NEW RESULTS FOR THE TWO-STAGE CONTACT PROCESS

ERIC FOXALL,* *University of Victoria*

Abstract

In this paper, we continue the work started by Steve Krone on the two-stage contact process. We give a simplified proof of the duality relation and answer most of the open questions posed in Krone (1999). We also fill in the details of an incomplete proof.

Keywords: Contact process; interacting particle systems

2010 Mathematics Subject Classification: Primary 60J25; 92B99

1. Introduction

The contact process is a stochastic spatial model of population growth that was introduced in the 1970s [8] and has since been widely studied. In its simplest form, each site is in one of two states, occupied or vacant, and a great deal is known about this model (see, for example, [12] and [13]). More recently, variants of the model have been studied in which there is more than one type of individual [14], [10], or more than one stage of development [9]. In [15] a multitype framework was used to model infection spread between households located at the vertices of a homogenous tree. In [11], a two-stage model of infection spread was studied on scale-free networks.

In this paper, we consider the two-stage contact process introduced in [9]. It is a natural generalization of the contact process in which there is an intermediate juvenile type that must mature before it can produce offspring. More precisely, it is a growth model on $\mathbb{Z}^d$ defined by the rates

$$
\begin{aligned}
0 &\rightarrow 1 \text{ at rate } \lambda n_2, \\
1 &\rightarrow 2 \text{ at rate } \gamma, \\
1 &\rightarrow 0 \text{ at rate } 1 + \delta, \\
2 &\rightarrow 0 \text{ at rate } 1,
\end{aligned}
$$

where $n_2(x)$ is the cardinality of the set $\{y \in \mathbb{Z}^d : 0 < \|y - x\|_\infty < r, y \text{ is in state } 2\}$ for some $r \geq 1$. The state space for the process is $\{0, 1, 2\}^{\mathbb{Z}^d}$, so each site is either unoccupied, recently occupied, or occupied by a mature organism that can give birth at other sites. Aside from the choice of neighbourhood, there are three parameters, λ, γ, and δ, which denote the transmission rate, the maturation rate, and the juvenile death rate, respectively.

A number of basic properties of the process are proved in [9], including additivity and monotonicity with respect to parameters (increasing in λ and γ and decreasing in δ), as well as a duality relation, and some bounds on the survival region (the set of parameters for which an initially finite population has a chance of surviving for all time).

Here we consider the process in the more general setting of a countable graph (V, E) with finite maximum degree, proving some results in this setting and more precise results on $\mathbb{Z}^d$. We simplify the proof of the duality relation given in [9] and answer most of the open questions posed in [9, Section 4]. As we will see, for the two-stage contact process there is a critical value of the maturation rate below which survival does not occur (see Theorem 1, below). Also, it shares many of the properties of the contact process; in particular, there is complete convergence (see Theorem 3, below). The following is a summary of the main results.

Our first result is an upper bound on the set of values γ so that the process dies out, i.e. reaches the all 0 state with probability 1. The bound depends only on the maximum degree $M = \max_x \deg x$ of the graph.

Theorem 1. *If $\gamma < 1/(2M - 1)$ then, starting from any finite number of occupied sites, the process dies out, no matter what the values of λ and δ are.*

This answers question 6 in [9], where the author supplies a bound for $\mathbb{Z}^1$ in the case of nearest neighbour interactions, and asks whether a bound exists for other interactions, or for $\mathbb{Z}^d$ with $d > 1$.

Our next result shows that two notions of survival for the two-stage contact process coincide, answering question 1 in [9] affirmatively. For terminology see the correction at the start of Section 3.1.

Theorem 2. *For the two-stage contact process on $\mathbb{Z}^d$, single-site survival occurs if and only if the upper invariant measure is nontrivial.*

The proof uses the construction of [2] to show that for both the process and its dual, single-site survival implies the upper invariant measure is nontrivial.

An important question for growth models is that of *complete convergence*, which we show is true for the two-stage contact process, answering question 3 in [9]. Here λ_c is the critical value for single-site survival as defined in Section 3.1 and ξ_t denotes the process. The notation '$\Rightarrow$' denotes weak convergence.

Theorem 3. *If $\lambda > \lambda_c$ then complete convergence holds, i.e.*

$$\xi_t \Rightarrow \alpha\delta_0 + (1 - \alpha)\nu,$$

where ν is the upper invariant measure, δ_0 concentrates on the configuration with all 0s, and $\alpha = \mathbb{P}(\xi_t$ dies out).

We now summarize the organization of this paper. In Section 2 we construct the process and prove the duality relation. In Section 3.1 we recall the critical values defined in [9]. In Section 3.2 we fill in some missing details in the proof of [9, Proposition 3.6]. In Section 3.3 we prove Theorem 1. In Section 3.4 we give a sufficient condition for the edge speed of the process in one dimension to characterize survival, providing a partial answer to question 2 in [9]. In Section 3.5 we prove Theorem 2 and in Section 3.6 we prove Theorem 3. We discuss the survival region in Section 3.7, and using the construction from the proof of Theorem 2 we find that the process dies out on the boundary of the survival region, providing a partial answer to question 5. We argue that question 4 appears not to have an affirmative answer, and we give some informal arguments as to why this should be so.

