CONTINUOUS TRACE $C^\ast$-ALGEBRAS WITH GIVEN DIXMIER-DOUADY CLASS

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Abstract

We give an explicit construction of a continuous trace $C^\ast$-algebra with prescribed Dixmier-Douady class, and with only finite-dimensional irreducible representations. These algebras often have non-trivial automorphisms, and we show how a recent description of the outer automorphism group of a stable continuous trace $C^\ast$-algebra follows easily from our main result. Since our motivation came from work on a new notion of central separable algebras, we explore the connections between this purely algebraic subject and $C^\ast$-algebras.

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Let $\mathcal{A}$ be a continuous trace $C^\ast$-algebra with paracompact spectrum $T$. Dixmier and Douady [4] constructed a cohomology class $\delta(\mathcal{A}) \in H^3(T, \mathbb{Z})$, now known as the Dixmier-Douady class of $\mathcal{A}$, which vanishes exactly when $\mathcal{A}$ is the $C^\ast$-algebra defined by a continuous field of Hilbert spaces over $T$ [3, 10.7.15]. This invariant has attracted considerable attention in recent years since in the case of separable algebras $\delta(\mathcal{A})$ determines $\mathcal{A}$ up to stable isomorphism. (This can easily be deduced from, for example, [2, Théorème 2], [8, Lemma 1.11] and [3, 10.8.4].)

Dixmier and Douady also showed in [4] that every class in $H^3(T, \mathbb{Z})$ is $\delta(\mathcal{A})$ for some $\mathcal{A}$. Their proof of this uses Zorn's lemma and the fact that when $H$ is an infinite-dimensional Hilbert space the sheaf of germs of $U(H)$-valued functions is...
soft, and all the irreducible representations of the resulting algebra are infinite-dimensional. We present here an explicit construction of a continuous trace $C^*$-algebra with prescribed Dixmier-Douady class. The irreducible representations of the $C^*$-algebra we construct are all finite-dimensional; as it is easy to see that $n \delta(A) = 0$ when $A$ is $n$-homogeneous [5, Proposition 1.4] it follows that our algebra is in general far from homogeneous.

This construction is the content of our first section. In Section 2 we discuss the automorphism groups of the algebras we have built. For any continuous trace $C^*$-algebra $A$ with spectrum $T$ and any automorphism $\alpha \in \text{Aut}_{C(T)} A$ there is a cohomology class $\xi(\alpha) \in H^2(T, \mathbb{Z})$ which vanishes when $\alpha$ is implemented by a multiplier, and the main theorem of [8] asserts that when $A$ is stable and separable every class in $H^2(T, \mathbb{Z})$ arises this way. The proof of this in [8] is modelled on the surjectivity argument of Dixmier-Douady, and is not constructive; however, for suitable algebras of the type in Section 1 we can write down automorphisms corresponding to given elements of $H^2(T, \mathbb{Z})$, and we use this to give a short proof of the theorem in [8].

The construction we describe here arose in connection with work on a notion of central separable algebra which does not require that the algebra have an identity [14], [11], and when the spectrum $T$ is compact our $C^*$-algebras are also central separable algebras in this sense. In our third section we discuss the relationship between these central separable algebras and continuous trace $C^*$-algebras.

Our notation concerning continuous trace $C^*$-algebras will more or less conform to that of [3, Chapter 10]. If $H$ is a continuous field of Hilbert spaces over $T$ with continuous sections $\Gamma(H)$, we denote the corresponding field of elementary $C^*$-algebras by $\mathfrak{A}(H)$ and write $\Gamma(\mathfrak{A}(H))$ for the $C^*$-algebra defined by $H$. In Section 3 it will be crucial that we are working with purely algebraic tensor products, so we shall always write $\otimes$ when we mean to take a completion.

1. The construction of continuous trace $C^*$-algebras with prescribed Dixmier-Douady class

Let $T$ be a paracompact space, and let $\mathcal{R}, \mathcal{S}$ respectively denote the sheaves of germs of continuous real and $S^1$-valued functions on $T$. Then the covering map $t \to \exp 2\pi it$ gives a short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{R} \to \mathcal{S} \to 0$$

of sheaves, which in turn gives a long exact sequence of cohomology

$$\ldots \to H^2(T, \mathcal{R}) \to H^2(T, \mathcal{S}) \to H^3(T, \mathbb{Z}) \to H^3(T, \mathcal{R}) \to \ldots$$
The sheaf $\mathcal{R}$ is fine, so $H^n(T, \mathcal{R}) = 0$ for $n > 0$, and the middle map is an isomorphism. The Dixmier-Douady class of a continuous trace $C^*$-algebra $A$ with paracompact spectrum $T$ is by definition the image of a class in $H^2(T, \mathcal{R})$ [3, Section 10.7], so we may as well start with a 2-cocycle $\lambda_{ijk}: N_{ijk} \to S^1$ relative to a locally finite open cover $\{N_i: i \in I\}$. We can always replace the cover by another $\{M_i: i \in I\}$ with $\overline{M}_i \subset N_i$, so we may also suppose that each $\lambda_{ijk}$ is defined on the closure $\overline{N}_{ijk}$.

