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ON JORDAN STRUCTURE IN SEMIPRIME RINGS

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A remarkable theorem of Herstein [**1**, Theorem 2] of which we have made several uses states: If R is a semiprime ring of characteristic different from 2 and if U is both a Lie ideal and a subring of R then either $U \subset Z$ (the centre of R) or U contains a nonzero ideal of R. In a recent paper [**3**] Herstein extends the above mentioned result significantly and has proved that if R is a semiprime ring of characteristic different from 2 and V is an additive subgroup of R such that $[V, U] \subset V$, where U is a Lie ideal of R, then either $[V, U] = 0$ or $V \supset [M, R] \neq 0$ where M is an ideal of R. In this paper our object is to prove the following.

THEOREM. *Let R be a semiprime ring of characteristic different from 2 and V be an additive subgroup of R such that $[U, V] \subset V$, where U is a Jordan ideal of R. Then either $[V, U] = 0$ or $V \supset [M, R] \neq 0$, where M is an ideal of R.*

Throughout this paper a ring will mean an associative ring. We say that a ring R is semiprime if it contains no nonzero nilpotent ideal. Z will denote the centre of R. For $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$. For any two given subsets A and B of R, $[A, B]$ denotes the additive subgroup of R generated by $ab - ba$ for $a \in A$ and $b \in B$ and $A \circ B$ denotes the additive subgroup of R generated by $ab + ba$ for $a \in A$ and $b \in B$. An additive subgroup U of R is said to be a Lie ideal (or Jordan ideal) of R if $[U, R] \subset U$ (or $(U \circ R \subset U)$.

We start this paper with the following theorem which may have some independent interest.

THEOREM 1. *Let R be a semiprime ring of characteristic different from 2, and U a Jordan ideal of R. Suppose that $t \in R$ commutes with u^2 for all $u \in U$. Then t commutes with every element of U.*

Proof. For $r \in R$, let $d(r) = tr - rt$; by hypothesis $d(u^2) = 0$ for every $u \in U$. On linearization, we get $d(uv + vu) = 0$ for $u, v \in U$, which yields on expansion

$$(1) \quad ud(v) + d(u)v + vd(u) + d(v)u = 0 \quad \text{for } u, v \in U.$$

In (1), replace v by $ur + ru$, $r \in R$. Then we have

$$(2) \quad ud(u)r + u^2d(r) + ud(r)u + urd(u) + d(u)ur + d(u)ru + urd(u) + rud(u) + d(u)ru + ud(r)u + d(r)u^2 + rd(u)u = 0 \quad \text{for } u \in U, r \in R.$$

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Replace r by ru in (2) and use (2) to get

(3) $(u^2r + 2uru + ru^2)d(u) = 0$ for $u \in U, r \in R$.

For $u \in U$ and $r \in R$ it follows that $ur + ru \in U$ and $uu + uu = 2u^2 \in U$. But

$$4uru = 2\{u(ur + ru) + (ur + ru)u\} - \{(2u^2)r + r(2u^2)\}.$$

The first and the second term on the right hand side are in U. Hence $4uru \in U$. Therefore, if we replace v by $4uru$ in (1), where $r \in R$, then

$$4\{ud(u)ru + u^2d(r)u + u^2rd(u) + d(u)uru + urud(u) + d(u)ru^2 + ud(r)u^2 + urd(u)u\} = 0$$

Since the characteristic of R is different from 2, we get

(4) $ud(u)ru + u^2d(r)u + u^2rd(u) + d(u)uru + urud(u) + d(u)ru^2 + ud(r)u^2 + urd(u)u = 0$ for $u \in U, r \in R$.

Replace r by ru in (4) and use (4) to get

(5) $(u^2ru + uru^2)d(u) = 0$ for $u \in U, r \in R$.

Since $0 = d(u^2) = ud(u) + d(u)u$, it follows that $ud(u) = -d(u)u$. Multiplying on the right of (3) by u, we obtain $(u^2ru + 2uru^2 + ru^3)d(u) = 0$. Subtracting Equation (5) from the last equation, we have

(6) $0 = (uru^2 + ru^3)d(u) = (ur + ru)u^2d(u)$ for $u \in U, r \in R$.

In (6) write r by rs where $s \in R$; since $u(rs) + (rs)u = [u, r]s + r(u \circ s)$, we get $\{[u, r]s + r(u \circ s)\}u^2d(u) = 0$. But again in view of (6) we get $[u, r]Ru^2d(u) = 0$. Replace r by $d(u)$ and use the fact that $ud(u) = -d(u)u$ (by hypothesis) to conclude that $2ud(u)Ru^2d(u) = 0$, as a consequence of which, we get $u^2d(u)Ru^2d(u) = 0$. This says that $[Ru^2d(u)]^2 = (0)$. Since R is semiprime, we get

(7) $u^2d(u) = 0$ for $u \in U$.