2. Construction and duality

We recall briefly the construction of the process. Here the process ξ_t lives on the state space $\{0, 1, 2\}^V$, where V is the vertex set of an undirected graph (V, E), with $V = \mathbb{Z}^d$ and $E = \{xy \colon 0 < \|x - y\|_\infty < r\}$ for some $r \geq 1$ being common choices. The state space is equipped with the partial order $\xi \leq \xi' \Leftrightarrow \xi(x) \leq \xi'(x)$ for each $x \in V$, where $0 < 1 < 2$ is the order of the state at each site. The process is *attractive* if there exists a coupling so that $\xi_0 \leq \xi'_0 \Rightarrow \xi_t \leq \xi'_t$ for $t > 0$. It is *additive* if $\xi_0 = \xi'_0 \vee \xi''_0 \Rightarrow \xi_t = \xi'_t \vee \xi''_t$, where $(\xi \vee \xi')(x) = \max(\xi(x), \xi'(x))$ for each x. It is *monotone increasing* (respectively *decreasing*) with respect to a parameter λ if $\xi_0 \leq \xi'_0$ and $\lambda \leq \lambda' \Rightarrow \xi_t \leq \xi'_t$ (respectively $\lambda \geq \lambda' \Rightarrow \xi_t \leq \xi'_t$). We will often use the word *active* to refer to a site or a point in spacetime where the state is not 0.

We can construct the process on any undirected graph (V, E) by taking $n_2(x)$ to be the cardinality of the set $\{y \in V \colon xy \in E, y \text{ is in state } 2\}$. Assign independent Poisson processes to each of the events:

- death of 1s and 2s at each site, at rate 1,

- death of 1s at each site at the additional rate δ,

- transmission across each edge at rate λ,

- maturation at each site at rate γ.

Place the events on the spacetime graph $V \times \mathbb{R}^+$ and fix a configuration at time 0. The configuration at later times can then be determined from the events on the graph. To ensure that it is well defined it suffices to work backwards from a point (x, t) on the spacetime graph and ensure that, with probability 1, only finitely many events occur that can influence the state of (x, t). For this to be true it suffices that the graph has finite maximum degree, i.e. for some M we have $\deg x \leq M < \infty$ for each $x \in V$; the desired property then follows by comparison with a branching process in which births occur at rate λM.

Additivity of the process is immediate from this construction and from the fact that each transition is additive. Monotonicity with respect to parameters can be established in the usual way; for example, to compare processes with identical values of γ and δ and transmission rates $\lambda < \lambda'$ on the same graph, simply add a point process at rate $\lambda' - \lambda$ for the extra transmission events in the second process and note that this tends to give larger configurations as the process evolves.

For each δ, there is a dual process ζ_t given by the rates

$$
\begin{aligned}
0 &\to 1 \text{ at rate } \lambda n_2, \\
1 &\to 2 \text{ at rate } \gamma, \\
2 &\to 1 \text{ at rate } \delta, \\
1, 2 &\to 0 \text{ at rate } 1;
\end{aligned}
$$

Krone called this the 'on–off' process because of the $2 \to 1$ transition. Note that the dual is similar to the original process in that type 0 represents a vacant state and type 2s give birth to type 1s. Define the compatibility relation $\xi \sim \zeta \Leftrightarrow \xi(x) \sim \zeta(x)$ for some x, where $1, 2 \sim 2$ and $2 \sim 1$. Note that type 2 in the dual process corresponds to type 1 or type 2 in the original process and that the dual of type 1 corresponds to the original type 2.

The interpretation of compatibility is that the configuration ξ is strong enough to be compatible with ζ at some site, and the stronger the dual type, the easier it is to match up. We give a simple proof of the following fact, the proof of which occupies several pages in [9].

Proposition 1. *The dual process has the property that*

$$\xi_t \sim \zeta_0 \quad \Leftrightarrow \quad \zeta_t \sim \xi_0,$$

with the dual running down the (same) spacetime graph from time t to time 0 so that ζ_s is on the time line $t - s$.

Proof. We start from the above condition to construct the dual, showing that it has the stated transitions and rates. The proof is given for the case $|V| < \infty$, that is, when the set of sites is finite, since only finitely many events occur in a finite time and we can proceed by induction on the events. To extend this to the case $|V| = \infty$ fix a finite subset V_0 and let $V_k = \{y \in V : \mathrm{d}(y, V_0) \le k\}$, where d is the graph distance. Denoting by $_{V_k}\xi_t$ the process constructed using the events on the subset $V_k \times \mathbb{R}^+$ of the spacetime graph, there is an almost surely finite value of k_0 so that $_{V_k}\xi_s(x) = \xi_s(x)$ for $x \in V_0$ and $0 \le s \le t$ when $k \ge k_0$, and this suffices to make the extension.

