Canad. Math. Bull. Vol. **20** (2), 1977

A NOTE ON CONVERGENCE OF FOURIER SERIES OF A FUNCTION ON WIENER'S CLASS V_p

BY
RAFAT N. SIDDIQI[(1)]

1. **Introduction.** Let f be a 2π-periodic function defined on $[0, 2\pi]$. We set

$$\overset{b}{\underset{a}{V_p}}(f) = \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p\right\}^{1/p} \qquad (1 \le p < \infty),$$

where suprema have been taken with respect to all partitions $P: a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of any segment $[a, b]$ contained in $[0, 2\pi]$. We call $V_p{}_a^b(f)$ the pth total variation of f on $[a, b]$. If we denote pth total variaiton of f on $[0, 2\pi]$ by $V_p(f)$, then we can define Wiener's class simply by

$$V_p = \{f : V_p(f) < \infty\}.$$

It is clear that V_1 is the ordinary class of functions of bounded variation, introduced by Jordan. The class V_p was first introduced by N. Wiener [5]. He [5] showed that functions of the class V_p could only have simple discontinuities. We note [3] that

(1) $$V_{p_1} \subset V_{p_2} \qquad (1 \le p_1 < p_2 < \infty)$$

is a strict inclusion. Hence for an arbitrary $1 \le p < \infty$, Wiener's class V_p is strictly larger than the class V_1.

2. Let $f \in V_p$ $(1 \le p < \infty)$ and let

$$S(f) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Wiener [5] proved the following theorem.

THEOREM A. *If $f \in V_p$ $(1 \le p < \infty)$, then $S(f)$ converges almost everywhere in $[0, 2\pi]$. In particular, if $f \in V_2$ then $S(f)$ converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at every $x \in [0, 2\pi]$.*

In this note we extend the above theorem for the class V_p for every p and we give the simple proof of our following main theorem.

Received by editors August 8, 1975 and, in revised form, May 12, 1976.

(1) This research was supported by National Research Council of Canada grant given to the Dept. of Phys.-Maths., Université de Moncton, Moncton, N.B., and by a fellowship of SRI of Canadian Mathematical Congress.

THEOREM. *If* $f\in V_p$ $(1\le p<\infty)$, *then* $S(f)$ *converges to* $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every* $x\in[0,2\pi]$.

3. **Proof.** In view of (1), it suffices to prove our Theorem for $p>2$. Suppose that

$$f(x)=\tfrac{1}{2}[f(x+0)+f(x-0)].$$

Also suppose that

$$f_x(t)=f(x+t)-2f(x)+f(x-t).$$

Then $f_x(t)$ is continuous at $t=0$ and hence

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le\varepsilon}\{f_x(t)\}=0. \tag{2}$$

If we denote $V_p{}_0^{2\pi}(f_x)$ the pth total variation of f_x on $[0,2\pi]$, it can easily be verified [5] that

$$\lim_{\varepsilon\to 0}\overset{\varepsilon}{\underset{0}{V_p}}(f_x)=0. \tag{3}$$

Let $S_n(x)$ denote the nth partial sum of the Fourier series of f at the point x then

$$\begin{aligned} S_n(x)-f(x)&=\pi^{-1}\int_0^{\pi}\frac{\sin nt}{t}f_x(t)\,dt+o(1)\\ &=\pi^{-1}\sum_{k=0}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n}\frac{\sin nt}{t}f_x(t)\,dt+o(1). \end{aligned} \tag{4}$$

By change of variable the above expression can be written

$$=(2\pi)^{-1}\sum_{j=1}^{n}\int_{(j-1)\pi/n}^{j\pi/n}\left(\frac{f_x(t+2j\pi/n)}{t+2j\pi/n}-\frac{f_x(t+(2j+1)\pi/n)}{t+(2j+1)\pi/n}\right)\sin nt\,dt+o(1)$$

$$=(\pi)^{-1}\int_0^{\pi/n}\left[\sum_{j=1}^{[n/2]}\left(\frac{f_x(t+2j\pi/n)}{t+2j\pi/n}-\frac{f_x(t+(2j+1)\pi/n)}{t+(2j+1)\pi/n}\right)\right]\sin nt\,dt+o(1)$$

$$=(\pi)^{-1}\int_0^{\pi/n}\left[\sum_{j=1}^{[\varepsilon n]}+\sum_{[\varepsilon n]+1}^{[n/2]}\right]\sin nt\,dt+o(1)=I_n(\varepsilon)+J_n(\varepsilon)+o(1).$$

We note here that

$$\int_0^{\pi/n} \frac{f_x\left(t+\dfrac{n-1}{n}\pi\right)}{t+\dfrac{n-1}{n}\pi} \sin nt\,dt = o(1), \qquad \int_0^{\pi/n} \frac{f_x\left(t+\dfrac{\pi}{n}\right)}{t+\dfrac{\pi}{n}} \sin nt\,dt = o(1).$$

