INTERSECTION OF CONJUGATE SOLVABLE SUBGROUPS
IN FINITE CLASSICAL GROUPS

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1. Statement of the problem

Consider some property $\Psi$ of a finite group inherited by all its subgroups. Important examples of such a property are the following: cyclicity; commutativity; nilpotence; solvability. A natural question arises: how large is a normal $\Psi$-subgroup in an arbitrary finite group $G$? A more precise formulation of this question is the following.

**Question 1.1.** Given a finite group $G$ with $\Psi$-subgroup $H$ of index $n$, is it true that $G$ has a normal $\Psi$-subgroup whose index is bounded by some function $f(n)$?

Since the kernel of the action of $G$ on the set of right cosets of $H$ by right multiplication is a subgroup of $H$ and such an action provides a homomorphism to the symmetric group $\text{Sym}(n)$, it always suffices to take $f(n) = n!$ for every such $\Psi$. We are interested in stronger bounds, in particular those of shape $f(n) = n^c$ for some constant $c$.

Lucchini [8] and, independently, Kazarin and Strunkov [6] proved the following result.

**Theorem 1.2.** If a finite group $G$ has a cyclic subgroup $C$ of index $n$, then $\bigcap_{g \in G} C^g$ has index at most $n^2 - n$.

The following theorem follows from results by Chermak and Delgado [4].

**Theorem 1.3.** Let $G$ be a finite group. If $G$ has an abelian subgroup of index $n$, then it has a normal abelian subgroup of index at most $n^2$.

Zenkov [10] proved the following result when $\Psi$ is nilpotence.
**Theorem 1.4.** Let $G$ be a finite group and let $F(G)$ be its maximal normal nilpotent subgroup. If $G$ has a nilpotent subgroup of index $n$, then $|G : F(G)| \leq n^3$.

Babai et al. [1] proved the following statement.

**Theorem 1.5.** There is an absolute constant $c$ such that, if a finite group $G$ has a solvable subgroup of index $n$, then $G$ has a solvable normal subgroup of index at most $nc$.

Although their proof does not yield an explicit value, they conjectured that $c \leq 7$.

This conjecture is closely related to the following problem [7, Problem 17.41(b)].

**Problem 1.6.** Let $H$ be a solvable subgroup of a finite group $G$ that has no nontrivial solvable normal subgroups. Do there always exist five conjugates of $H$ whose intersection is trivial?

Before we explain how Problem 1.6 is related to Question 1.1, we need to introduce some notation. Problem 1.6 can be reformulated using the notion of base size.

**Definition 1.7.** Assume that a finite group $G$ acts on a set $\Omega$. A point $\alpha \in \Omega$ is $G$-regular if its stabiliser in $G$ is trivial. Define the action of $G$ on $\Omega$ by

$$(\alpha_1, \ldots, \alpha_k)g = (\alpha_1g, \ldots, \alpha_kg).$$

If $G$ acts faithfully and transitively on $\Omega$, then the minimal number $k$ such that the set $\Omega^k$ contains a $G$-regular point is the base size of $G$ and is denoted by $b(G)$. For a positive integer $m$, a regular point in $\Omega^m$ is a base for the action of $G$ on $\Omega$. Denote the number of $G$-regular orbits on $\Omega^m$ by $\text{Reg}(G,m)$ (this number is 0 if $m < b(G)$). If $G$ acts by right multiplication on the set $\Omega$ of right cosets of a subgroup $H$, then $G/H_G$ acts faithfully and transitively on $\Omega$. (Here $H_G = \bigcap_{g \in G} H^g$.) In this case, we denote

$$b_H(G) := b(G/H_G) \quad \text{and} \quad \text{Reg}_H(G,m) := \text{Reg}(G/H_G,m).$$

Therefore, for $G$ and $H$, as in Problem 1.6, the existence of five conjugates of $H$ whose intersection is trivial is equivalent to the statement that $b_H(G) \leq 5$. We remark that 5 is the best possible bound for $b_H(G)$ because $b_H(G) = 5$ when $G = \text{Sym}(8)$ and $H = \text{Sym}(4) \wr \text{Sym}(2)$. This can be easily verified. In fact, there are infinitely many examples with $b_H(G) = 5$ (see [2, Remark 8.3]).

Let $G$ act transitively on $\Omega$ and let $H$ be a point stabiliser, so $|\Omega| = |G : H|$. If $(\beta_1, \ldots, \beta_n)$ is a base for the natural action of $G/H_G$ on $\Omega$, then

$$|(\beta_1, \ldots, \beta_n)^G| \leq |\Omega| \cdot (|\Omega| - 1) \cdots (|\Omega| - n + 1) < |\Omega|^n = |G : H|^n.$$}

Therefore,

$$|G : H_G| < |G : H|^n,$$

and if Problem 1.6 has a positive answer, then $c \leq 5$ in Theorem 1.5.
Problem 1.6 is essentially reduced to the case when $G$ is almost simple by Vdovin [9]. In particular, to solve Problem 1.6, it is sufficient to prove

$$\text{Reg}_H(G, 5) \geq 5$$

for every almost simple group $G$ and each of its maximal solvable subgroups $H$.