**Theorem 1.** Let $\{N_i: i \in I\}$ be a locally finite cover of a locally compact paracompact space $T$ by relatively compact open sets, and suppose that $\lambda_{ijk}: \overline{N}_{ijk} \to S^1$ is a 2-cocycle. Let

$$A_1 = \left\{ \sum_{j, k \in I} \phi_{jk} e_{jk} | \phi_{jk} \in C(T), \phi_{jk} \equiv 0 \text{ outside } N_{jk} \right\}$$

have the obvious structure as a $C(T)$-module, multiplication defined by

$$(\phi_{jk} e_{jk})(\psi_{lm} e_{lm}) = \delta_{kl} \psi_{jm},$$

where $\delta_{kl}$ is the Kronecker delta and $\psi$ is given by

$$\psi(t) = \begin{cases} \lambda_{jkm}(t) \phi_{jk}(t) \psi_{km}(t) & \text{for } t \in N_{jk}, \\ 0 & \text{for } t \notin N_{jk}, \end{cases}$$

and involution defined by

$$(\phi_{jk} e_{jk})^* = \overline{\phi_{jk}} e_{kj}.$$

For $t \in T$ let $I(t) = \{i \in I: t \in N_i\}$; note that $n_i = |I(t)|$ is finite. If $t \in N_i$, then for the usual $n_i \times n_i$ matrix norm we have

$$\| (\lambda_{ikl}(t) \phi_{kl}(t))_{k, l \in I(t)} \| = \| (\lambda_{jkl}(t) \phi_{kl}(t))_{k, l \in I(t)} \|,$$

so that for each $t$ we have a semi-norm $\| \cdot \|_t$ on $A_1$. Let $A$ be the set of $a \in A_1$ such that $t \to \|a\|_t$ vanishes at infinity, and set $\|a\| = \sup_t \|a\|_t$. Then $A$ is a continuous trace $C^*$-algebra with spectrum $T$ whose Dixmier-Douady class $\delta(A) \in H^3(T, \mathbb{Z}) = H^2(T, \mathcal{R})$ is represented by the cocycle $\{N_i, \lambda_{ijk}\}$. The dimension of the irreducible representation corresponding to $t \in T$ is $n_i$.

**Proof.** Simple calculations using the cocycle identity show that $A$ is a $*$-algebra with the above operations, and that for $t \in N_i$

$$\pi_{i,t}\left( \sum_{j, k \in I} \phi_{jk} e_{jk} \right) = \left( \frac{\lambda_{ijk}(t)}{\phi_{jk}(t)} \right)_{j, k \in I(t)}$$

defines a $*$-representation of $A$ into $M_{n_i}(\mathbb{C})$. If $t \in N_{ij}$ and $D(\mu_k)$ is the diagonal matrix with entries $\mu_k$, then the cocycle identity yields

$$D\left( \frac{\lambda_{ijk}(t)}{\phi_{jk}(t)} \right) \left( \left( \frac{\lambda_{ikl}(t)}{\phi_{kl}(t)} \right)_{k, l \in I(t)} \right) D\left( \frac{\lambda_{jkl}(t)}{\phi_{kl}(t)} \right) = \left( \left( \frac{\lambda_{jkl}(t)}{\phi_{kl}(t)} \right)_{k, l \in I(t)} \right).$$
The diagonal matrices are unitary, so we deduce that the norms of the matrices \( \pi_{i,t}(a) \) and \( \pi_{j,t}(a) \) are always equal, and we have well-defined semi-norms

\[
\|a\|_t = \|\pi_{i,t}(a)\| \quad \text{for } t \in N_t,
\]

as claimed. The norm on \( A \) satisfies the \( C^* \)-condition \( \|aa^*\| = \|a\|^2 \) since each \( \| \cdot \|_t \) does, and it is not hard to see that \( A \) is complete, so \( A \) is a \( C^* \)-algebra.

For each \( t \in T \) we define an ideal in \( A_1 \) by

\[
J_t = \left\{ \sum_{j,k} \phi_{jk}e_{jk} \in A | \phi_{jk}(t) = 0 \text{ for all } j, k \right\}.
\]