We consider now

$$|I_n(\varepsilon)| = \left|(\pi)^{-1}\left[\int_0^{\pi/n}\left(\sum_{j=1}^{[\varepsilon n]} \frac{f_x(t+2j\pi/n)}{t+2j\pi/n} - \frac{f_x(t+(2j+1)\pi/n)}{t+(2j+1)\pi/n}\right)\right] \sin nt\,dt\right|$$

$$\leq (\pi)^{-1}\int_0^{\pi/n}\left[\sum_{j=1}^{[\varepsilon n]} \frac{|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n|}{t+2j\pi/n}\right] \sin nt\,dt$$

$$+\frac{1}{n}\int_0^{\pi/n}\left[\sum_{j=1}^{[\varepsilon n]} \frac{|f_x(t+(2j+1)\pi/n)|}{(t+2j\pi n)(t+(2j+1)\pi/n)}\right] \sin nt\,dt$$

$$= I_{n_1}(\varepsilon) + I_{n_2}(\varepsilon).$$

For $1/p+1/q=1$, if we apply Hölder's inequality on the sum of integrand of $I_{n_1}(\varepsilon)$, we obtain,

$$I_{n_1}(\varepsilon) \leq (\pi)^{-1}\left[\int_0^{\pi/n}\left(\sum_{j=1}^{[\varepsilon n]} |f_x(t+2j\pi/n) - f_x(t+(2j+1)\pi/n)|^p\right)^{1/p}\right.$$

$$\left.\left(\sum_{j=1}^{[\varepsilon n]}\left|\frac{1}{2j\pi/n}\right|^q\right)^{1/q}\right] \sin nt\,dt.$$

Using the definition of $V_p{}_a^b(f)$ we obtain

(5) $$I_{n_1}(\varepsilon) \leq \pi^{-1}\int_0^{\pi/n} \overset{3\varepsilon\pi}{\underset{0}{V_p}}(f_x)\left(\sum_{j=1}^{[\varepsilon n]}\left|\frac{1}{2j\pi/n}\right|^q\right)^{1/q} \sin nt\,dt.$$

Now consider the sum

$$\sum_{j=1}^{[\varepsilon n]}\left(\frac{1}{|2j\pi/n|}\right)^q = \left(\frac{n}{2\pi}\right)^q\left\{1+\frac{1}{2^q}+\frac{1}{3^q}+\cdots+\frac{1}{[\varepsilon n]^q}\right\} \leq \left(\frac{n}{2\pi}\right)^q \sum_{n=1}^{\infty}\frac{1}{n^q},$$

but the series on right hand side converges for $q>1$, hence we can find a positive number M such that

(6) $$\sum_{j=1}^{[\varepsilon n]}\left(\frac{1}{2j\pi/n}\right)^q \leq Mn^q.$$

From (5) and (6) we obtain,

(7) $$I_{n_1}(\varepsilon) \leq \pi^{-1}\int_0^{\pi/n} \overset{3\varepsilon\pi}{\underset{0}{V_p}}(f_x) M^{1/q} n \sin nt\,dt = 0(1)\left(\overset{3\pi\varepsilon}{\underset{0}{V_p}}(f_x)\right);$$

and also

$$I_{n_2}(\varepsilon)=\frac{1}{n}\int_0^{\pi/n}\left[\sum_{j=1}^{[\varepsilon n]}\frac{|f_x(t+(2j+1)\pi/n)|}{(t+2j\pi/n)(t+(2j+1)\frac{\pi}{n}}\right]\sin nt\,dt$$

(8)
$$=o(1)\left[\sup_{0\le t\le 3\varepsilon\pi}\{f_x(t)\}\right]$$

Hence from (7) and (8), we obtain

$$|I_n(\varepsilon)|=o(1)\left(\overset{3\pi\varepsilon}{\underset{0}{V_p}}(f_x)+\sup_{0\le t\le 3\varepsilon\pi}\{f_x(t)\}\right).$$

Taking limit as $n\to\infty$ and using (2) and (3) we obtain,

(9) $|I_n(\varepsilon)|\le K(\varepsilon)$ if $n\ge n(\varepsilon)$, $K(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Similarly we can show that

(10)
$$|J_n(\varepsilon)|=o(1)\left[\overset{\pi}{\underset{0}{V_p}}(f_x)\frac{1}{(\varepsilon n)^{1/p}}+\frac{1}{n}\right]\quad\text{as}\quad n\to\infty.$$

Therefore taking limits as $n\to\infty$, we obtain,

$$\lim_{n\to\infty}\sup|S_n(x)-f(x)|\le\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup[|I_n(\varepsilon)|+|J_n(\varepsilon)|+o(1)]$$

which tends to zero from (9), (10), and (4). This completes the proof of our main Theorem.

ACKNOWLEDGEMENT. I would like to express my sincere thanks to Professor Thu Pham-Gia for his helpful suggestions concerning this paper. My thanks are also due to the referee for his suggestions to improve its presentation.

REFERENCES

1. N. Bari, *A treatise on trigonometric series*, Vol. **1**, Oxford, Pergamon Press, 1964.
2. W. Rudin, *Fourier Analysis on groups*, New York, Interscience Publishers, 1962.
3. R. N. Siddiqi, *The order of Fourier coefficients of a function of higher variation*, Proc. Japan Acad., **48** (1972), 569–572.
4. R. N. Siddiqi, *Some properties of Fourier-Stieltjes coefficients of a function of Wiener's class* V_p, Bull. Math. De Roumanie, Tome **16** (**64**), nr. **1** (1972), 105–112.

5. N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, Massachusetts J. Math. **3** (1924), 72–94.

6. A. Zygmund, *Trigonometric series*, *Vol.* **1,** Cambridge, 1959.

DEPARTMENT OF PHYSICS-MATHEMATICS
UNI√ERSITÉ DE MONCTON
MONCTON, N.B., CANADA

and

SUMMER RESEARCH INSTITUTE OF
CANADIAN MATHEMATICAL CONGRESS
DALHOUSIE UNIVERSITY
HALIFAX, N.S., CANADA

Present address:
DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY
KUWAIT