Say that a set of (forward) configurations Λ is *dualisable* if there is a dual configuration ζ so that

$$\Lambda = \{\xi : \xi \sim \zeta\}.$$

Note that ζ is unique, if it exists. For fixed ζ_0 and $0 \le s \le t$ let $\Lambda_s = \{\xi_{t-s} : \xi_t \sim \zeta_0\}$. Clearly, Λ_0 is dualisable with dual configuration ζ_0. If Λ_s is dualisable, denote by ζ_s its dual configuration. Suppose that there is an event at time s, and use the notation ξ_{t-s+} and Λ_{s+} to denote the state just prior to its occurrence (which corresponds to just after its occurrence in backward time, which is why we denote it s^+ rather than s^-). Note that

$$\Lambda_{s+} = \{\xi_{t-s+} : \xi_{t-s} \in \Lambda_s\}.$$

Suppose that Λ_s is dualisable with dual configuration ζ_s, then $\Lambda_{s+} = \{\xi_{t-s+} : \xi_{t-s} \sim \zeta_s\}$. We show that Λ_{s+} is dualisable by producing its dual configuration ζ_{s+}. A type 2 death at x (i.e. a rate-1 death event) kills both active types, so $\zeta_{s+}(x) = 0$ whatever the value of $\zeta_s(x)$; this causes the dual $1, 2 \to 0$ transition at rate 1. A type 1 death at x (i.e. a rate-δ death event) kills only type 1. If $\zeta_s(x) = 2$, i.e. a 1 or a 2 is sufficient for compatibility after the event, then a 2 is required for compatibility before, so $\zeta_{s+}(x) = 1$; this is the dual $2 \to 1$ transition at rate δ. A (forward) transmission event from $y \to x$ leads to a 1 at x after the event, if y is in state 2 just before the event, so $\zeta_{s+}(y) = 1$ if $\zeta_s(x) = 2$; this is the dual transmission event. A maturation event at x causes a $1 \to 2$ transition so that $\zeta_{s+}(x) = 2$ if $\zeta_s(x) = 1$; this is the dual $1 \to 2$ transition at rate γ. For values of $\zeta_s(x)$ not mentioned, or for sites that aren't involved in the transition, it is easily verified that $\zeta_{s+}(x) = \zeta_s(x)$. This finishes the induction step and establishes the dual transitions, completing the proof.

Before moving on, we note that the dual process is also additive and monotone increasing in λ and γ, and monotone decreasing in δ, a fact which is noted in [9] and which we use later.

3. Main results

3.1. Critical values for survival

Denoting by ξ_t^o the process starting a single mature site (the 'o' stands for 'origin', which if the process lives on the lattice, we can without loss of generality set to be the initially occupied site), we say that ξ_t^o survives or has single-site survival if

$$\mathbb{P}(\text{for all } t > 0, \text{ there exists } x \colon \xi_t^o(x) \neq 0) > 0,$$

and dies out otherwise. Defining the critical value

$$\lambda_c(\gamma, \delta) = \inf\{\lambda > 0 \colon \xi_t^o \text{ survives}\},$$

it follows by monotonicity that λ_c is an increasing function of δ and a decreasing function of γ and that ξ_t^o survives if $\lambda > \lambda_c$. For each δ, by taking γ and λ large enough and comparing it to a (suitably scaled in time) 1-dependent bond percolation diagram, it is possible to show that ξ_t survives, which implies that $\lambda_c(\delta, \gamma) < \infty$ if γ is large enough. The first proof of this type was given by Harris for the contact process in [8]; its application to the two-stage process was noted in [9].

For fixed δ the parameter space for the process is the quadrant $\{(\lambda, \gamma) \colon \lambda \geq 0, \gamma \geq 0\}$, and by identifying the survival region $\mathcal{S} = \{(\lambda, \gamma) \colon \xi_t^o \text{ survives}\}$ we obtain a phase diagram for survival. We can define the critical lines

$$\lambda_*(\delta) = \inf\{\lambda \colon \xi_t^o \text{ survives for some } (\lambda, \gamma)\},$$
$$\gamma_*(\delta) = \inf\{\gamma \colon \xi_t^o \text{ survives for some } (\lambda, \gamma)\}$$

that bound the survival region below, and on the left. From monotonicity it follows that $\lambda_*(\delta) \geq \lambda_c(\infty, 0)$, the critical value for the contact process, and also that $\gamma_*(\delta) \geq \gamma_*(0)$, the left-hand critical line when $\delta = 0$. We will have more to say about the survival region in Section 3.7. First, we complete a proof given in [9] that characterizes λ_* for any value of δ.