We denote the quotient \( C^* \)-algebra \( A/J_t \) by \( A(t) \), and we write \( a(t) \) for the image of \( a \in A_1 \) in \( A(t) \). Note that if \( t \in N_t \) the representation \( \pi_{i,t} \) induces an isomorphism of \( A(t) \) onto \( M_{n_t} \), so each \( A(t) \) is an elementary \( C^* \)-algebra. In fact, \( \mathfrak{A} = \{ A(t), A_1 \} \) is a continuous field of elementary \( C^* \)-algebras over \( T \) such that \( A \) is the \( C^* \)-algebra of continuous sections vanishing at infinity. For by definition \( \{ a(t) : a \in A \} \) is all of \( A(t) \), and the continuity of

\[
t \to \|a(t)\| = \|\pi_{i,t}(a)\|
\]

follows from the continuity of the matrix norm (note that if \( t_a \to t \) then \( I(t_a) \) eventually contains \( I(t) \)). Further, if \( x = (x(t)) \in \prod A(t) \) then there are unique scalars \( \nu_{jk}(t) \) such that

\[
\pi_{i,t}(x(t)) = \left( \lambda_{ijk}^{(t)} \nu_{jk}(t) \right)_{j, k \in I(t)}.
\]

If \( x \) is locally uniformly approximable by elements of \( A_1 \), then standard arguments show that the \( \nu_{jk} \) are continuous and vanish off \( N_{jk} \), so that \( x \) is the section defined by \( \sum \nu_{jk}e_{jk} \in A \). Thus \( \mathfrak{A} \) is a continuous field as asserted and \( A = \Gamma_0(\mathfrak{A}) \) has spectrum \( T \). It is easy to see that \( \mathfrak{A} \) satisfies Fell's condition (for example, if \( t \in N_t, \rho = 1 \text{ near } t \text{ and } \rho = 0 \text{ off } N_t \), then \( (\rho e_{ii})(s) \) is a rank one projection for \( s \) near \( t \) and hence \( A \) has continuous trace.

To compute the Dixmier-Douady class of \( A \) we build fields \( H_t \) of Hilbert spaces over \( \overline{N_t} \) and isomorphisms of the associated fields of elementary \( C^* \)-algebras \( \mathfrak{A}(H_t) \) onto \( \mathfrak{A}|_{\overline{N_t}} \). For \( t \in \overline{N_t} \) we define

\[
H_t(t) = \left\{ \sum_{k \in I(t)} \lambda_k e_k : \lambda_k \in \mathbb{C} \right\},
\]

with the usual inner product \( (e_k|e_l) = \delta_{kl} \), and take as our space of continuous sections

\[
\Gamma(H_t) = \left\{ \sum_{k \in I_t} \phi_k e_k : \phi_k \in C(\overline{N_t}), \phi \equiv 0 \text{ off } N_k \right\},
\]

where \( I_t = \{ k : N_k \cap \overline{N_t} \neq \{ \emptyset \} \} \). It is routine to check that this does define a continuous field of Hilbert spaces. The corresponding field \( \mathfrak{A}_t = \mathfrak{A}(H_t) \) of
elementary $C^*$-algebras is that generated by fields of the form $e \otimes \overline{f}$ for $e, f \in \Gamma(H_i)$, where $x \otimes \overline{y}$ denotes the rank one operator $z \rightarrow (z|y)x$. We define a linear map from $\Gamma(H_i) \otimes \Gamma(H_j)$ to $A|_{N_i}$ by

$$h_i(e_j \otimes e_k) = \theta e_{jk},$$

where $\theta(t) = \begin{cases} \lambda_{ijk}(t) \phi(t) \overline{\psi(t)} & \text{for } t \in \overline{N_{ijk}}, \\ 0 & \text{for } t \in \overline{N_i \setminus N_{jk}} \end{cases}$

a standard Urysohn's lemma argument shows that $\theta e_{jk}$ is in fact the restriction of an element of $A$. Further, $h_i$ is a $*$-homomorphism, isometric from the usual norm on $\Gamma(\mathcal{A}_i)$ to the given one on $A$, and is easily seen to be surjective; hence it extends to an isomorphism of $\mathcal{A}_i$ onto $\mathcal{A}|_{N_i}$. (In fact, every $\theta e_{jk}$ is the image of an elementary tensor so if the index set $I$ is finite $h_i$ defines an isomorphism of the algebraic tensor product $\Gamma(H_i) \otimes_{\mathcal{C}(\overline{N_i})} \Gamma(H_j)$ onto $A|_{N_i}$. As the latter algebra is complete so is the algebraic tensor product, which therefore equals $\Gamma(\mathcal{A}_i)$. We now define isomorphisms $g_{ij}: H_j|_{N_i} \rightarrow H_j|_{N_{ij}}$ by

$$g_{ij}(t)(\phi_k(t)e_k) = \begin{cases} \lambda_{ijk}(t)\phi_k(t)e_k & \text{if } t \in \overline{N_{ijk}}, \\ 0 & \text{if } t \in \overline{N_i \setminus N_k}. \end{cases}$$

