)[H']\Big[E', F'; \sigma', a = \frac{1}{4}\big(\lambda_n - H'(H' + 2)\big)\Big] \\ &\simeq \mathbb{K}(Z) \otimes U'/(\Delta' - \lambda_n).\end{aligned}$$

So, either $P = (C' - \lambda_n Z)$ or, otherwise, $P = \mathbb{I}'_n$ since the ideal I'_n is the only proper ideal of U' that contains the element $\Delta' - \lambda_n$. If $P' \neq (C' - \lambda_n Z)$ for all $n \in \mathbb{N}_+$ then $P' \in \mathcal{H}_1$ and $S^{-1}(P/(P')) = 0$ by the simplicity of the GWA $\Gamma[H'][E', F'; \sigma', a']$. Hence, $P = (P')$ by the choice of S and P'. So, we have proved that the picture (3.2) represents all the prime ideals of $\mathcal{S}$. The inclusions in (3.2) are obvious apart from $\mathbb{I}'_n \subseteq (X)$. The latter follows from the explicit description of generators of the ideal I_n (see [3, Section 4]) and the facts that $\mathcal{S}_Z = \mathcal{U}_Z \otimes A_1 = \oplus_{i,j \geqslant 0} \mathcal{U}_Z X^i Y^j$, the elements X, Y, Z, e, f and h belong to (X) and $\mathcal{U}_Z \cap \mathcal{S} = \mathcal{U}$.

Let $\mathscr{L}$ be the prime ideals over (Z) including (Z) and $\mathscr{R} = \operatorname{Spec}(\mathcal{S}) \smallsetminus \{\mathscr{L} \cup \{0\}\}$. By Proposition 3.1, none of the ideals of $\mathscr{L}$ contains an ideal of $\mathscr{R}$. The containments of prime ideals in $\operatorname{Spec}(A) = \mathscr{L}$ are described in [5]. Straightforward arguments show that there are no more new containments from $\mathscr{R}$ to $\mathscr{L}$. ■

The next corollary classifies all the maximal and completely prime ideals of the algebra $\mathcal{S}$.

Corollary 3.4 (i) *We have that* $\operatorname{Max}(\mathcal{S}) = \big\{(\mathfrak{r}, Z), (\mathfrak{d}, X), (I_n, X), (\mathfrak{q}, I'_n), (\mathfrak{m}) \mid n \in \mathbb{N}_+, \mathfrak{r} \in \mathcal{R}, \mathfrak{d} \in \mathcal{D}, \mathfrak{q} \in \mathcal{Q}, \mathfrak{m} \in \mathcal{H}_2\big\}$, *where* $\mathcal{R}, \mathcal{D}, \mathcal{Q}$ *and* $\mathcal{H}_2$ *are defined in Theorem* 3.3.

(ii) *The set* $\operatorname{Spec}_c(\mathcal{S})$ *of completely prime ideals of* $\mathcal{S}$ *is equal to*

$$\operatorname{Spec}(\mathcal{S}) \smallsetminus \{(I_n, X), (\mathfrak{q}, I'_n) \mid n \geqslant 2\}.$$

Proof (i) Statement (i) follows from (3.2).

(ii) Statement (ii) follows from (3.1), Lemma 3.2, Theorem 3.3, and its proof. ■

4 The Primitive Ideals and a Classification of Singular Whittaker Modules over the Schrödinger Algebra

In this section, $\mathbb{K}$ is an algebraically closed field. The aim of this section is to give a classification of primitive ideals of the algebra $\mathcal{S}$ (Theorem 4.4), to prove the existence of simple singular Whittaker $\mathcal{S}$-modules (Proposition 4.6), and to classify all of them (Theorem 4.7).