3.2. Correction to [9, Proposition 3.6]

In [9, Proposition 3.6] it was claimed that $\lambda_*(\delta) = \lambda_c(\infty, 0)$ for the process on $\mathbb{Z}^d$, for any dimension d. However, the proof given covers only the case $d = 1$. This is because the paper to which it refers gives a finite spacetime condition for survival only when $\lambda > \lambda_c^{(1)}$, the critical value for the contact process in one dimension. Here we use the more general construction of [2], plus a perturbation argument, to show that $\lambda_*(\delta) \leq \lambda_c(\infty, 0)$ in any dimension, which combined with the previous inequality implies the desired result.

In [2] it was shown for the contact process that if $\lambda > \lambda_c$ and $\varepsilon > 0$, we can place a latticework structure over an effectively two-dimensional region in $\mathbb{Z}^d \times \mathbb{R}^+$ and make a $1:1$ correspondence between certain spacetime boxes contained in this structure and the set $\{(x, y) \in \mathbb{Z}^2 \colon y \geq 0, x + y \text{ is even}\}$, with the property that when the process starts with a large disc of active sites in the box corresponding to (x, y), then with probability $> 1 - \varepsilon$ it can produce a large disc of active sites in the boxes corresponding to both $(x - 1, y + 1)$ and $(x + 1, y + 1)$. It was then shown in [2] that if one decreases λ slightly, this property still holds, and using results for oriented percolation in two dimensions, conclude that the process still survives.

In our case it suffices to show that the property still holds when γ is decreased slightly from ∞, i.e. when γ is large enough. From this we may then conclude that if $\lambda > \lambda_c(\infty, 0)$ then $\lambda > \lambda_c(\gamma)$ for some γ, which implies that $\lambda > \lambda_*$, or $\lambda_c(\infty, 0) \geq \lambda_*$, and combining the inequalities, $\lambda_c(\infty, 0) = \lambda_*$.

It is sufficient to show that on a bounded spacetime region, when γ is large enough and the two processes are started from the same configuration (with mature sites in the place of active sites in the two-stage process), with high probability,

- between any two transmission events incident at a given site, there is a maturation event, and

- if at a fixed time the contact process has a certain set of active sites, then in the two-stage process those sites are all mature sites.

The first condition ensures that no connections are cut due to a juvenile site being unable to give birth at a neighbouring site. The second condition ensures that if the contact process has produced a large disc of active sites then the two-stage process has produced a large disc of mature sites.

To satisfy both conditions, it suffices to ensure that maturation events occur arbitrarily often, since on a finite spacetime region $B \subset \mathbb{Z}^d \times \mathbb{R}^+$, for each $\varepsilon > 0$ there is a $\delta > 0$ so that with probability $> 1 - \varepsilon$, the waiting time between transmission events is $\geq \delta$ everywhere on B. However, for each $\delta > 0$ and $\varepsilon > 0$ there is a γ_0 so that if $\gamma > \gamma_0$, with probability $> 1 - \varepsilon$ the waiting time between maturation events is $< \delta$ everywhere on B; thus, for $\gamma > \gamma_0$ the conditions hold.

The two assertions of the last paragraph (those regarding waiting times) require proof, and it suffices to consider a spacetime region which is a single interval of length L. To prove the first assertion, note that with high probability a finite number N of events occur in the interval, and, with probability $e^{-\delta\lambda N}$ which tends to 1 as $\delta \to 0$, each event takes time $\geq \delta$ to occur. To prove the second assertion, break up the interval into pieces of length δ so that the number of events on each piece is distributed like a Poisson random variable with mean δ. The probability that on each interval at least one event has occurred is $(1 - e^{-\delta\gamma})^{L/\delta}$ which tends to 1 as $\gamma \to \infty$, for fixed δ.

3.3. Critical maturation rate (question 6 in [9])

In [9], a lower bound on $\gamma_*(\delta)$ is given for the process on $\mathbb{Z}$ with nearest-neighbour interactions which is about $\frac{1}{4}$ when $\delta = 0$ and increases towards 1 as $\delta \to \infty$. Here we answer question 6 in [9], which asks for lower bounds on $\gamma_*(\delta)$ in other settings; we obtain here a simple lower bound on $\gamma_*(0)$ (and by monotonicity, on $\gamma_*(\delta)$) that works for any graph of finite maximum degree, and depends only on the maximum degree. A graph has finite maximum degree if there is a number M so that $\deg x \leq M < \infty$ for each $x \in V$.

