USING THE POISSON INVERSE GAUSSIAN IN BONUS-MALUS SYSTEMS

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ABSTRACT

In this paper, we will cover the bonus-malus system in automobile insurance. Bonus-malus systems are based on the distribution of the number of car accidents. Therefore, the modelling and fitting of that distribution are considered. Fitting of data is done using the Poisson inverse Gaussian distribution, which shows a good fit. Building the bonus system is done by minimizing the insurer's risk, according to LEMAIRE's (1985) bonus system.

KEYWORDS

Mixed Poisson distribution; Poisson inverse Gaussian; generalized inverse Gaussian; Bayes theorem; quadratic loss function; exponential utility function.

As LEMAIRE (1976) put it, bonus-malus systems are based on the random variable number of claim N (frequency), irrespective of their amount. To begin with, we must adapt a Poisson process which is not homogeneous. The heterogeneity aspect is introduced by mixing the Poisson distribution. The parameter λ in the Poisson distribution is considered a random variable. A similar contention is made by BESSON and PARTRAT (1992).

Let us assume that the expected frequency of claims varies within the portfolio. Let us further assume that any particular risk in the portfolio has a Poisson distribution of claim frequencies with mean Λ, where Λ is itself a random variable with distribution representing the expected risks inherent in the given portfolio. The distribution function of Λ is given by $U(\Lambda)$ and the unconditional distribution of claim frequencies of an individual drawn from the portfolio is mixed Poisson. N has a Poisson distribution with probability function

$$f_N(n|\Lambda) = e^{-\Lambda}\frac{\Lambda^n}{n!} \qquad n = 0, 1, 2, \dots$$

with

$$E(N) = E(\Lambda) \qquad \mathrm{Var}\,(N) = \mathrm{Var}\,(\Lambda)+E(\Lambda)$$

It is obvious that mixed Poisson variates have a variance exceeding the mean (unlike the Poisson where mean and variance are equal). This state of fact, which is usually the case in practical situations, is normally desirable from the insurer's standpoint in that mixed distribution can be thought of as being "safer" than the original Poisson.

One interesting distribution for Λ is the inverse Gaussian. It has thick tails and it also provides the advantage of having closed form expression for the moment generating function. It is a reasonable distribution for modeling in many insurance situations. The probability density function of an inverse Gaussian distribution is,

$$f(\lambda) = \frac{\mu}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}}, \qquad \lambda > 0$$

The distribution function is,

$$F(\lambda) = \Phi\left[\frac{(\lambda-\mu)}{\sqrt{\beta\lambda}}\right] + e^{\frac{2\mu}{\beta}} \Phi\left[\frac{-(\lambda+\mu)}{\sqrt{\beta\lambda}}\right], \qquad \lambda > 0$$

where $\Phi(\cdot)$ is the standard Normal (with mean 0 and variance 1) distribution function.

The mean and variance are

$$E(\Lambda) = \mu \qquad \text{Var}(\Lambda) = \mu\beta$$

The Poisson mixed over the inverse Gaussian is thus obtained and called the Poisson inverse Gaussian. One can obtain the probabilities from the probability generating function which is

$$P(z) = e^{\frac{\mu}{\beta}[1-\sqrt{1+2\beta(1-z)}]}$$

The mean and variance are thus obtained,

$$E(N) = \mu$$
$$\text{Var}(N) = \mu(1+\beta)$$

The Poisson inverse Gaussian has two parameters.

Regarding the particular case of liability policies (private cars) which have had k claims, we shall use the results found by PANJER and WILLMOT (1987). BÜHLMANN (1970) published the data that had been gathered in Switzerland in 1961 (see Table 1.1).

We will use the maximum likelihood estimator to estimate a parameter $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ from our set of independent and identically distributed data $(X_1, X_2, \ldots, X_k)$ with probability function $p_n(\theta)$.