The induced isomorphism $\text{Ad } g_{ij}$ of $\mathcal{A}(H_j)$ into $\mathcal{A}(H_j)$ is given on elementary tensors by

$$(\text{Ad } g_{ij})(t)(\phi_k(t)e_k \otimes \overline{\psi_l(t)}e_l) = g_{ij}(t)(\phi_k(t)e_k \otimes g_{ij}(t)\overline{\psi_l(t)}e_l)$$

$$= \begin{cases} \lambda_{ijkl}(t)\phi_k(t)e_k \otimes \lambda_{ijk}(t)\overline{\psi_l(t)}e_l & \text{if } t \in \overline{N_{ijkl}}, \\ 0 & \text{if } t \in \overline{N_i \setminus N_{kl}}. \end{cases}$$

so that routine calculations using the cocycle identity give

$$h_i(t) \circ (\text{Ad } g_{ij})(t) = h_j(t) \quad \text{for } t \in \overline{N_i}.$$ 

Thus $g_{ij}$ defines the isomorphism $h_i^{-1}h_j$ as in [3, 10.7.11], and for $t \in \overline{N_{ij}}$ we have

$$g_{ij}(t)g_{jk}(t) = \lambda_{ijk}(t)g_{ik}(t),$$

so that the class $\delta(A) = \gamma(\mathcal{A})$ in $H^2(T, \mathcal{S})$ is represented by the cocycle $\{N_i, \lambda_{ijk}\}$ as claimed (see [3, 10.7.12–14]).

**Remarks.** 1. If $T$ has covering dimension $n$, then we can realise any class in $H^2(T, \mathcal{S})$ as a cocycle relative to a cover where at most $n + 1$ different sets intersect. Because the Dixmier-Douady class determines a separable continuous trace $C^*$-algebra up to stable isomorphism, our theorem implies that every such algebra with spectrum $T$ is stably isomorphic to an algebra whose irreducible representations have dimension $\leq n + 1$. This has already been shown by Brown [1, Corollary 2.11] using different reasoning.
2. The last part of the proof could be simplified a bit by constructing the class \( \delta(A) \) as in [9, 2.6–2.9] using local rank one projections and intertwining partial isometries rather than fields of Hilbert spaces and isomorphisms. However, in our present proof we also showed that, when \( T \) is compact, the algebraic tensor product \( \Gamma(H_i) \otimes_{C(T)} \Gamma(H_j) \) is complete, and this has some interesting algebraic consequences, which we shall discuss in Section 3.

3. The \( C^* \)-algebra \( A \) we construct in Theorem 1 can also be viewed as a twisted groupoid \( C^* \)-algebra. For simplicity we suppose \( T \) is compact and \( \{ N_i; i \in I \} \) is a finite cover. Then we let \( \psi \) be the local homeomorphism of the disjoint union \( X = \bigcup N_i \) onto \( T \), let \( \mathcal{R}(\psi) \) be the equivalence relation induced on \( X \) by \( \psi \) as in [6, Section 4], and let \( G \) be the corresponding topological groupoid with left Haar system induced by counting measure on the fibres of \( \psi \) (see [12, Section 1.2]). Given a cocycle \( \lambda_{ijk}: N_{ijk} \rightarrow S^1 \) we define a 2-cocycle \( \sigma: G^2 \rightarrow S^1 \) by

\[
\sigma((x, y), (y, z)) = \lambda_{i(x)(y)(z)}(\psi(x)),
\]

where \( G^2 \) denotes the set of composable elements of \( G \) and \( i: X \rightarrow I \) is defined by \( x \in N_{i(x)} \). We define \( \Phi: C_c(G) \rightarrow A \) by \( \Phi f = \sum \phi_{jk} e_{jk} \), where

\[
\phi_{jk}(t) = \begin{cases} f(x, y) & \text{if } x \in N_j, y \in N_k \text{ and } \psi(x) = \psi(y) = t, \\ 0 & \text{if } t \notin N_{jk}. \end{cases}
\]

It is routine to verify that \( \Phi \) defines a \( * \)-monomorphism of the convolution algebra \( C_c(G, \sigma) \) (see [12, Section 4.1]) onto a dense subalgebra of \( A \), and hence gives an isomorphism of \( C^*(G, \sigma) \) with \( A \).

In particular, when \( \lambda_{ijk} = 1 \) the algebra \( A \) is the \( C^* \)-algebra \( C^*(\psi) \) associated by Kumjian [6] to the local homeomorphism \( \psi \). This can also be seen directly: his imprimitivity bimodule \( l^2(\psi) \) is isomorphic to

\[
H = \left\{ \sum_k \phi_k e_k: \phi_k \in C(T), \phi_k = 0 \text{ outside } N_k \right\}
\]

with \( C(T) \)-valued inner product given by

\[
\left( \sum \phi_k e_k | \sum \psi_l e_l \right) = \sum \overline{\phi_k} \psi_l,
\]

and \( A \) acts on \( H \) by

\[
\left( \sum \phi_{jk} e_{jk} \right) \left( \sum \psi_l e_l \right) = \sum_j \left( \sum_k \phi_{jk} \psi_k e_j \right).
\]

This is, of course, the same construction as we carried out locally to prove our theorem, modulo changes in convention regarding inner products.