Proposition 2. *If $\delta = 0$ and $\gamma < 1/(2M - 1)$ then the process dies out for any value of λ.*

Proof. It suffices to show this for $\lambda = \infty$, i.e. when the $0 \to 1$ transition at x is instantaneous if $n_2(x) > 0$. The result is obtained by estimating the average number of offspring of a site x in state 1. The transition $1 \to 2$ occurs with probability $\gamma/(1 + \gamma)$, since $1 \to 0$ at rate 1 and $1 \to 2$ at rate γ. If the $1 \to 2$ transition occurs at x then each unoccupied neighbour of x becomes occupied. In order for x to send a second offspring to a neighbour y, the existing offspring at y has to die off. Denoting by N_t a Poisson process with rate 1 (representing the number of deaths at y, starting from the moment the $1 \to 2$ transition occurs at x) and by X_t an independent exponential random variable (representing death of the mature organism at x), the number of additional offspring produced at y is equal to N_{X_t}. Intuitively, we might expect

$\mathbb{E}N_{X_t} = \mathbb{E}N_{\mathbb{E}X_t} = 1$, and computing, we confirm that

$$\mathbb{E}N_{X_t} = \int_0^\infty \sum_k kx^k \frac{\mathrm{e}^{-x}}{k!} \mathrm{e}^{-x}\, \mathrm{d}x$$

$$= \int_0^\infty \sum_k k \frac{x^k}{k!} \mathrm{e}^{-2x}\, \mathrm{d}x$$

$$= \sum_k k \int_0^\infty \frac{x^k}{k!} \mathrm{e}^{-2x}\, \mathrm{d}x$$

$$= \sum_k k 2^{-(k+1)}$$

$$= 1.$$

Thus, the expected number of offspring at each initially unoccupied neighbouring site is $1 + \mathbb{E}N_{X_t} = 1 + 1 = 2$, so the expected number of offspring of a site in state 2 is at most $2M$. Since the probability of making the $1 \to 2$ transition before dying is $\gamma/(1+\gamma) = 1/(1+1/\gamma)$ the expected number of offspring of a site in state 1 is at most $2M/(1+1/\gamma)$. Setting this < 1 and comparing to a branching process gives the result.

From Proposition 2 we conclude that $\gamma_*(0) \geq 1/(2M-1)$ so that $\gamma_*(\delta) \geq \gamma_*(0) \geq 1/(2M-1)$ for any δ, proving Theorem 1. For the nearest-neighbour process on $\mathbb{Z}^d$ we have $M = 2d$, giving $\gamma_*(\delta) \geq 1/(4d-1)$, which is $\frac{1}{3}$ for $d = 1$, $\frac{1}{7}$ for $d = 2$, etc.

3.4. Single-site survival and edge speed (question 2 in [9])

Let ξ_t^- denote the process starting from the type 2s on $\mathbb{Z}^-$, and let $r_t = \sup\{x : \xi_t^-(x) \neq 0\}$ denote the right-hand edge of ξ_t^-. A result of Durrett [4] showed that r_t/t converges to a constant, α, as $t \to \infty$. It was asked in question 2 of [9] whether $\lambda_c = \inf\{\lambda : \alpha(\gamma, \delta) > 0\}$. Here we do not prove this, but we give a sufficient condition for it to be true. To get a sense of what it means, note that this property is equivalent to the property that ξ_t is supercritical (i.e. $\lambda > \lambda_c$) if and only if the right-hand edge of the process started from a half-line of mature sites has a positive spreading speed. For the equivalence of these statements, note that α is upper-semicontinuous in λ, since it is the infimum of a family of continuous functions as described in [4].

One side of the result is easy; letting ξ_t^+ denote the process starting from the type 2s on $\mathbb{Z}^+$ and ℓ_t its left-hand edge, by attractiveness $\xi_t^o \leq \min\xi_t^+, \xi_t^-$, so $\xi_t^o(x) = 0$ for $x > r_t$ and for $x < l_t$. If $\alpha < 0$ then by symmetry $\ell_t/t \to -\alpha > 0$. Since $r_t \to -\infty$ and $l_t \to \infty$, eventually $r_t < l_t$ and $\xi_t^o(x) = 0$ for all x, i.e. ξ_t^o dies out.

For the converse, for $x \in \mathbb{Z}$ denote by C_x the 'active cluster' of x, i.e. the set of spacetime points (y, t) such that if site x is initially in state 2 then site y is active at time t, and denote by $|C_x|$ its width, that is, $|C_x| = \sup\{|y - x| : (y, t) \in C_x \text{ for some } y, t\}$. On the event that the process does not die out, the number of active sites tends to ∞, so if ξ_t^o survives, $\mathbb{P}(|C_x| = \infty) > 0$ and so $\mathbb{E}|C_x| = \infty$ for each x; thus, if $\mathbb{E}|C_x| < \infty$ then $\lambda \leq \lambda_c$. By analogy with percolation theory [7] we might guess that the converse holds, i.e. that if $\lambda < \lambda_c$ then $\mathbb{E}|C_x| < \infty$; this is proved, for example, for the contact process in [3]. We do not pursue this here, but instead show that if $\mathbb{E}|C_x| < \infty$ then $\alpha \leq 0$. Thus, a sufficient condition for edge speed to characterize single-site survival is for the subcritical process to have a finite expected size.

Proposition 3. *If* $\mathbb{E}|C_x| < \infty$ *then* $\alpha \leq 0$.

