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INTERSECTION OF CONJUGATE SOLVABLE SUBGROUPS IN FINITE CLASSICAL GROUPS

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1. Statement of the problem

Consider some property Ψ of a finite group inherited by all its subgroups. Important examples of such a property are the following: cyclicity; commutativity; nilpotence; solvability. A natural question arises: how large is a normal Ψ-subgroup in an arbitrary finite group G? A more precise formulation of this question is the following.

QUESTION 1.1. Given a finite group G with Ψ-subgroup H of index n, is it true that G has a normal Ψ-subgroup whose index is bounded by some function $f(n)$?

Since the kernel of the action of G on the set of right cosets of H by right multiplication is a subgroup of H and such an action provides a homomorphism to the symmetric group $\mathrm{Sym}(n)$, it always suffices to take $f(n) = n!$ for every such Ψ. We are interested in stronger bounds, in particular those of shape $f(n) = n^c$ for some constant c.

Lucchini [8] and, independently, Kazarin and Strunkov [6] proved the following result.

THEOREM 1.2. *If a finite group G has a cyclic subgroup C of index n, then $\bigcap_{g\in G} C^g$ has index at most $n^2 - n$.*

The following theorem follows from results by Chermak and Delgado [4].

THEOREM 1.3. *Let G be a finite group. If G has an abelian subgroup of index n, then it has a normal abelian subgroup of index at most n^2.*

Zenkov [10] proved the following result when Ψ is nilpotence.

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THEOREM 1.4. *Let G be a finite group and let $\boldsymbol{F}(G)$ be its maximal normal nilpotent subgroup. If G has a nilpotent subgroup of index n, then $|G : \boldsymbol{F}(G)| \leq n^3$.*

Babai *et al.* [1] proved the following statement.

THEOREM 1.5. *There is an absolute constant c such that, if a finite group G has a solvable subgroup of index n, then G has a solvable normal subgroup of index at most n^c.*

Although their proof does not yield an explicit value, they conjectured that $c \leq 7$. This conjecture is closely related to the following problem [7, Problem 17.41(b)].

PROBLEM 1.6. Let H be a solvable subgroup of a finite group G that has no nontrivial solvable normal subgroups. Do there always exist five conjugates of H whose intersection is trivial?

Before we explain how Problem 1.6 is related to Question 1.1, we need to introduce some notation. Problem 1.6 can be reformulated using the notion of *base size*.

DEFINITION 1.7. Assume that a finite group G acts on a set Ω. A point $\alpha \in \Omega$ is *G-regular* if its stabiliser in G is trivial. Define the action of G on Ω^k by

$$(\alpha_1, \ldots, \alpha_k)g = (\alpha_1 g, \ldots, \alpha_k g).$$

If G acts faithfully and transitively on Ω, then the minimal number k such that the set Ω^k contains a G-regular point is the *base size* of G and is denoted by $b(G)$. For a positive integer m, a regular point in Ω^m is a *base* for the action of G on Ω. Denote the number of G-regular orbits on Ω^m by $\mathrm{Reg}(G, m)$ (this number is 0 if $m < b(G)$). If G acts by right multiplication on the set Ω of right cosets of a subgroup H, then G/H_G acts faithfully and transitively on Ω. (Here $H_G = \bigcap_{g \in G} H^g$.) In this case, we denote

$$b_H(G) := b(G/H_G) \quad \text{and} \quad \mathrm{Reg}_H(G, m) := \mathrm{Reg}(G/H_G, m).$$

Therefore, for G and H, as in Problem 1.6, the existence of five conjugates of H whose intersection is trivial is equivalent to the statement that $b_H(G) \leq 5$. We remark that 5 is the best possible bound for $b_H(G)$ because $b_H(G) = 5$ when $G = \mathrm{Sym}(8)$ and $H = \mathrm{Sym}(4) \wr \mathrm{Sym}(2)$. This can be easily verified. In fact, there are infinitely many examples with $b_H(G) = 5$ (see [2, Remark 8.3]).

Let G act transitively on Ω and let H be a point stabiliser, so $|\Omega| = |G : H|$. If $(\beta_1, \ldots, \beta_n)$ is a base for the natural action of G/H_G on Ω, then

$$|(\beta_1, \ldots, \beta_n)^G| \leq |\Omega| \cdot (|\Omega| - 1) \cdots (|\Omega| - n + 1) < |\Omega|^n = |G : H|^n.$$

Therefore,

$$|G : H_G| < |G : H|^n,$$

and if Problem 1.6 has a positive answer, then $c \leq 5$ in Theorem 1.5.

Problem 1.6 is essentially reduced to the case when G is almost simple by Vdovin [9]. In particular, to solve Problem 1.6, it is sufficient to prove

$$\mathrm{Reg}_H(G, 5) \geq 5$$

for every almost simple group G and each of its maximal solvable subgroups H.