According to PANJER and WILLMOT (1987), the likelihood function is

$$L(\theta) = \prod_{k=0}^{\infty} [p_k(\theta)]^{N_k}$$

TABLE 1.1

CLAIM FREQUENCY DATA

Number of claims per policy (k)	Number of policies (n)	Fitted Values
0	103,704	103,710.03
1	14,075	14,054.65
2	1,766	1,784.65
3	255	254.49
4	45	40.42
5	6	6.94
6	2	1.26
Total	119,853	119,852.44

The log-likelihood is defined to be

$$l(\theta) = \log L(\theta) = \sum_{k=0}^{\infty} N_k \cdot \log p_k(\theta)$$

where $N_k = \{\text{number of } X_i\text{'s for which } X_i = k\}$, $k = 0, 1, 2, \ldots$ are the observed frequencies. The maximum likelihood estimator is the random variable $\hat{\theta}$ for which the likelihood (or equivalently the log likelihood) is a maximum.

The values of the maximum likelihood estimator of μ and β are $\hat{\mu} = 0{,}15514$ and $\hat{\beta} = 0.15527$. Then, multiplying the probabilities by 119,853 yields the fitted values of Table 1.1.

The value of $\bar{x}$ is 0.15514 and s^2 is 0.24174.

The goodness of fit statistic D Chi-squared distributed with $(k-r-l)$ degrees of freedom (k intervals and r parameters) defined as

$$D = \sum_{i=0}^{k-1} \frac{(N_i - np_i)^2}{np_i}$$

the value of which is 0.78 on 3 degrees of freedom, yielding a significance level of 85%, which is good. This is a better fit than the Poisson-Gamma model, as one can compare with BICHSEL's (1964) results calculated on the same set of data.

We have now a frequency distribution, the Poisson inverse Gaussian, that is fitted to our data. We can now build our bonus-malus system in the same manner than LEMAIRE (1985) did. We will minimize the average total risk of the insurer, since the insurer is at risk. Let us consider an insured observed during t years and let us call n_j the number of accidents in fault reported during the jth year. For each insured, we have the information vector $(n_1, \ldots, n_t)$. Each n_j is a realization of the random variable N_j, for which we assumed independent and identical distributions. For each observation set $n_1, \ldots, n_t$ we have to relate a number $\lambda_{t+1}(n_1, \ldots, n_t)$, which is the best estimator of λ at time $t+1$. We also choose a quadratic loss function $(\lambda_{t+1} - \lambda)^2$ which yields

$$\lambda_{t+1}(n_1, \ldots, n_t) = E(\Lambda | n_1, \ldots, n_t)$$

Hence, we need to determine the posterior mean of Λ. We already know that $\mu(\lambda)$ is an inverse Gaussian with parameters β and μ. We had N/λ as Poisson distributed, hence the likelihood distribution is

$$P(n_1, \ldots, n_t | \lambda) = \frac{\lambda^n e^{-t\lambda}}{\prod_{j=1}^{t} (n_j!)}, \qquad \text{where} \qquad n = \sum_{i=1}^{t} n_i .$$

Next, the joint distribution of the number of accidents is

$$\begin{aligned}
\bar{P}(n_1, \ldots, n_t) &= \int_0^\infty P(n_1, \ldots, n_t | \lambda) \cdot \mu(\lambda) \cdot d\lambda \\
&= \int_0^\infty \frac{\lambda^n e^{-t\lambda}}{\prod_{j=1}^{t} (n_j!)} \cdot \frac{\mu}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}} d\lambda \\
&= \frac{\mu}{\prod_{j=1}^{t} (n_j!)} \int_0^\infty \frac{\lambda^n}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}} d\lambda
\end{aligned}$$

The Bayes theorem for the posterior distribution of Λ is

$$\begin{aligned}
\mu(\lambda | n_1, \ldots, n_t) &= \frac{\dfrac{\lambda^n e^{-t\lambda}}{\prod_{j=1}^{t} (n_j!)} \cdot \dfrac{\mu}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}}}{\dfrac{\mu}{\prod_{j=1}^{t} (n_j!)} \displaystyle\int_0^\infty \frac{\lambda^n}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}} d\lambda} \\
&= \frac{\lambda^n \cdot e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}}}{\sqrt{2\pi\beta\lambda^3} \displaystyle\int_0^\infty \frac{\lambda^n}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}} d\lambda}
\end{aligned}$$

One can see that the integral term is not a function of λ once it is solved. We can then find to which distribution $\mu(\lambda | n_1, \ldots, n_t)$ is proportional by omitting