Proof. If $\mathbb{E}|C_x| < \infty$ but $\alpha > 0$ then each C_x is bounded almost surely, but for each $\varepsilon > 0$ eventually $r_t/t > \alpha - \varepsilon$, which means with probability 1 there is an infinite sequence of sites (x_k) in $\mathbb{Z}^-$ and (y_k) in $\mathbb{Z}^+$ with $x_{k+1} < x_k$ for each k, and an infinite sequence of times (t_k) with $t_k \to \infty$ such that for each k, $(x_k, 0) \to (y_k, t_k)$. This is because the cluster of any finite collection of sites is almost surely bounded, which means that later activity of the process must originate from sites which are progressively further to the left; note that this also implies that the stated paths must be disjoint, although we will not need this here. In any case, the event $|C_x| \geq |x|$ occurs for infinitely many $x \in \mathbb{Z}^-$. However,

$$\sum_{x \in \mathbb{Z}^-} \mathbb{P}(|C_x| \geq |x|) = \mathbb{P}(|C_0 \geq 0|) + \mathbb{E}|C_x| < \infty,$$

so applying the Borel–Cantelli lemma, $|C_x| \geq |x|$ occurs infinitely often with probability 0, contradicting our assumption.

3.5. Equality of critical values (question 1 in [9])

For any attractive growth model there is at least one other characterization of survival aside from single-site survival or divergence of the expected cluster size, which is the existence of a nontrivial upper invariant measure ν, obtained as the weak limit of the distribution of the process started from its largest initial configuration (in the case of ξ_t, when started from all sites in state 2). For either the two-stage contact process or the on–off process, this weak limit exists by attractiveness, and from the Feller property is an invariant measure for the system; see [12, Theorem 2.3, p. 135] for a proof. The proof is for spin systems but generalizes without modification to any attractive system with a largest configuration.

It is possible that $\nu = \delta_0$, the measure that concentrates on the configuration with all 0s; we say that ν is nontrivial if $\nu \neq \delta_0$, equivalently, if ν assigns positive density at each site, that is, $\nu(\{\xi : \xi(x) \neq 0\}) > 0$ for each x. In question 1 of [9], it was asked whether single-site survival is equivalent to this property. First we show that single-site survival of either the two-stage contact process or of the on–off process, implies that $\nu \neq \delta_0$, which supplies one direction of the proof. We then use the duality relation to observe that

$$\nu(\{\xi : \xi(\mathrm{o}) \neq 0\}) = \mathbb{P}(\zeta_t^{\mathrm{o}} \text{ survives}),$$

where ξ_t is the two-stage contact process and ζ_t is the on–off process, and that the same property holds when ξ_t and ζ_t are exchanged in the formula. Thus, if the two-stage contact process has a nontrivial stationary distribution, then the on–off process has single-site survival, which means that the on–off process has a nontrivial stationary distribution. This means that the two-stage contact process has single-site survival, which supplies the other direction of the proof, and shows that the two notions of survival are in fact equivalent, proving Theorem 2. Therefore, it suffices to show that single-site survival of the two-stage process or of the on–off process, implies that $\nu \neq \delta_0$.

For the (single-stage) contact process η_t on $\mathbb{Z}^d$, if $\lambda > \lambda_c$ then the method described in [2] allows us to conclude that, under a suitable renormalization and started from a finite number of active sites, η_t dominates a supercritical 1-dependent oriented site percolation process in two dimensions. For this process it is known that the origin is active for a positive fraction of the time. From this fact it follows that $\liminf_t \mathbb{P}(\eta_t^{\mathrm{o}}(0) \neq 0) > 0$, which since $\eta_0^{\mathrm{o}} \leq \eta_0^1$ and by attractiveness implies that $\nu(\{\eta : \eta(0) \neq 0\}) = \lim_{t \to \infty} \mathbb{P}(\eta_t^1(0) \neq 0) > 0$, where η_t^1 is the

contact process started from all sites active. The following lemma allows us to conclude the same fact for the two-stage contact process, whenever the interaction neighbourhood is symmetric about permutation and sign change of coordinates, and such that with some probability, any site can infect any other site; the first condition we call coordinate symmetry and the second we call irreducibility. Note that the interaction neighbourhood must, of course, be finite.

Lemma 1. *The construction in [2] is valid for the two-stage contact process and for the on–off process on $\mathbb{Z}^d$, for any coordinate-symmetric and irreducible interaction neighbourhood.*

Proof. The basic strategy is to repeat the construction, replacing 'infected site' with 'active site'. Most of the steps go through without modification and the required changes are addressed below.

In the construction in [2], nearest-neighbour interactions are assumed. This condition can be relaxed by redefining the 'sides' of the box to be a region whose width is equal to the interaction range of the process. In this way, we can control transmission from the sides of the rectangle to the outside world as is done in the nearest-neighbour case; see [1] for an example where this modification is carried out.