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2. Automorphisms

For any continuous trace $C^*$-algebra $A$ with paracompact spectrum $T$ there is an exact sequence

$$0 \rightarrow \text{Inn } A \rightarrow \text{Aut}_{C(T)} A \rightarrow H^2(T, \mathbb{Z}),$$

where $\text{Inn } A$ denotes the group of $A$ determined by multipliers of $A$ (see [13, Section 5]). We shall now investigate the range of the homomorphism $\zeta_A$ for the algebra $A$ constructed in Theorem 1.

So let $\{N_i, \lambda_{ij}\}$ and $A$ be as in Theorem 1, and let $c \in H^2(T, \mathbb{Z})$ be given. If $c$ can be represented by a 1-cocycle $\{N_t, \mu_{ij}\}$ with values in $\mathcal{S}$ relative to the same cover $\{N_i\}$, then we can define an automorphism $\alpha$ of $A$ by

$$\alpha\left(\sum \phi_{jk} e_{jk}\right) = \sum \psi_{jk} e_{jk} \text{ where } \psi_{jk}(t) = \begin{cases} \mu_{jk}(t) \phi_{jk}(t) & \text{if } t \in N_{jk}, \\ 0 & \text{otherwise.} \end{cases}$$

This is easily seen to be a $C(T)$-module automorphism: we compute its class $\xi(\alpha)$ in $H^1(T, \mathcal{S}) \cong H^2(T, \mathbb{Z})$. Let $\{M_i\}$ be an open cover of $T$ with $M_i \subset N_i$, and choose continuous functions $\rho_i: T \rightarrow [0,1]$ such that $\rho_i \equiv 1$ on $M_i$ and $\rho_i \equiv 0$ off $N_i$. We can now define multipliers $m_i$ of $A$ by

$$m_i = \sum_{j \in I} \rho_i \mu_{ij} e_{jj},$$

note that although $\rho_i \mu_{ij}$ is not defined on all of $T$, whenever we have $\phi \equiv 0$ off $N_{jk}$ the function $\rho_i \mu_{ij} \phi$ does extend to be continuous on $T$, and simple calculations show that under the usual multiplication rule this gives a multiplier of $A$. Further, the cocycle identity shows that for $t \in M_i$

$$\alpha\left(\sum \phi_{jk} e_{jk}\right)(t) = \left(m_i\left(\sum \phi_{jk} e_{jk}\right) m_i^*\right)(t),$$

so that $m_i$ implements $\alpha$ over $M_i$. Moreover, the same cocycle identity also gives

$$\mu_{ij}(t) m_j(t) = m_i(t) \text{ for } t \in M_{ij},$$

and we deduce that $\xi(\alpha)$ is represented by the cocycle $\{M_i, \mu_{ij}\}$ (see [13, Section 5]). This defines the same class as the one we started with, and therefore $\xi(\alpha) = c$.

Of course, we cannot expect to represent an arbitrary class in $H^2(T, \mathbb{Z})$ relative to a fixed open cover (and we will come back to this question later), but if we start with classes $d \in H^3(T, \mathbb{Z})$ and $c \in H^2(T, \mathbb{Z})$ then we can always represent them as $S^1$-valued cocycles relative to the same cover. Hence the argument in the preceding paragraph proves the following result:

**Proposition 2.** Let $T$ be a locally compact paracompact space, and let $d \in H^3(T, \mathbb{Z})$, $c \in H^2(T, \mathbb{Z})$. Then there are a continuous trace $C^*$-algebra $A$ with spectrum $T$, with $\delta(A) = d$, and with only finite-dimensional irreducible representations, and an automorphism $\alpha \in \text{Aut}_{C(T)} A$ such that $\zeta_A(\alpha) = c$. 

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Remark. Alex Kumjian has noticed independently that, given a cocycle \( \mu_{ij}: N_{ij} \to S^1 \), one can write down an automorphism \( \alpha \) of the \( C^* \)-algebra \( C^*(\psi) \) associated to the local homeomorphism \( \psi: \bigcup_i N_i \to T \) such that \( \xi(\alpha) \) is represented by \( \{ N_i, \mu_{ij} \} \). In fact, it was his observation that alerted us to the realisation of \( A \) as the twisted groupoid \( C^* \)-algebra \( C^*(G, \sigma) \) (see Remark (3) in Section 1). The automorphism \( \alpha \) can be conveniently viewed in this realisation too: define a continuous 1-cocycle \( c: G \to S^1 \) by
\[
c(x, y) = \mu_{i(x)(y)}(\psi(x)),
\]
and then the automorphism \( \alpha \) of \( C^*(G, \sigma) \) is defined by
\[
\alpha(f)(x, y) = c(x, y)f(x, y)
\]
for \( f \in C_c(G) \), as in [12, Proposition II.5.1].

Corollary 3 ([8, Theorem 2.1]). Let \( A \) be a separable stable continuous trace \( C^* \)-algebra with spectrum \( T \). Then the homomorphism \( \xi_A \) is surjective.