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that constant term (the integral);

$$\mu(\lambda|n_1, \ldots, n_t) \propto \frac{\lambda^n e^{\left\{-t\lambda - \frac{\lambda}{2\beta} - \frac{\mu^2}{2\beta\lambda}\right\}} e^{\frac{\mu}{\beta}}}{(2\pi\beta)^{1/2}\lambda^{3/2}}.$$

Similarly, we omit all terms that do not depend on λ, since $\mu(\lambda|n_1, \ldots, n_t)$ is a function of λ,

$$\mu(\lambda|n_1, \ldots, n_t) \propto \lambda^{\left(n - \frac{3}{2}\right)} e^{\left\{-\lambda\left(\frac{1}{2\beta} + t\right) - \frac{1}{\lambda}\left(\frac{\mu^2}{2\beta}\right)\right\}}$$

in which

$$a = n - \frac{3}{2} \qquad b = \frac{1}{\frac{1}{2\beta} + t} \qquad c = \frac{\mu^2}{2\beta}$$

to obtain

$$\mu(\lambda|n_1, \ldots, n_t) \propto \lambda^a e^{\left\{-\frac{\lambda}{b} - \frac{c}{\lambda}\right\}}.$$

The above expression is the probability density function of Generalized inverse Gaussian distribution, which can also be presented as

$$P(x) = \frac{\mu'^{(-\upsilon)} x^{(\upsilon-1)} e^{-\left\{\frac{x^2 + \mu'^2}{2\beta' x}\right\}}}{2K_\upsilon(\mu'/\beta')}, \qquad x > 0$$

$$= \frac{\mu'^{(-\upsilon)} x^{(\upsilon-1)} e^{\left(-\frac{x}{2\beta'} - \frac{\mu'^2}{2\beta' x}\right)}}{2K_\upsilon(\mu'/\beta')}$$

where $\mu' > 0$, $\beta' > 0$, $-\infty < \upsilon < \infty$ and $K_\upsilon(x)$ is the modified Bessel function of the third kind with index υ. JORGENSEN (1982) shows several results concerning the Generalized inverse Gaussian. One of interest to us is the mean, given by

$$E(\text{G.I.G.}) = \mu' \frac{K_{\upsilon+1}(\mu'/\beta')}{K_\upsilon(\mu'/\beta')}$$

in our case

$$\upsilon = a + 1 = n - \frac{1}{2}$$

$$\beta' = \frac{b}{2} = \frac{1}{\frac{1}{\beta} + 2t}$$

$$\mu' = \sqrt{2\beta' c} = \mu \sqrt{\frac{1}{1+2\beta t}}\,.$$

When the prior distribution of Λ is inverse Gaussian and the conditional probability function $(N_1, \ldots, N_t)$ given $\Lambda = \lambda$ a t-dimension Poisson distribution, we then see that the posterior distribution of Λ is a Generalized inverse Gaussian.

We already estimated μ and β, then we can write

$$v = n - \frac{1}{2}$$

$$\beta' = \frac{1}{6.44039 + 2t}$$

$$\mu' = 0.15514 \times \sqrt{\frac{1}{1+0.31054\,t}}$$

and

$$\lambda_{t+1}(n_1, \ldots, n_t) = \mu' \frac{K_{v_{t+1}}(\mu'/\beta')}{K_v(\mu'/\beta')}$$

In our case, the pure premium to be charged is related to the frequency of accidents. That is

$$P_{t+1}(n_1, \ldots, n_t) = \lambda_{t+1}(n_1, \ldots, n_t) = \mu' \frac{K_{v_{t+1}}(\mu'/\beta')}{K_v(\mu'/\beta')}$$

Hence, we can now build a table of premiums to be charged as a function of accidents (n) and number of years (t). To estimate the modified Bessel functions, we use

$$\frac{K_{v_{t+1}}(\mu'/\beta')}{K_v(\mu'/\beta')} = Q_v(\mu'/\beta')\,.$$

We estimate Q by a Newton Raphson approach. Then we normalize the posterior premium in a way that the premium for a new insured is 100 ($n = 0$ and $t = 0$). We then obtain

$$P_{t+1}(n_1, \ldots, n_t) = 100 \cdot \frac{\mu' Q_v\,(\mu'/\beta')}{\mu_0\, Q_{v_0}(\mu_0/\beta_0)}$$