For $\lambda \in \mathbb{K}$, let $\mathcal{S}(\lambda) := \mathcal{S}/\mathcal{S}(Z - \lambda)$. Then $\mathcal{S}(0) \simeq A$. If $\lambda \neq 0$, then by (2.8),

$$\mathcal{S}(\lambda) = \mathcal{S}_Z/\mathcal{S}_Z(Z - \lambda) = U'_\lambda \otimes A_1 \tag{4.1}$$

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is a tensor product of algebras. The algebra U'_λ, which is isomorphic to the enveloping algebra $U(\mathrm{sl}_2)$, is generated by the elements

$$H'_\lambda = H + \lambda^{-1}XY - \frac{1}{2}, \quad E'_\lambda = E - \frac{1}{2}\lambda^{-1}X^2, \quad F'_\lambda = F + \frac{1}{2}\lambda^{-1}Y^2.$$

The elements H'_λ, E'_λ, and F'_λ are the canonical generators for the Lie algebra sl_2 ($[H'_\lambda, E'_\lambda] = 2E'_\lambda$, $[H'_\lambda, F'_\lambda] = -2F'_\lambda$, and $[E'_\lambda, F'_\lambda] = H'_\lambda$; see Lemma 2.2 for details). The algebra A_1 is the Weyl algebra which is generated by the elements $\lambda^{-1}X$ and Y. So, $\mathcal{S}(\lambda)$ is a Noetherian domain of Gelfand–Kirillov dimension 5, and the ideal of $\mathcal{S}$ generated by $Z - \lambda$ is completely prime. Furthermore, $Z(\mathcal{S}(\lambda)) = K[\Delta'_\lambda]$, where

$$\Delta'_\lambda := 4E'_\lambda F'_\lambda + H'^2_\lambda - 2H'_\lambda = (4FE + H^2 + H) + 2\lambda^{-1}(EY^2 + HXY - FX^2) - \frac{3}{4}$$

is the Casimir element of the algebra U'_λ.

For any $\mu \in \mathbb{K}$, let $\mathcal{S}(\lambda, \mu) := \mathcal{S}(\lambda)/\mathcal{S}(\lambda)(\Delta'_\lambda - \mu)$. The following lemma is a simplicity criterion for the algebra $\mathcal{S}(\lambda, \mu)$.

Lemma 4.1 *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$.*

(i) $Z(\mathcal{S}(\lambda, \mu)) = \mathbb{K}$.
(ii) *The algebra $\mathcal{S}(\lambda, \mu)$ is a simple algebra if and only if the algebra $U'_\lambda/(\Delta'_\lambda - \mu)$ is simple if and only if $\mu \neq n^2 + 2n$ for all $n \in \mathbb{N}$.*
(iii) *If $\mu = n^2 + 2n$ for some $n \in \mathbb{N}$, then $\mathcal{S}(\lambda, \mu)$ has a unique proper two-sided ideal that is the tensor product of the annihilator of the unique simple $(n+1)$-dimensional sl_2-module and the Weyl algebra A_1. This ideal is an idempotent ideal.*

Proof Statement (i) follows from (4.1). By (4.1), the algebra $\mathcal{S}(\lambda, \mu)$ is simple if and only if the algebra $U'_\lambda/(\Delta'_\lambda - \mu)$ is so. Then statements (ii) and (iii) follows from [10, 4.9.22]. ∎

Primitive ideals of the algebra $\mathcal{S}$ Proposition 4.2 provides classifications of prime, completely prime, maximal, and primitive ideals of the algebra $\mathcal{S}(\lambda)$ where $\lambda \neq 0$, it also gives explicit generators for them.

Let $\mathcal{J}(\mathcal{S}(\lambda))$ and $\mathcal{J}(U'_\lambda)$ be the sets of ideals of the algebras $\mathcal{S}(\lambda)$ and U'_λ, respectively. The sets $\mathcal{J}(\mathcal{S}(\lambda))$ and $\mathcal{J}(U'_\lambda)$ are partially ordered sets (posets) with respect to inclusion. Since the algebra A_1 is a central simple algebra, the map

$$\mathcal{J}(U'_\lambda) \longrightarrow \mathcal{J}(\mathcal{S}(\lambda)), \quad I \longmapsto I \otimes A_1,$$