Proof. Let \( c \in H^2(T, \mathbb{Z}) \). Then by the proposition there are an algebra \( B \) and an automorphism \( \alpha \in \text{Aut}_{C(T)} B \) such that \( \delta(B) = \delta(A) \) and \( \xi_B(\alpha) = c \). Since the Dixmier-Douady class determines a separable continuous trace \( C^* \)-algebra up to stable isomorphism, we have
\[
A \cong A \boxtimes K(H) \cong B \boxtimes K(H),
\]
and we may assume that this isomorphism induces the identity map from \( T = \hat{A} \) to \( T = \hat{B} = (B \boxtimes K(H))^\wedge \) (see [10, Lemma 4.3]). If \( m \in M(B) \) implements \( \alpha \) over \( N \) then
\[
m \otimes 1 \in M(B) \boxtimes M(K(H)) \subset M(B \boxtimes K(H))
\]
implements \( \alpha \otimes \text{id} \) over \( N \), so \( \xi_B \circ K(H)(\alpha \otimes \text{id}) = \xi_B(\alpha) = c \), and the corresponding automorphism \( \beta \) of \( A \) therefore satisfies \( \xi_A(\beta) = c \).

Remark. Corollary 3 holds for arbitrary \( C^* \)-algebras with paracompact spectrum \( T \) [10, Corollary 3.12]. However, the proof given there involves establishing surjectivity for \( A = C_0(T, K(H)) \) first, and this seems to be more complicated than our Proposition 2.

As we have seen above, if \( A \) is the \( C^* \)-algebra of Theorem 1 corresponding to a cocycle \( \lambda_{ijk}: \overline{N}_{ijk} \to S^1 \) then the range of \( \xi_A \) contains the subgroup \( H^1(\mathcal{N}, \mathcal{P}) \) of \( H^1(T, \mathcal{P}) \cong H^2(T, \mathbb{Z}) \) consisting of those classes realisable on the cover \( \mathcal{N} = \{ N_i \} \). It is quite easy to see that, while this range need not be all of \( H^2(T, \mathbb{Z}) \), it may contain more than \( H^1(\mathcal{N}, \mathcal{P}) \). First of all, take the one set cover \( \{ T \} \) of a space with \( H^2(T, \mathbb{Z}) \neq 0 \); then \( A \cong C_0(T) \) and the range of \( \xi_A \) is \( \{ 0 \} \neq H^2(T, \mathbb{Z}) \).
This example is more general than it appears, since if \( Y \) is a compact set contained in only one member of the cover, then \( A|_Y \cong C(Y) \) and no element of \( H^2(T, \mathbb{Z}) \) whose image in \( H^2(Y, \mathbb{Z}) \) is non-zero can come from a \( C(T) \)-automorphism of \( A \). On the other hand, if we take the trivial cover \( N_1 = N_2 = T \) consisting of two sets, then \( A = C(T, M_2) \) and there can be automorphisms \( \alpha \) for which \( \xi(\alpha) \neq 0 \) and hence does not belong to \( H^1(\mathcal{M}, \mathcal{P}) \). This argument is also more general than it first appears, since if \( Y \) is a compact subset of \( N_1 \cap N_2 \) which meets no other \( N_k \), then \( A|_Y \cong C(Y, M_2) \) and there could be automorphisms of \( A \) which do not trivialise over \( N_1 \cap N_2 \).

### 3. Central separable algebras

Let \( R \) be a commutative ring with identity, let \( A \) be an \( R \)-algebra (not necessarily with an identity) and let \( Z(A) \) denote the ring of \( A \rightarrow A \) bimodule endomorphisms of \( A \). There is always a natural map \( i: R \rightarrow Z(A) \) and we call \( A \) central if this is an isomorphism. Following [14, Section 2] we say \( A \) is separable if \( A^2 = A \), \( A \) is projective as an \( A \rightarrow A \) bimodule, and for each maximal ideal \( M \) of \( Z(A) \) we have \( MA \neq A \). An immediate property of such algebras is that the multiplication map: \( A \otimes_R A \rightarrow A \) is split as an \( A \rightarrow A \) bimodule homomorphism. Our main theorem can be strengthened as follows.

**Proposition 4.** Let \( T \) be a compact Hausdorff space and let \( d \in H^3(T, \mathbb{Z}) \). Then there is a continuous trace \( C^* \)-algebra \( A \) with \( \delta(A) = d \) which is also a central separable \( C(T) \)-algebra.