In (7) replace u by $u + v$ where $v \in U$ and use (7). Then

$$u^2d(v) + v^2d(u) + (uv + vu)\{d(u) + d(v)\} = 0.$$

Replace v by $-v$ to obtain

$$-u^2d(v) + v^2d(u) + (uv + vu)\{-d(u) + d(v)\} = 0.$$

Adding last two equations and dividing by 2, we have

$$v^2d(u) + (uv + vu)d(v) = 0$$

for all $u, v \in U$. Replace v by $2v^2$; by hypothesis $d(v^2) = 0$, so that $4v^4d(u) = 0$ from which we get

(8) $v^4d(u) = 0$ for u, $v \in U$.

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For $u \in U$ and $r \in R$, $u \circ (ru - ur) = ru^2 - u^2r \in U$ and so $2(ru^2 - u^2r) \in U$. Since $2u^2 \in U$, $2(ru^2 + u^2r) \in U$. It follows that $4u^2r$ and $4ru^2$ are in U. Therefore, replacing u by $4u^2r$ where $r \in R$ in (8), we obtain $4v^4\{u^2d(r) + d(u^2)r\} = 0$ for all $u, v \in U$ and $r \in R$. In view of (8), $2v^4d(u^2) = 0$. Hence $4v^4u^2d(r) = 0$. In particular,

(9) $\quad u^6d(r) = 0 \quad$ for $u \in U, r \in R$.

Let $M = \{x \in R | xd(r) = 0$ for all $r \in R\}$. Clearly M is a left ideal of R. Now, if $m \in M$ and $x \in R$, then for any $r \in R$, $0 = md(xr) = mxd(r) + md(x)r = mxd(r)$. Hence M is an ideal of R. Let $\bar{R} = R/M$. Clearly, the characteristic of $\bar{R}$ is different from 2, because the characteristic of R is different from 2. Now we shall show that $\bar{R}$ is semiprime. Let $\bar{N}$ be a nilpotent ideal of $\bar{R}$ with $\bar{N}^t = (\bar{0})$, that is, $N^t \subset M$, where N is an ideal of R and the inverse image of $\bar{N}$. Hence $N^td(r) = 0$. But $(Nd(r))^t = Nd(r)Nd(r) \ldots Nd(r) \subset N^td(r) = 0$. Therefore $Nd(r)$ is a nilpotent left ideal of R, hence $Nd(r) = (0)$, that is $N \subset M$ and $\bar{N} = (\bar{0})$. Hence $\bar{R} = R/M$ is semiprime. By (9), $\bar{u}^6 = \bar{0}$ for all $\bar{u} \in \bar{U}$, where $\bar{U}$ is a Jordan ideal of $\bar{R}$.

We shall show that if $u^6 = 0$ for all $u \in U$, where U is a Jordan ideal of a semiprime ring R of characteristic different from 2, then $u = 0$ for all $u \in U$. For $u \in U$ and $r \in R$ we have $2u^2 \in U$, and so $(2u^2)(2u^2) + (2u^2)(2u^2) = 8u^4 \in U$. Therefore $8(u^4r + ru^4) \in U$ and hence $0 = 8^6(u^4r + ru^4)^6$. Multiply on the right by u^4r to obtain $8^6(u^4r)^7 = 0$. Hence $(u^4r)^7 = 0$. If for some u, $u^4 \neq 0$, then u^4R is a nonzero right ideal of R, in which the seventh power of every element is zero. By Levitzki's theorem [**2**, Lemma 1.1] R would have a nonzero nilpotent ideal, which is impossible for a semiprime ring. Hence $u^4 = 0$ for all $u \in U$. By repeating the above argument twice we can show that $u = 0$ for all $u \in U$. Hence $U = 0$.

By the above argument we conclude that $\bar{U} = \bar{0}$, that is, $U \subset M$ and, by definition of M, $ud(r) = 0$ for $u \in U$ and $r \in R$. Replace r by rx for $x \in R$; then $uRd(r) = 0$. As $tR \subset R$, $utRd(r) = 0$. But $tuRd(r) = 0$. Thus we conclude that $d(u)Rd(r) = 0$ for $r \in R$, $u \in U$. In particular, $d(u)Rd(u) = 0$ for $u \in U$. This says that $[d(u)R]^2 = (0)$ which implies that $d(u) = 0$ for all $u \in U$, because R is semiprime. Hence t commutes with every element of U, and this completes the proof of Theorem 1.

Now we shall prove the following corollaries of the above theorem.