When widening the sides, it is necessary to make sure that a large finite disc can be produced at an extra distance corresponding to the range of the interaction, but this can be prescribed. Irreducibility is required to ensure that, starting from a single infectious site, all sites in a large finite disc can be made infectious with a certain probability.

Coordinate symmetry is implicit in the construction, and it is assumed when proving that the process reaches each orthant of the top and sides of the box with high probability.

When defining the event where a single site produces a large finite disc, it doesn't matter too much which type is produced in the disc; arbitrarily, we pick mature sites. However, we should require that a juvenile site can produce the disc, since whatever a juvenile site can produce, a mature site can produce. The construction uses the property that the process dies out if its population dips below a certain value infinitely often. This is a property that holds for any growth model in which there is a finite number of active types, and each active type dies before reproducing with a certain probability.

When we apply the argument along the sides of the box, it is also important that active sites die at a certain uniform rate; for example, the argument would not carry over directly if juvenile sites had to mature before being subject to death events. Note that the argument in [2] where it applies along the sides of the box is somewhat convoluted; for a clearer proof of this part see [13, Part 1, Proposition 2.8].

For the (usual) contact process, the application of the Fortuin—Kasteleyn—Ginibre inequality given in the construction follows from the invariance of positive correlations (see [12, Theorem 2.14, p. 80]), for which it is sufficient that the process be attractive and that its transitions occur only between comparable states, a property true of the two-stage contact process and of the on–off process.

We have addressed the changes needed to adapt the construction of [2] to the two-stage and on–off processes. The rest of the construction consists of geometrical or 'resetting' arguments, and no modification is required.

It follows from Lemma 1 and from the discussion preceding it that for the two-stage contact process or the on–off process on $\mathbb{Z}^d$, whenever there is single-site survival ($\lambda > \lambda_c$) the upper invariant measure assigns a positive density at each site, i.e. $\nu(\{\xi : \xi(x) \neq 0\}) > 0$ for each x. The proof of question 1 is now complete.

3.6. Complete convergence (question 3 of [9])

Let δ_0 be the measure that concentrates on the configuration with all 0s, and let ν be the upper invariant measure introduced in Section 3.5. For an attractive growth model, complete convergence means that

$$\xi_t \quad \Rightarrow \quad \alpha\delta_0 + (1-\alpha)\nu,$$

as $t \to \infty$, where '$\Rightarrow$' denotes weak convergence and $\alpha = \mathbb{P}(\xi_t \text{ dies out})$. In [9] it was asked whether complete convergence holds for the two-stage contact process when $\lambda > \lambda_c$. We follow [5, Section 5], where the argument is used for the contact process; the idea is originally due to Griffeath [6]. Fix an arbitrary configuration ξ_0 and a dual configuration ζ_0 with finitely many active sites; doing this for all such ζ_0 we will recover the finite-dimensional distributions of the upper invariant measure. We have that

$$\xi_{2t} \sim \zeta_0 \quad \Leftrightarrow \quad \xi_t \sim \zeta_t,$$

where $\zeta_s, 0 \leq s \leq t$, is constructed on the same spacetime graph as ξ_t and run from time $2t$ down to time t, with initial configuration ζ_0. Use the notation $\xi \neq 0$ to denote 'not identically zero'. Then observe that

$$\mathbb{P}(\xi_t \sim \zeta_t) = \mathbb{P}(\xi_t \neq 0, \zeta_t \neq 0) - \mathbb{P}(\xi_t \neq 0, \zeta_t \neq 0, \xi_t \nsim \zeta_t).$$

Since $\xi_s, 0 \leq s \leq t$, and $\zeta_s, 0 \leq s \leq t$, are built over disjoint parts of the graph, they are independent, so

$$\mathbb{P}(\xi_t \neq 0, \zeta_t \neq 0) = \mathbb{P}(\xi_t \neq 0)\mathbb{P}(\zeta_t \neq 0),$$

for each $t > 0$. Using the duality relation, $\mathbb{P}(\zeta_t \neq 0) = \mathbb{P}(\xi_t \sim \zeta_0)$ with ξ_0 in this case being the configuration with all 2s. Letting $t \to \infty$, we have

$$\mathbb{P}(\xi_t \neq 0)\mathbb{P}(\zeta_t \neq 0) \to (1-\alpha)\nu(\{\xi : \xi \sim \zeta_0\}).$$

To have complete convergence, it therefore suffices to show that

$$\mathbb{P}(\xi_t \neq 0, \zeta_t \neq 0, \xi_t \nsim \zeta_t) \to 0,$$

as $t \to \infty$. A method for doing this is outlined in [5] for a certain class of growth models. In [5] a restart argument was used to show that whenever the process survives and is suitably rescaled, it eventually dominates a two-dimensional oriented percolation process, which is known to have a positive density of sites. Using this fact it is then argued that if run for long enough, the process and its dual intersect with high probability. In [2, Section 5] it was noted that, using the construction from [2] and the ideas from [5], the same can be concluded for the contact process in $\mathbb{Z}^d$. Noting the equality of critical values proved in Section 3.5 if $\lambda > \lambda_c$ the construction of [2] can be applied to both the two-stage contact process and the on–off process, and the same arguments apply to show that the two processes eventually intersect with high probability, which proves Theorem 3.