2. Results

We study the situation when $G_0$ is a simple classical group of Lie type isomorphic to $\text{PSL}_n(q), \text{PSU}_n(q)$ or $\text{PSP}_n(q)'$ for some $(n, q)$ and $G$ is an almost simple classical group with socle isomorphic to $G_0$. In particular, we identify $G_0$ with its group of inner automorphisms, so

$$G_0 \leq G \leq \text{Aut}(G_0).$$

When $G_0 = \text{PSL}_n(q)$ or $G_0 = \text{PSP}_n(q)$, we also assume that $G$ contains neither graph nor graph-field automorphisms (see [5, Definition 2.5.10]). Therefore, in our results, $G$ is an arbitrary almost simple group with socle $G_0$ if

(1) $G_0 = \text{PSU}_n(q)$;
(2) $G_0 = \text{PSP}_n(q)$ for $n > 4$ and for $n = 4$ if $q$ is odd.

If $q$ is even, then $\text{PSP}_4(q)$ has a graph automorphism [3, Proposition 12.3.3].

If $X$ is $\Gamma L_n(q), \Gamma U_n(q)$ or $\Gamma S_n(q)$, and $N$ is the subgroup of all scalar matrices in $X$, then $X/N$ is isomorphic to a subgroup of $\text{Aut}(G_0)$ where $G_0$ is equal to $\text{PSL}_n(q), \text{PSU}_n(q)$ and $\text{PSP}_n(q)'$, respectively. Hence $G$ can be considered as a subgroup of $X/N$. A maximal solvable subgroup $H$ of $G$ lies in some maximal solvable subgroup $H_1$ of $X/N$. Assume that $b_{H_1}(H_1 \cdot G_0) \leq c$, so there exist $a_1, \ldots, a_c \in H_1 \cdot G_0$ such that

$$H_1^{a_1} \cap \cdots \cap H_1^{a_c} = 1.$$ 

Since $a_i \in H_1 \cdot G_0$, $a_i = h_i x_i$ for $h_i \in H_1, x_i \in G_0$ and $i = 1, \ldots, c$,

$$H_1^{x_1} \cap \cdots \cap H_1^{x_c} = 1$$

and, finally,

$$H_1^{x_1} \cap \cdots \cap H_1^{x_c} = 1.$$ 

Thus, it suffices to consider the situation when $H$ is a maximal solvable subgroup of $X/N$ and $G = H \cdot G_0$.

If $S$ and $\hat{G}$ are the full preimages of $H$ and $G$ in $X$, then $S$ is solvable and

$$b_H(G) = b_S(\hat{G}).$$

It is convenient to work with matrix groups, so we formulate our main results as follows.
THEOREM 2.1. Let \( X = \Gamma \mathrm{L}_n(q) \), \( n \geq 2 \), where \((n, q)\) is neither \((2, 2)\) nor \((2, 3)\). If \( S \) is a maximal solvable subgroup of \( X \), then \( \mathrm{Reg}_S(S \cdot \mathrm{SL}_n(q), 5) \geq 5 \), and in particular \( b_S(S \cdot \mathrm{SL}_n(q)) \leq 5 \).

THEOREM 2.2. Let \( X = \Gamma \mathrm{U}_n(q) \), \( n \geq 3 \) where \((n, q)\) is not \((3, 2)\). If \( S \) is a maximal solvable subgroup of \( X \), then one of the following holds:

1. \( b_S(S \cdot \mathrm{SU}_n(q)) \leq 4 \), so \( \mathrm{Reg}_S(S \cdot \mathrm{SU}_n(q), 5) \geq 5 \);
2. \((n, q) = (5, 2)\) and \( S \) is the stabiliser in \( X \) of a totally isotropic subspace of dimension 1, \( b_S(S \cdot \mathrm{SU}_n(q)) = 5 \) and \( \mathrm{Reg}_S(S \cdot \mathrm{SU}_n(q), 5) \geq 5 \).

THEOREM 2.3. Let \( X = \Gamma \mathrm{S}_n(q) \) and \( n \geq 4 \). If \( S \) is a maximal solvable subgroup of \( X \), then \( b_S(S \cdot \mathrm{Sp}_n(q)) \leq 4 \), so \( \mathrm{Reg}_S(S \cdot \mathrm{Sp}_n(q), 5) \geq 5 \).

References


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