Corollary 1. *Let R be a semiprime ring of characteristic different from* 2 *and, U a Jordan ideal of R. If $[U, U] \subset Z$, then $U \subset Z$.*

Proof. If $[U, U] = 0$, then for $t \in R$, $v(tv + vt) = (tv + vt)v$ for all $v \in U$; that is, $tv^2 = v^2t$. Hence by Theorem 1, t commutes with every element of U. Since t is an arbitrary element of R, $[R, U] = 0$; that is, $U \subset Z$. Thus suppose that $[U, U] \neq 0$, so we can find an $\alpha \in [U, U] \subset Z$ such that $\alpha \neq 0$ and $\alpha = uv - vu$ for some $u, v \in U$. For $r \in R$, let $d(r) = ur - ru$. So $d(v) =$

$\alpha\ (\neq 0) \in Z$. For $x \in R$, $ud(x) + d(x)u = u(ux - xu) + (ux - xu)u = u(ux + xu) - (ux + xu)u \in [U, U] \subset Z$. Hence $ud(x) + d(x)u \in Z$ for all $x \in R$. Replace x by v; then $ud(v) + d(v)u = 2\alpha u \in Z$. On commuting this by v, we get $2\alpha[u, v] = 0$; that is, $\alpha^2 = 0$. However, $\alpha \in Z$ and is nilpotent, which is a contradiction in semiprime rings. Hence $\alpha = 0$, which proves the corollary.

COROLLARY 2. *Let R be a semiprime ring of characteristic different from 2 and U be a Jordan ideal of R. Suppose that $t \in R$ commutes with every element of $[U, U]$. Then t commutes with every element of U.*

Proof. For $u, v \in U$ and $r \in R$, $(uv + vu)r - r(uv + vu) = \{u(vr + rv) - (vr + rv)u\} + \{v(ur + ru) - (ur + ru)v\}$. Thus,

(10) $[U \circ U, R] \subset [U, U]$.

Again, for u, $v \in U$ and $r \in R$, $(uv - vu)r - r(uv - vu) = \{u(vr + rv) + (vr + rv)u\} - \{v(ur + ru) + (ur + ru)v\}$, which gives that

(11) $[[U, U], R] \subset U \circ U$.

Let $T = U \circ U + [U, U]$, which is clearly an additive subgroup of R. In view of (10) and (11), we get

$$[T, R] = [U \circ U + [U, U], R] = [U \circ U, R] + [[U, U], R] \subset [U, U] + U \circ U = T.$$

Hence T is a Lie ideal of R. Equation (10) and the hypothesis yield

$$[[U \circ U, t], t] \subset [[U, U], t] = (0)$$

Therefore, $[[T, t], t] = [[U \circ U], t], t] + [[U, U], t]t] = 0$. It follows from Theorem 1 of Herstein [**3**], that $[t, T] = 0$. But $[t, U \circ U] \subset [t, [U, U] + U \circ U] = (0)$. Hence for $u \in U$, $[t, 2u^2] = 0$; that is, $[t, u^2] = 0$. Therefore, by Theorem 1, we conclude that $[t, u] = 0$ for all $u \in U$ and so t commutes with every element of U.

We now prove the main theorem of this paper.

THEOREM 2. *Let R be a semiprime ring of characteristic different from 2 and V be additive subgroup of R such that $[V, U] \subset V$, where U is a Jordan ideal of R. Then either $[V, U] = 0$ or there exists an ideal M of R such that*

$$V \supset [M, R] \neq (0).$$

Proof. Consider $T = [U, U] + U \circ U$. We know in the proof of Corollary 2 that T is a Lie ideal of R. Using the Jacobi identity and $[V, U] \subset V$, we get

$$\begin{aligned}[V, T] &= [V, [U, U] + U \circ U] \\ &= [V, [U, U]] + [V, U \circ U] \\ &\subset [[V, U], U] + [V, U] \\ &\subset [V, U] + V \subset V + V \subset V.\end{aligned}$$

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Thus $[V, T] \subset V$. Hence by Theorem 5 of Herstein [**3**], either $[V, T] = 0$ or $V \supset [M, R] \neq 0$, where M is a nonzero ideal of R. Suppose that $[V, T] = (0)$; then $[V, U \circ U] \subset [V, [U, U] + U \circ U] = [V, T] = (0)$. Thus $[V, U \circ U] = (0)$ and so for all $u \in U$, $[V, 2u^2] = 0$; that is, $[V, u^2] = (0)$. Hence, by Theorem 1, $[V, u] = 0$, for $u \in U$. Hence $[V, U] = (0)$, which proves the theorem.