2. Results

We study the situation when G_0 is a simple classical group of Lie type isomorphic to $\mathrm{PSL}_n(q), \mathrm{PSU}_n(q)$ or $\mathrm{PSp}_n(q)'$ for some (n, q) and G is an almost simple classical group with socle isomorphic to G_0. In particular, we identify G_0 with its group of inner automorphisms, so

$$G_0 \leq G \leq \mathrm{Aut}(G_0).$$

When $G_0 = \mathrm{PSL}_n(q)$ or $G_0 = \mathrm{PSp}_n(q)$, we also assume that G contains neither graph nor graph-field automorphisms (see [5, Definition 2.5.10]). Therefore, in our results, G is an arbitrary almost simple group with socle G_0 if

(1) $G_0 = \mathrm{PSU}_n(q)$;
(2) $G_0 = \mathrm{PSp}_n(q)$ for $n > 4$ and for $n = 4$ if q is odd.

If q is even, then $\mathrm{PSp}_4(q)$ has a graph automorphism [3, Proposition 12.3.3].

If X is $\Gamma\mathrm{L}_n(q), \Gamma\mathrm{U}_n(q)$ or $\Gamma\mathrm{S}_n(q)$, and N is the subgroup of all scalar matrices in X, then X/N is isomorphic to a subgroup of $\mathrm{Aut}(G_0)$ where G_0 is equal to $\mathrm{PSL}_n(q), \mathrm{PSU}_n(q)$ and $\mathrm{PSp}_n(q)'$, respectively. Hence G can be considered as a subgroup of X/N. A maximal solvable subgroup H of G lies in some maximal solvable subgroup H_1 of X/N. Assume that $b_{H_1}(H_1 \cdot G_0) \leq c$, so there exist $a_1, \ldots, a_c \in H_1 \cdot G_0$ such that

$$H_1^{a_1} \cap \cdots \cap H_1^{a_c} = 1.$$

Since $a_i \in H_1 \cdot G_0$, $a_i = h_i x_i$ for $h_i \in H_1$, $x_i \in G_0$ and $i = 1, \ldots, c$,

$$H_1^{x_1} \cap \cdots \cap H_1^{x_c} = 1$$

and, finally,

$$H^{x_1} \cap \cdots \cap H^{x_c} = 1.$$

Thus, it suffices to consider the situation when H is a maximal solvable subgroup of X/N and $G = H \cdot G_0$.

If S and $\hat{G}$ are the full preimages of H and G in X, then S is solvable and

$$b_H(G) = b_S(\hat{G}).$$

It is convenient to work with matrix groups, so we formulate our main results as follows.

THEOREM 2.1. *Let $X = \Gamma L_n(q)$, $n \geq 2$, where (n, q) is neither $(2, 2)$ nor $(2, 3)$. If S is a maximal solvable subgroup of X, then $\mathrm{Reg}_S(S \cdot SL_n(q), 5) \geq 5$, and in particular $b_S(S \cdot SL_n(q)) \leq 5$.*

THEOREM 2.2. *Let $X = \Gamma U_n(q)$, $n \geq 3$ where (n, q) is not $(3, 2)$. If S is a maximal solvable subgroup of X, then one of the following holds:*

(1) *$b_S(S \cdot SU_n(q)) \leq 4$, so $\mathrm{Reg}_S(S \cdot SU_n(q), 5) \geq 5$;*
(2) *$(n, q) = (5, 2)$ and S is the stabiliser in X of a totally isotropic subspace of dimension 1, $b_S(S \cdot SU_n(q)) = 5$ and $\mathrm{Reg}_S(S \cdot SU_n(q), 5) \geq 5$.*

THEOREM 2.3. *Let $X = \Gamma S_n(q)$ and $n \geq 4$. If S is a maximal solvable subgroup of X, then $b_S(S \cdot Sp_n(q)) \leq 4$, so $\mathrm{Reg}_S(S \cdot Sp_n(q), 5) \geq 5$.*

References

[1] L. Babai, A. Goodman and L. Pyber, 'Groups without faithful transitive permutation representations of small degree', *J. Algebra* **195** (1997), 1–29.

[2] T. Burness, 'Base sizes for primitive groups with soluble stabilisers', *Algebra Number Theory* **15** (2021), 1755–1807.

[3] R. W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters* (John Wiley and Sons, Inc., New York, 1985).

[4] A. Chermak and A. Delgado, 'A measuring argument for finite groups', *Proc. Amer. Math. Soc.* **107**(4) (1999), 907–914.

[5] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups, Number 3: Part I, Chapter A*, Mathematical Surveys and Monographs, 40 (American Mathematical Society, Providence, RI, 1998).

[6] L. S. Kazarin and S. P. Strunkov, 'On intersections of subgroups in groups', *J. Math. Sci. (N.Y.)* **100**(1) (2000), 1888–1892.

[7] E. I. Khukhro and V. D. Mazurov (eds.), *The Kourovka Notebook: Unsolved Problems in Group Theory*, 19th edn, Sobolev Institute of Mathematics (Russian Academy of Sciences, Siberian Branch, Novosibirsk, 2019).

[8] A. Lucchini, 'On the order of transitive permutation groups with cyclic point-stabilizer', *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **9**(4) (1999), 241–243.

[9] E. P. Vdovin, 'On the base size of a transitive group with solvable point stabilizer', *J. Algebra Appl.* **11**(1) (2012), Article no. 1250015, 14 pages.

[10] V. I. Zenkov, 'Intersections of three nilpotent subgroups in a finite group', *Sib. Math. J.* **62**(4) (2021), 621–637.

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