is an *isomorphism of posets*. We identify the posets $\mathcal{J}(U'_\lambda)$ and $\mathcal{J}(\mathcal{S}(\lambda))$ via the map above. As a result, the first two equalities of Proposition 4.2 are obvious. Notice that the set of completely prime ideals of the algebra U'_λ is equal to $\mathrm{Spec}_c(U'_\lambda) = \mathrm{Spec}(U'_\lambda) \setminus \{\mathrm{ann}(V_n) | n = 2, \dots\}$ where V_n is a unique simple n-dimensional U'_λ-module. The set of primitive ideals of U'_λ is

$$\mathrm{Prim}(U'_\lambda) = \mathrm{Spec}(U'_\lambda) \setminus \{0\} = \{\mathrm{ann}(V_n) \mid n = 1, \dots\} \sqcup \{(\Delta'_\lambda - \mu) \mid \mu \in \mathbb{K}\}.$$

Proposition 4.2 *Let $\lambda \in \mathbb{K}^*$. Then*

$$\mathrm{Spec}(\mathcal{S}(\lambda)) = \mathrm{Spec}(U'_\lambda), \qquad \mathrm{Max}(\mathcal{S}(\lambda)) = \mathrm{Max}(U'_\lambda),$$
$$\mathrm{Spec}_c(\mathcal{S}(\lambda)) = \mathrm{Spec}_c(U'_\lambda), \qquad \mathrm{Prim}(\mathcal{S}(\lambda)) = \mathrm{Prim}(U'_\lambda).$$

Proof Let B be an algebra. Then the tensor product of algebras $B \otimes A_1$ is a domain if and only if B is so. Hence, $\mathrm{Spec}_c(\mathcal{S}(\lambda)) = \mathrm{Spec}_c(U'_\lambda)$. Clearly, $\mathrm{Prim}(U'_\lambda) \subseteq \mathrm{Prim}(\mathcal{S}(\lambda))$ and $\mathrm{Prim}(U'_\lambda) = \mathrm{Spec}(U'_\lambda) \smallsetminus \{0\} = \mathrm{Spec}(\mathcal{S}(\lambda)) \smallsetminus \{0\}$. Since $\mathrm{Spec}(\mathcal{S}(\lambda)) = \mathrm{Spec}(U'_\lambda)$ and 0 is not a primitive ideal of $\mathcal{S}(\lambda)$ (since $Z(\mathcal{S}(\lambda)) = \mathbb{K}[\Delta'_\lambda]$), we must have $\mathrm{Prim}(U'_\lambda) = \mathrm{Prim}(\mathcal{S}(\lambda))$. ∎

Primitive ideals of the algebra $\mathcal{S}$ and their explicit generators The primitive ideals of the algebra $A = \mathcal{S}/(Z)$ were classified in [5, Theorem 2.10]; see Theorem 4.3 below. For each of the primitive ideals an explicit set of generators is given. The centre of the algebra A is a polynomial algebra $\mathbb{K}[\mathsf{c}]$ [5, Lemma 2.2.(1)] where $\mathsf{c} = FX^2 - HXY - EY^2$. The algebra $U := U(\mathrm{sl}_2)$ is isomorphic to $A/(X) \simeq \mathcal{S}/(Z, X)$, [5, Lemma 2.3.(2)]. Therefore, we can write $\mathrm{Spec}(U) \subseteq \mathrm{Spec}(A) \subseteq \mathrm{Spec}(\mathcal{S})$.

Theorem 4.3 ([5, Theorem 2.10]) *We have*

$$\mathrm{Prim}(A) = \mathrm{Prim}(U) \sqcup \left\{ A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}(\mathbb{K}[\mathsf{c}]) \smallsetminus \{0\} \right\}.$$

The primitive ideals of $\mathcal{S}(\lambda)$ for $\lambda \neq 0$ were described in [12, Corollary 30]: Each of them is the annihilator of a simple highest weight modules.

The next theorem together with Theorem [5, Theorem 2.10], gives an explicit description of the set of primitive ideals of $\mathcal{S}$ and their generators.