where

$$v_0 = -\frac{1}{2} \qquad \beta_0 = 0.15527 \qquad \mu_0 = 0.15514$$

But we realize that

$$\mu_0 Q_{v_0}(\mu_0/\beta_0)$$

is the mean of the prior distribution of Λ. The Generalized inverse Gaussian includes the inverse Gaussian when $\lambda = -1/2$. So

$$P_{t+1}(n_1, \ldots, n_t) = 100 \cdot \frac{\mu' Q_v(\mu'/\beta')}{0.15514}$$

Table 1.2 gives the results for P_{t+1} for various t and n. We limited ourselves to $n = 10$ since $n > 10$ accidents is most unlikely to occur. An APL program was used (for further calculations also).

This is an equitable system: each insured always pays a premium proportional to the estimation of his frequency of accidents, according to the information accumulated during t years. For example, if our insured had 1 accident during his first year, he would have a surcharge of 67.72% ((163.72 − 100) ÷ 100). But if he had no accident during that first year, he is entitled to a reduction of 12.65% ((100 − 87.35) ÷ 100). Later, if he has 1 accident during the second year (and none in his first), he will be penalized 60.60% ((140.28 − 87.35) ÷ 87.35). Similarly, if he has no accident during the second year (and 1 in the first year), he will be granted a bonus of 14.32% ((163.72 − 140.28) ÷ 163.72). This process can be continued indefinitely.

TABLE 1.2

BONUS-MALUS TABLE

(based on a Poisson inverse Gaussian frequency distribution)

t \ n	0	1	2	3	4	5	6	7	8	9	10
0	100	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
1	87.35	163.72	275.71	409.52	553.21	701.11	850.94	1001.76	1153.14	1304.88	1456.85
2	78.54	140.28	229.19	335.61	450.55	569.34	689.96	811.55	933.69	1056.17	1178.88
3	71.95	123.76	197.27	285.31	380.84	479.91	580.73	682.49	784.79	887.42	990.29
4	66.78	111.42	173.94	248.83	330.38	415.23	501.75	589.17	677.13	765.42	853.93
5	62.59	101.80	156.10	221.13	292.16	366.27	441.97	518.56	595.66	673.10	750.75
6	59.10	94.05	142.00	199.37	262.20	327.91	395.15	463.25	531.86	600.80	669.95
7	56.13	87.67	130.54	181.81	238.07	297.05	357.49	418.76	480.54	542.64	604.96
8	53.57	82.30	121.05	167.33	218.22	271.69	326.53	382.20	438.37	494.86	551.56
9	51.33	77.71	113.03	155.18	201.60	250.43	300.63	351.62	403.10	454.89	506.89
10	49.35	73.73	106.17	144.84	187.47	232.40	278.65	325.66	373.16	420.96	468.98
20	37.24	51.12	67.13	86.24	110.43	137.23	160.11	185.51	212.31	239.24	269.44
50	24.60	30.65	36.32	45.23	55.62	65.41	73.76	84.42	94.61	106.22	120.60
100	17.66	20.79	24.21	28.85	33.02	37.60	40.24	44.31	49.51	56.92	64.62

Let us now introduce a utility function of the insurer, as LEMAIRE (1979) did. The insurer assesses the risk according to a utility function $\mu(x)$, and determines the premium by equating the utility function of his present situation and the expected utility of the risk: that is, he is indifferent between his present situation and being at risk.

$$\mu(R) = E\{\mu(R+P-x)\} = \int_0^\infty \mu(R+P-x)\cdot dG(x)$$

where R is the reserve of the insurer, P the premium to be charged and $G(x)$ the distribution function of claims in a portfolio. This null utility principle has numerous interesting properties when one uses an exponential utility function,

$$\mu(x) = \frac{1}{c}(1-e^{-cx}), \qquad c > 0.$$

The parameter c characterizes the risk aversion of the insurer. We can then evaluate the premium

$$\frac{1}{c}(1-e^{-cR}) = \int_0^\infty \frac{1}{c}(1-e^{-c(R+P-x)})\, dG(x)$$

which yields

$$P = \frac{1}{c}\log M(c)$$

where $M(c)$ is the moment generating function of the claims distribution. In our situation of a bonus-malus system based on a Poisson inverse Gaussian, we have

$$P = \frac{1}{c}\log \int_0^\infty M(c,\lambda)\, dU(\lambda)$$

where

$$M(c,\lambda) = e^{\lambda(e^c-1)}$$

is the moment generating function of the Poisson distribution, and $U(\lambda)$ is inverse Gaussian. Then

$$P = \frac{1}{c}\log\left[\int_0^\infty e^{\lambda(e^c-1)}\frac{\mu}{\sqrt{2\pi\beta\lambda^3}}\, e^{\left\{-\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}}\, d\lambda\right].$$