**Proof.** Let \( \{ N_i \} \) be a finite open cover of \( T \) such that \( d \) is represented by a cocycle \( \lambda_{ijk}: \overline{N}_{ijk} \rightarrow S^1 \), and let \( A \) be the algebra constructed in Theorem 1. Choose another cover \( \{ M_i \} \) with \( M_i \subset N_i \), and functions \( \rho_i \in C_0(N_i) \) with \( \rho_i \equiv 1 \) on \( M_i \). We define \( A \rightarrow A \) bimodule homomorphisms \( \omega_i \) on \( A \otimes_{C(T)} A \) by interchanging the two copies of \( \Gamma(H_i) \) in

\[
A \otimes_{C(T)} A|_{\overline{N}_i} \cong \Gamma(H_i) \otimes_{C(T)} \Gamma(H_i) \otimes_{C(T)} \Gamma(H_i) \otimes_{C(T)} \Gamma(H_i)
\]

and multiplying by \( \rho_i \). Let \( \Omega \) be the module of \( A \rightarrow A \) homomorphisms generated by the \( \omega_i \), and let \( \text{tr}: A \otimes_{C(T)} \Omega \rightarrow Z(A) \) be as in [14, page 174]. We have \( Z(A) = C(T) \), since \( Z(A) \) is by definition the centre of the multiplier algebra of \( A \). Thus it will follow from [14, Proposition 3.8] that \( A \) is central separable if we can prove that the range of \( \text{tr} \) is not contained in any maximal ideal of \( C(T) \). However, a straightforward calculation shows that if \( t \in M_i \) then \( \text{tr}(\rho_i e_{ii} \otimes \omega_i) \) cannot vanish at \( t \), and the result is proved.
This last result raises an obvious question: what is the relationship between the classes of $C^*$-algebras with compact spectrum $T$ and central separable algebras over $C(T)$? It is well-known that the central separable $C(T)$-algebras with identity are precisely the locally homogeneous $C^*$-algebras with spectrum $T$, but the situation for algebras without identity is rather more complicated. For example, the algebra $FR(H)$ of finite rank operators on a separable (!) Hilbert space $H$ is central separable over $C$ (we have $FR(H) = H \otimes C \tilde{H}$) but it is not a $C^*$-algebra unless $H$ is finite-dimensional. Our next proposition gives an answer to this question—and shows, among other things, that Proposition 4 does say more than Theorem 1.

PROPOSITION 5. (1) Let $A$ be a $C^*$-algebra with compact Hausdorff spectrum $T$ which is also a central separable $C(T)$-algebra for the natural action of $C(T)$ on $A$. Then $A$ is a continuous trace $C^*$-algebra whose irreducible representations have finite, bounded dimensions.

(2) There are continuous trace $C^*$-algebras $A$ with compact spectrum and $\{\text{dim } \pi: \pi \in \hat{A}\}$ bounded which are not central separable algebras.

The proof of this result will depend on a series of simple lemmas. We begin with a purely algebraic result which is implicit in [14].

LEMMA 6. Let $B$ be a central separable algebra over a commutative ring $R$, and suppose that $p \in B$ is a rank one idempotent (i.e., $pBp = Rp$). Then the map $ap \otimes pb \rightarrow apb$ induces an isomorphism of $Bp \otimes R pB$ onto $B$.

PROOF. This is a consequence of the proofs of Proposition 4.2 and 4.3 of [14] with $N = Bp$, $M = pB$ and $\lambda: N \otimes_R M \rightarrow B$ given by the multiplication in $B$; the regularity of $N$, $M$ follows from [14, Propositions 1.1 and 1.6]. The proof of Proposition 4.2 shows that if $A = M \otimes_B N$, then $\lambda$ induces an isomorphism of $N \otimes_A M$ onto $B$. However, the last argument in the proof of Proposition 4.3 shows that the multiplication also induces an isomorphism of $A = pB \otimes_B pB$ onto $pBp$, which is just $Rp$ since $p$ is rank one. We therefore deduce that $B \cong Bp \otimes_R pB$ as claimed.

LEMMA 7. Let $A$ be a $C^*$-algebra which is also a central separable $C(T)$-algebra. Then every closed 2-sided ideal in $A$ is regular, and has the form $IA$ for some 2-sided ideal $I$ in $C(T)$.

PROOF. The multiplication map $A \otimes_{C(T)} A \rightarrow A$ is split and $A$ is therefore a regular 2-sided $A$-module. If $M$ is a 2-sided ideal in $A$, then we have $MA = M$
LEMMA 8. Let $A$ be a $C^*$-algebra which is central separable over $C$. Then $A \cong M_n(C)$.

**Proof.** Let $\pi: A \to B(H)$ be an irreducible representation; by Lemma 7 $A$ has no non-trivial ideals so $\pi$ must be faithful. By [14, Proposition 4.8] $A$ must contain a rank one idempotent $p$: we claim that $P = \pi(p) \in B(H)$ is also rank one. For suppose $\xi \in PH, \xi \neq 0$ and $\eta \in P^*H$ satisfies $(\xi|\eta) = 0$. Then for any $a \in A$

$$(\eta|\pi(a)\xi) = (P^*\eta|\pi(a)P\xi) = (\eta|\pi(pap)\xi) \in C(\eta|\xi) = 0.$$ 

Since $P^*H = ((1 - P)H)^\perp$ this says that

$$\eta \perp \xi, \eta \perp (1 - P)H \Rightarrow \eta \perp \pi(A)\xi$$

and because $\pi$ is irreducible it follows that $\xi$ and $(1 - P)H$ span $H$. Thus $PH = C\xi$ and $P$ is rank one. The irreducibility of $\pi$ implies that $\pi(A) \supseteq K(H)$, and as $A$ has no ideals $\pi(A) = K(H)$. But the latter consists of finite rank operators so $H$ must be finite-dimensional. This will be a $*$-isomorphism if $P$ is chosen so that $P^* = P$.