Theorem 4.4 *Suppose that $\mathbb{K}$ is an algebraically closed field. Then*

$$\mathrm{Prim}(\mathcal{S}) = \left\{ (Z-\lambda, \mathfrak{p}) \mid \lambda \in \mathbb{K}^*, \mathfrak{p} \in \mathrm{Spec}(U'_\lambda) \smallsetminus \{0\} \right\} \sqcup \left\{ (Z, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Prim}(A) \right\}.$$

Proof Since Z is a central element of $\mathcal{S}$ and $\mathbb{K}$ is algebraically closed, any primitive ideal of $\mathcal{S}$ contains $Z - \lambda$ for some $\lambda \in \mathbb{K}$. Hence, $\mathrm{Prim}(\mathcal{S}) = \sqcup_{\lambda \in \mathbb{K}^*} \mathrm{Prim}(\mathcal{S}(\lambda)) \sqcup \mathrm{Prim}(A) = \{(Z-\lambda, \mathfrak{p}) \mid \lambda \in \mathbb{K}^*, \mathfrak{p} \in \mathrm{Prim}(U'_\lambda)\} \sqcup \{(Z, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Prim}(A)\}$, as required. Notice that $\mathrm{Prim}(U'_\lambda) = \mathrm{Spec}(U'_\lambda) \smallsetminus \{0\}$. ∎

Singular Whittaker $\mathcal{S}$-modules Simple, non-singular, Whittaker modules of the Schrödinger algebra were classified in [18]. It was conjectured that there is no simple singular Whittaker module for the Schrödinger algebra [18, Conjecture 4.2]. Proposition 4.6 shows that there exist simple singular Whittaker A-modules (these are Whittaker Schrödinger modules of *zero level*), hence the conjecture is not true, in general. But we prove that the conjecture is true for Whittaker Schrödinger modules of *non-zero level*.

Let $R = \mathcal{S}$ or $\mathcal{S}(\lambda)$ for some $\lambda \in \mathbb{K}$ and let V be an R-module. A non-zero element $w \in V$ is called a *Whittaker vector* of type (μ, δ) if $Ew = \mu w$ and $Xw = \delta w$ where $\mu, \delta \in \mathbb{K}$. An R-module V is called a *Whittaker module* of type (μ, δ) if V is generated by a Whittaker vector of type (μ, δ). An R-module V is called a *singular Whittaker module* if V is generated by a Whittaker vector $w \in V$ of type $(0, 0)$ and $Hw \notin \mathbb{K}w$.

Using the decomposition (4.1), we can give easily a classification of simple Whittaker $\mathcal{S}$-module of non-zero level (*i.e.,* the simple Whittaker $\mathcal{S}(\lambda)$-module where $\lambda \neq 0$).

Whittaker $\mathcal{S}(\lambda)$-modules where $\lambda \neq 0$ Let $\mu, \delta \in \mathbb{K}$. The *universal* Whittaker $\mathcal{S}(\lambda)$-module of type (μ, δ) is $W := W(\mu, \delta) := \mathcal{S}(\lambda)/\mathcal{S}(\lambda)(E - \mu, X - \delta)$. So, any Whittaker $\mathcal{S}(\lambda)$-module of type (μ, δ) is a homomorphic image of W. By (4.1),

$$
\begin{aligned}
(4.2)\quad W &= \frac{\mathcal{S}(\lambda)}{\mathcal{S}(\lambda)(E'_\lambda + 1/2\lambda^{-1}X^2 - \mu, X - \delta)} = \frac{\mathcal{S}(\lambda)}{\mathcal{S}(\lambda)(E'_\lambda + 1/2\lambda^{-1}\delta^2 - \mu, X - \delta)}\\
&= U'_\lambda/U'_\lambda(E'_\lambda + 1/2\lambda^{-1}\delta^2 - \mu) \otimes A_1/A_1(X - \delta).
\end{aligned}
$$