3.7. Structure of the survival region (questions 5 and 4 of [9])

Continuing the analysis of the survival region $\mathcal{S}$ begun in Section 3.1, we show that the process dies out on the boundary ∂S. By monotonicity, $\lambda_c(\gamma)$ can have only jump discontinuities, which means that the boundary of the survival region is the set

$$\{(\gamma, \lambda) : \gamma \geq \gamma_*, \lambda_c^-(\gamma) \geq \lambda \geq \lambda_c^+(\gamma)\},$$

where $\lambda_c^-(\gamma)$ and $\lambda_c^+(\gamma)$ are, respectively, the left-hand and right-hand limits of λ_c at γ; set $\lambda_c^-(\gamma_*) = \infty$.

It follows from Lemma 1 that survival of the two-stage contact process is given by a finite spacetime condition of the form 'a certain event happens with probability $> 1 - \varepsilon$', where ε is sufficiently small. Moreover, the probability of this event is continuous in λ and γ (also δ, but we will not use this fact here). This is because, by a small enough change in parameters, on a finite spacetime region we can ensure that the probability of even one more or one fewer transmission/maturation event can be made arbitrarily small. This implies that $\mathcal{S}$ is an open subset of the plane in any dimension and for any value of δ. Since $\mathcal{S}$ is open it follows that the process dies out on its boundary ∂S, which includes the critical values $\lambda_c(\gamma)$.

It seems that question 4 in [9], namely, whether there is a formula for λ_c in terms of $\lambda_c(\infty, 0)$, γ, and δ, should be false. One good reason to believe this is that for the contact process, a sequence of approximants is known that converges to the critical value, and these are roots of successively more complicated rational functions, as shown in [16]. There is no obvious reason to believe why the introduction of an additional stage to the process should lead to a critical value that is any simpler to determine, even if the critical value of the contact process is used in the expression.

Remaining questions for the survival region include whether $\lambda_c(\gamma)$ is continuous, whether it is strictly decreasing on $\{\gamma > \gamma_*\}$, and whether

$$\lim_{\gamma \to \gamma_*^+} \lambda_c(\gamma) = \infty.$$

We believe the answers are, respectively, yes, yes, and yes, but we are not sure how to prove this.

References

[1] BEZUIDENHOUT, C. AND GRAY, L. (1994). Critical attractive spin systems. *Ann. Prob.* **22,** 1160–1194.
[2] BEZUIDENHOUT, C. AND GRIMMETT, G. (1990). The critical contact process dies out. *Ann. Prob.* **18,** 1462–1482.
[3] BEZUIDENHOUT, C. AND GRIMMETT, G. (1991). Exponential decay for subcritical contact and percolation processes. *Ann. Appl. Prob.* **19,** 984–1009.
[4] DURRETT, R. (1980). On the growth of one-dimensional contact processes. *Ann. Prob.* **8,** 890–907.
[5] DURRETT, R. AND SCHONMANN, R. H. (1987). Stochastic growth models. In *Percolation Theory and Ergodic Theory of Infinite Particle Systems* (IMA Vol. Math. Appl. **8**), Springer, New York, pp. 85–119.
[6] GRIFFEATH, D. (1978). Limit theorems for nonergodic set-valued Markov processes. *Ann. Prob.* **6,** 379–387.
[7] GRIMMETT, G. (1999). *Percolation*, 2nd edn. Springer, Berlin.
[8] HARRIS, T. E. (1974). Contact interactions on a lattice. *Ann. Prob.* **2,** 969–988.
[9] KRONE, S. M. (1999). The two-stage contact process. *Ann. Appl. Prob.* **9,** 331–351.
[10] LANCHIER, N. (2011). Two-scale multitype contact process: coexistence in spatially explicit metapopulations. *Markov Process. Relat. Fields* **17,** 151–186.
[11] LI, Y. AND HAN, D. (2013). A two-stage contact process on scale-free networks. *J. Statist. Phys.* **153,** 312–324.
[12] LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.
[13] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, Berlin.
[14] NEUHAUSER, C. (1992). Ergodic theorems for the multitype contact process. *Prob. Theory Relat. Fields* **91,** 467–506.
[15] SUGIMINE, N., MASUDA, N., KONNO, N. AND AIHARA, K. (2008). On global and local critical points of extended contact process on homogenous trees. *Math. Biosci.* **213,** 13–17.
[16] ZIEZOLD, H. AND GRILLENBERGER, C. (1988). On the critical infection rate of the one-dimensional basic contact process: numerical results. *J. Appl. Prob.* **25,** 1–8.