We close this paper by proving a generalization of a well known result of Herstein, namely: *Let R be a semiprime ring of characteristic different from* 2. *If $t \in R$ such that $[t, [t, r]] = 0$ for all $r \in R$, then $t \in Z$* ([**2**, p. 5]). The generalization to Jordan ideals of the just mentioned result is as follows:

THEOREM 3. *Let R be a semiprime ring of characteristic different from* 2, *and U a Jordan ideal of R. If $t \in R$ such that $[t, [t, U]] = 0$ for all $u \in U$, then t commutes with every element of U.*

Proof. For $r \in R$, let $d(r) = tr - rt$; by hypothesis, $d^2(u) = 0$ for all $u \in U$. If $u \in U$ and $r \in R$, then $ur + ru \in U$ and $0 = d^2(ur + ru) = ud^2(r) + 2d(u)d(r) + d^2(u)r + rd^2(u) + 2d(r)d(u) + d^2(r)u$ or,

$$(12) \quad ud^2(r) + 2d(u)d(r) + 2d(r)d(u) + d^2(r)u = 0 \quad \text{for } r \in R \text{ and } u \in U.$$

Replace r by rt in (12) and use $d(t) = 0$, to obtain

$$(13) \quad ud^2(r)t + 2d(u)d(r)t + 2d(r)td(u) + d^2(r)tu = 0 \quad \text{for } r \in R, u \in U.$$

Multiply on the right hand side of (12) by t and subtract Equation (13) from the resulting expression to get $2d(r)(d(u)t - td(u)) + d^2(r)(ut - tu) = 0$; that is, $2d(r)d^2(u) + d^2(r)d(u) = 0$. But $d^2(u) = 0$, so that

$$(14) \quad d^2(r)d(u) = 0 \quad \text{for } r \in R, u \in U.$$

In (14), replace u by $u(xy) + (xy)u$, where x and y are in R, then

$$\begin{aligned} 0 &= d^2(r)d(u \circ xy) \\ &= d^2(r)d\{[u, x]y + x(u \circ y)\} \\ &= d^2(r)\{d([u, x])y + [u, x]d(y) + d(x)(u \circ y) + xd(u \circ y)\}. \end{aligned}$$

Replace x by $d(v)$ where $v \in U$. By (14), the fourth term is zero, while by hypothesis, $d^2(u) = 0$ for all $u \in U$. Thus the third term is zero and the first term is equal to

$$\begin{aligned} d^2(r)d([u, d(v)]).\, y = d^2(r)\{[d(u), d(v)] + [u, d^2(v)]\}y \\ = d^2(r)[d(u), d(v)]y. \end{aligned}$$

Therefore, we get

$$d^2(r)\{[(d(u), d(v)]y + (ud(v) - d(v)u)d(y)\} = 0.$$

In view of (14) the last equation reduces to

$$(15) \quad d^2(r)ud(v)d(y) = 0 \quad \text{for } r, y \in R, \text{ and } u, v \in U$$

Replace y by xy in (15) for $x \in R$ and use (15), to obtain $d^2(r)ud(v)Rd(y) = 0$. This says that $[d^2(r)ud(v)R]^2 = 0$. Since R is semiprime, it follows that

(16) $d^2(r)ud(v) = 0$ for $r \in R$, u and $v \in U$.

Replace r by rw, where $w \in U$ in (14). Then

$$\{d^2(r)w + 2d(r)d(w) + rd^2(w)\}d(u) = 0.$$

By (16) the first term is zero, while by (14) the third term is zero. Therefore, $2d(r)d(w)d(u) = 0$, that is, $d(r)d(w)d(u) = 0$, for all $r \in R$ and $u, w \in U$. Replace r by rx for $x \in R$. Then $d(r)Rd(w)d(u) = 0$, which gives that $[d(w)d(u)R]^2 = (0)$. Since R is semiprime, we get

(17) $d(w)d(u) = 0$ for all $u, w \in U$.

We have seen in the proof of Theorem 1 that for $u \in U$ and $r \in R$, $4u^2r$ and $4ru^2$ are in U. Therefore, if we replace u by $4u^2r$ in (17), then $d(w)\{4u^2\mathrm{d}(r) + d(4u^2)r\} = 0$. But by (17), $d(w)d(2u^2) = 0$ and so $d(w)u^2d(r) = 0$ for u, $w \in U$ and $r \in R$. Replace r by rx for $x \in R$. Then $d(w)u^2Rd(x) = 0$; in particular, $d(w)u^2Rd(w) = 0$ and so $[d(w)u^2R]^2 = (0)$ which implies that, as R is semiprime,

(18) $d(w)u^2 = 0$ for all w and $u \in U$.

Replace w by $4rv^2$ for $v \in U$ in (18) and use (18) to get

$$0 = \{d(r)(4v^2) + rd(4v^2)\}u^2 = 4d(r)v^2u^2,$$

as a consequence of which we get

(19) $d(r)u^4 = 0$ for all $u \in U$ and $r \in R$.

By using the parallel argument, which we have used in passing from Equation (9), during the proof of Theorem 1, we get $d(u) = 0$ for all $u \in U$. Hence t commutes with every element of U. This proves the theorem.

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