The module $W_{U'_\lambda} := U'_\lambda/U'_\lambda(E'_\lambda + 1/2\lambda^{-1}\delta^2 - \mu)$ is a Whittaker U'_λ-module of type $(-1/2\lambda^{-1}\delta^2 + \mu)$. Note that $A_1/A_1(X - \delta)$ is a *simple* A_1-module with $\mathrm{End}_{A_1}(A_1/A_1(X - \delta)) = \mathbb{K}$, we call this module the Whittaker A_1-module of type (δ). For an algebra R, the set of isomorphism classes of simple R-modules is denoted by $\widehat{R}$. If $\mathscr{P}$ is an isomorphism invariant property then $\widehat{R}(\mathscr{P})$ is the set of isomorphism classes of simple R-modules that satisfy the property $\mathscr{P}$. The simple Whittaker U'_λ-modules are easily classified. Namely, by (4.2), for $\lambda \neq 0$,

$$
\begin{aligned}
&\widehat{\mathcal{S}(\lambda)}(\text{Whittaker module of type } (\mu, \delta)) =\\
&\widehat{U'_\lambda}(\text{Whittaker module of type } (\mu - 1/2\lambda^{-1}\delta^2)) \otimes \widehat{A_1}(\text{Whittaker module of type } (\delta)),
\end{aligned}
$$

i.e., when $\lambda \neq 0$, each simple Whittaker $\mathcal{S}(\lambda)$-module of type (μ, δ) is isomorphic to the tensor product of a simple Whittaker U'_λ-module of type $(-1/2\lambda^{-1}\delta^2 + \mu)$ and the simple A_1-module $A_1/A_1(X - \delta)$ and these two modules uniquely (up to isomorphism) determine the Whittaker module.

The next proposition shows that there is no simple singular Whittaker $\mathcal{S}$-module of nonzero level; *i.e.,* if $\lambda \neq 0$, then all the simple Whittaker $\mathcal{S}(\lambda)$-modules of type $(0, 0)$ are weight modules.

Proposition 4.5 *If $\lambda \in \mathbb{K}^*$, then there is no simple singular Whittaker $\mathcal{S}(\lambda)$-module.*

Proof By (4.2), $W = U'_\lambda/U'_\lambda E'_\lambda \otimes A_1/A_1X$. Notice that A_1/A_1X is a simple A_1-module and $\mathrm{End}_{A_1}(A_1/A_1X) = \mathbb{K}$. Hence, each simple factor module L of W is equal to $M \otimes A_1/A_1X$ where M is a simple factor module of the U'_λ-module $U'_\lambda/U'_\lambda E'_\lambda$. Then by [18, Theorem 6.10.(i)], M is a (highest) H'_λ-weight U'_λ-module; *i.e.,* M is a simple factor module of $U'_\lambda/U'_\lambda(H'_\lambda - \mu, E'_\lambda)$ for some $\mu \in \mathbb{K}$. Then L is a simple factor module of $U'_\lambda/U'_\lambda(H'_\lambda - \mu, E'_\lambda) \otimes A_1/A_1X \simeq \mathcal{S}(\lambda)/\mathcal{S}(\lambda)(H + \frac{1}{2} - \mu, E, X)$. Hence, L is a weight module. This completes the proof. ∎

Recall that $\mathcal{S}(0) = A$. Let $\mathcal{W} := A/A(X, E)$, a left A-module. Then any singular Whittaker A-module is an epimorphic image of $\mathcal{W}$. For any $\nu \in \mathbb{K}^*$, we define the A-module

$$V(\nu) := A/A(X, E, Y - \nu) = \bigoplus_{i,j \in \mathbb{N}} \mathbb{K}H^iF^j\bar{1} \quad \text{where } \bar{1} = 1 + A(X, E, Y - \nu).$$

Clearly, $V(\nu)$ is a singular Whittaker A-module. Then next proposition shows that $V(\nu)$ is a simple A-module. Hence, [18, Conjecture 4.2] does not hold in this case.