The expression in brackets is the moment generating function of the inverse Gaussian distribution valued at e^c-1. For the Generalized inverse Gaussian,

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we have

$$M(z) = (1-2\beta z)^{-\frac{\lambda}{2}} \frac{K_\lambda\left(\frac{\mu}{\beta}\sqrt{1-2\beta z}\right)}{K_\lambda(\mu/\beta)}, \qquad z \leq \frac{1}{2\beta}$$

and in particular for the inverse Gaussian when $\lambda = -1/2$. Hence, we replace in P

$$P = \frac{1}{c}\log\left[[1-2\beta(e^c-1)]^{-\frac{\lambda}{2}} \frac{K_\lambda\left(\frac{\mu}{B}\sqrt{1-2\beta(e^c-1)}\right)}{K_\lambda(\mu/\beta)}\right].$$

This formula is valid for all values of parameters λ, β and μ, and in particular for our values of υ, β' and μ' of the posterior distribution. So,

$$P_{t+1}(n_1, \ldots, n_t) = \frac{1}{c}\log\left[[1-2\beta'(e^c-1)]^{-\frac{\upsilon}{2}} \frac{K_\upsilon\left(\frac{\mu'}{\beta'}\sqrt{1-2\beta'(e^c-1)}\right)}{K_\upsilon(\mu'/\beta')}\right].$$

This premium is a non-decreasing continuous function of c. A choice of $c = 0.25$ (risk aversion) yields a reasonable initial premium $P_1 = 0.18032$; since the pure premium is 0.15514, it corresponds to a safety loading of about 16%. The results are in Table 1.3.

TABLE 1.3

BONUS-MALUS TABLE

(based on a Poisson inverse Gaussian frequency distribution)
(with exponential utility function, $c = 0.25$)

t \ n	0	1	2	3	4	5	6	7	8	9	10
0	100	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
1	86.87	164.15	277.80	413.55	559.18	708.98	860.68	1013.36	1166.59	1320.16	1473.90
2	77.84	139.89	229.47	336.66	452.35	571.85	693.15	815.39	938.18	1061.31	1184.66
3	71.15	122.99	196.67	284.93	380.63	479.84	580.76	682.60	784.98	887.68	990.61
4	65.93	110.44	172.89	247.70	329.13	413.81	500.13	587.34	675.08	763.14	851.42
5	61.72	100.72	154.82	219.60	290.35	364.12	439.46	515.67	592.39	669.43	746.70
6	58.22	92.92	140.58	197.63	260.07	325.36	392.16	459.79	527.92	596.38	665.05
7	55.25	86.51	129.07	179.95	235.78	294.28	354.23	414.99	476.25	537.82	599.61
8	52.70	81.14	119.54	165.42	215.85	268.80	323.14	378.28	433.90	489.84	545.98
9	50.47	76.55	111.52	153.26	199.20	247.53	297.20	347.65	398.57	449.80	501.24
10	48.50	72.59	104.67	142.92	185.08	229.51	275.22	321.70	368.64	415.88	463.33
20	36.51	50.16	69.26	89.63	111.75	134.89	158.72	183.61	209.27	236.41	264.95
50	24.08	30.01	37.81	45.21	54.47	64.08	73.11	83.32	94.47	106.03	118.19
100	17.28	20.33	23.47	27.32	31.76	36.79	39.97	43.31	48.92	55.28	63.24

The Table 1.3 slightly differs from the preceding one (Table 1.2). It can be shown that even for very unreasonable values of c, the differences are small. Finally, these results compare favourably with LEMAIRE's (1985) results where he used a Negative Binomial distribution (Λ was Gamma distributed).

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