Proposition 4.6 *For any $\nu \in \mathbb{K}^*$, the module $V(\nu)$ is a simple singular Whittaker A-module.*

Proof We have to show that for any $0 \neq v = \sum_{i,j\in\mathbb{N}} \alpha_{i,j} H^i F^j \bar{1} \in V(\nu)$ where $\alpha_{i,j} \in \mathbb{K}$, there exists an element $a \in A$ such that $av \in \mathbb{K}^* \bar{1}$. If $j > 0$, then

$$Xv = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}(-\nu) j (H-1)^i F^{j-1} \bar{1} \neq 0.$$

Therefore, $0 \neq X^n v \in \mathbb{K}[H]\bar{1}$ for some $n \in \mathbb{N}$. So, we may assume that $v = \sum_{i=0}^{m} \alpha_i H^i \bar{1}$ where $\alpha_i \in \mathbb{K}$, $m \in \mathbb{N}$ and $\alpha_m \neq 0$. Then $0 \neq (Y-\nu)v = \sum_{i=0}^{m} \alpha_i \nu((H+1)^i - H^i)\bar{1}$. By induction on m, we have $(Y-\nu)^m v \in \mathbb{K}^* \bar{1}$, as required. ■

Classification of simple singular Whittaker A-modules Consider the following subalgebras of A: $R = \mathbb{K}[H][Y;\sigma]$ where $\sigma(H) = H+1$; $T = \mathbb{K}\langle F, H, Y\rangle$; and $\mathcal{A} = \mathbb{K}\langle H, E, X, Y\rangle$. Clearly, $R \subset T$, $R \subset \mathcal{A}$ and the left ideal $\mathcal{A}(E,X) := \mathcal{A}E + \mathcal{A}X$ of the algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$, *i.e.,*

$$\mathcal{A}(E,X) = (E,X) \quad \text{and} \quad R \simeq \mathcal{A}/(E,X).$$

Hence, every R-module is automatically an $\mathcal{A}$-module. Moreover, R-modules are precisely $\mathcal{A}$-modules that are annihilated by the ideal (E,X). In particular, every simple R-module is a simple epimorphic image of the $\mathcal{A}$-module $\mathcal{A}/\mathcal{A}(E,X) = \mathcal{A}/(E,X)$, and vice versa.

The next theorem together with Theorem 4.8 gives a classification of simple singular Whittaker A-modules.

Theorem 4.7 *The map $\widehat{R}(\mathbb{K}[H]\text{-torsionfree}) \to \widehat{A}(\text{singular Whittaker})$, $[M] \mapsto [\widetilde{M} := A \otimes_{\mathcal{A}} M]$ is a bijection with the inverse $[\mathcal{M}] \mapsto [\ker(X_{\mathcal{M}}) \cap \ker(E_{\mathcal{M}})]$, and $\widetilde{M} = \bigoplus_{i\geqslant 0} F^i \otimes M$.*

Proof (i) *The map $[M] \mapsto [\widetilde{M}]$ is well defined*: Since $A = \bigoplus_{i\geqslant 0} F^i \mathcal{A}$, we must have $\widetilde{M} = \bigoplus_{i\geqslant 0} F^i \otimes M$. Since M is a simple $\mathbb{K}[H]$-torsionfree R-module (*i.e.,* an $\mathcal{A}/(E,X)$-module), the A-module $\widetilde{M}$ belongs to $\widehat{A}(\text{singular Whittaker})$ provided it is simple. Since Y is a normal element of the ring R and $R/RY \simeq \mathbb{K}[H]$, the map $Y_M\colon M \to M$, $m \mapsto Ym$ is an injection (since $[M] \in \widehat{R}(\mathbb{K}[H]\text{-torsionfree})$). Suppose that V is a nonzero submodule of $\widetilde{M}$; we aim to show that $V = \widetilde{M}$. Fix a nonzero element $v = \sum_{i=0}^{n} F^i m_i$ of V (where $m_i \in M$ such that $m_n \neq 0$ and n is the least possible). We claim that $n = 0$, otherwise, the element of V, $Xv = \sum_{i=1}^{n} iF^{i-1} Y m_i$, is nonzero. This contradicts the choice of n since $0 \neq Ym_n \in M$. So, $V \cap M \neq 0$, and so $M \subseteq V$ and $V = \widetilde{M}$, *i.e.,* $\widetilde{M}$ is a simple singular Whittaker A-module.

(ii) *The map $[M] \to [\widetilde{M}]$ is a surjection*: Let $[N] \in \widehat{A}(\text{singular Whittaker})$. Then $N = AM$ where $M := \ker(E_N) \cap \ker(X_N)$ is a nonzero $\mathbb{K}[H]$-torsionfree R-module/$\mathcal{A}$-module. In particular, $N = AN = \sum_{i\geqslant 0} F^i \mathcal{A} M = \sum_{i\geqslant 0} F^i M$ (since $A = \bigoplus_{i\geqslant 0} F^i \mathcal{A}$).

Claim 1: *The map $Y_M\colon M \to M$, $m \mapsto Ym$ is an injection.* The kernel $M_0 = \ker(Y_M)$ is an R-submodule/$\mathcal{A}$-submodule of M. If $M_0 \neq 0$, then the A-module $N = AM_0 = \sum F^i M_0$ is annihilated by the ideal (X,Y) of A (since $XM_0 = YM_0 = 0$ and $(X,Y) =$

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$AX + AY$). So, N is a simple, highest weight sl_2-module, but each such sl_2-module is $\mathbb{K}[H]$-torsion. This contradicts the fact that M is $\mathbb{K}[H]$-torsionfree.

Claim 2: $N = \bigoplus_{i\geqslant 0} F^i M$. Suppose this is not true, *i.e.*, $u = \sum_{i=0}^m F^i u_i = 0$ for some elements $u_i \in M$ such that $u_m \neq 0$. We may assume that m is the least possible. Then $m \geqslant 1$ and

$$0 = Xu = \sum_{i=1}^{m} iF^{i-1}Yu_i = mF^{m-1}Yu_m + \cdots,$$

a contradiction (since $Yu_m \neq 0$, by Claim 1).

By Claim 2, $N = \widetilde{M}$. Since N is a simple A-module, the R-module/$\mathcal{A}$-module M must be simple (by Claim 2). The proof of statement (ii) is complete.

(iii) *For all* $M \in \widehat{R}(\mathbb{K}[H]\text{-torsionfree})$, $\ker(E_{\widetilde{M}}) \cap \ker(X_{\widetilde{M}}) = M$: This is obvious.

(iv) *The map* $[M] \to [\widetilde{M}]$ *is an injection*: This follows from statement (iii).

Now the theorem follows from the statements (ii)–(iv). ∎

The set $\widehat{R}(\mathbb{K}[H]$-torsionfree) The algebra $R = \mathbb{K}[H][Y;\sigma]$ is a subalgebra of the algebra $B = \mathbb{K}(H)[Y;\sigma]$, which is a localization of R at $\mathbb{K}[H] \setminus \{0\}$. The algebra B is a (left and right) principle ideal domain. We denote by $\mathrm{Irr}(B)$ be the set of its irreducible elements.

In [4], a classification is given of simple modules over an arbitrary Ore extension $D[X;\sigma,\delta]$, where D is a Dedekind ring, σ is an automorphism and δ is a σ-derivation of D. The ring R is a very special case of such a ring.

Theorem 4.8 ([4], [5, Theorem 4.10]) *We have*

(i) $\widehat{R}(\mathbb{K}[H]\text{-torsion}) = \widehat{R}(Y\text{-torsion}) = \widehat{R/(Y)} = \{[R/R(H-\nu, Y)] | \nu \in \mathbb{K}\}$,

(ii) $$\widehat{R}(\mathbb{K}[H]\text{-torsionfree}) = \widehat{R}(Y\text{-torsionfree}) = \left\{ [M_b] \mid b \in \mathrm{Irr}(B), R = RY + R \cap Bb \right\},$$

where $M_b := R/R \cap Bb$; $M_b \simeq M_{b'}$ if and only if the elements b and b' are similar (if and only if $B/Bb \simeq B/Bb'$ as B-modules).

Another approach for classifying the simple R-modules was taken by Block [8].

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