**Proof of Proposition 5(1).** Let $\pi: A \to B(H)$ be irreducible. By Lemma 7 $\ker \pi$ is a regular ideal of the form $IA$ for some ideal $I$ in $C(T)$. The extension $\bar{\pi}$ of $\pi$ to the multiplier algebra restricts to a representation of $C(T)$ in $\pi(A)' = C1$, and hence $\ker \bar{\pi}$ is the ideal $I_t$ of functions vanishing at some point $t$ of $T$. We then have $IA \subseteq \ker \pi$ and the maximality of $I_t$ shows that $IA = \ker \pi$. Thus by [14, Propositions 2.7 and 3.5] $A/\ker \pi$ is central separable over $C(T)/I_t \cong C$, hence isomorphic to $M_n(C)$ by Lemma 8, and $\pi$ is finite-dimensional.

We now prove that $A$ satisfies Fell’s condition. Let $\pi \in \hat{A}$ and choose $a \in A$ such that $\pi(a)$ is a rank one projection. The map $\rho \to \|\rho(a)\|$ is continuous on $\hat{A}$, so $\|\rho(a)^2 - \rho(a)\|$ is small for $\rho$ near $\pi$, and if $f \equiv 1$ near 1, $f \equiv 0$ near 0 then $p = f(a)$ will satisfy $\rho(p)^2 = \rho(p) = \rho(p)^*$ for $\rho$ near $\pi$. If $N$ is a compact neighbourhood of $\pi$ then $pAp|_N = A/I_N\ A$ is central separable over $C(N)$ and has an identity; therefore by shrinking $N$ we may suppose $pAp|_N \cong C(N, M_n)$ for some $n$. As $\pi(p)$ is rank one, $n = 1$ and $\rho(p)$ is rank one throughout $N$. Thus $A$ has continuous trace.

Suppose now that $\pi_n \in \hat{A}$ and $\dim \pi_n \geq n$ for all $n$. As $\hat{A}$ is compact, we may assume $\pi_n \to \pi$ (technically, we might have to pass to a subnet, but the idea’s the same). Pick $a_n \in A$ with $a_n \geq 0$ and rank $\pi_n(a_n) \geq n$, and let $a = \sum_{k=1}^\infty 2^{-k}a_k$. 

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Then \( a \geq 0 \) and we have
\[
\text{rank } \pi_n(a) = \text{rank} \left( \sum_{k=1}^{\infty} \pi_n(2^{-k}a_k) \right) \geq \text{rank } \pi_n(2^{-n}a_n) \geq n.
\]
By restricting to a compact neighbourhood \( N \) of \( \pi \) we may suppose that \( A \) has a rank one idempotent \( p \), and then by Lemma 6 multiplication gives an isomorphism \( \mu : Ap \otimes_{C(N)} PA \to A \). In particular, we can write
\[
a = \mu \left( \sum_{i=1}^{m} a_i p \otimes pb_i \right) = \sum_{i=1}^{m} a_i pb_i;
\]
but this is impossible since the rank of \( \pi_n(\sum a_ipb_i) \) is at most \( m \) for each \( n \).

The proof of the second part of Proposition 5 consists of building an example. We are grateful to Shaun Disney for providing the following topological lemma.

**Lemma 9.** Let \( L_n \) be the canonical complex line bundle over complex projective space \( \mathbb{C}P^n \). Then any \( n \) sections of \( L_n \) have a common zero.

**Proof.** Let \( \xi_1, \ldots, \xi_n \) be \( n \) sections of \( L = L_n \), and suppose they do not simultaneously vanish. Then the direct sum \( nL \) of \( n \) copies of \( L \) has a non-vanishing section, and so can be decomposed as \( nL = 1 \oplus F \), where \( 1 \) denotes the trivial line bundle. The first Chern class of \( 1 \) is 0, so
\[
c_1(L)^n = c_n(nL) = c_1(1)c_{n-1}(F) = 0.
\]
But the cohomology ring \( H^*(\mathbb{C}P^n, \mathbb{Z}) \) is a truncated polynomial ring generated by \( c_1(L) \), and in particular \( H^{2n}((\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z} \) is generated by \( c_1(L)^n \). We therefore have a contradiction, and the \( \xi_i \) must vanish simultaneously.

**Corollary 10.** Let \( L_n \) be the field of Hilbert spaces over \( \mathbb{C}P^n \) obtained by putting a Hermitian structure on the canonical line bundle, and let \( A_n = \Gamma(\mathbb{H}(L_n)) \) be the C*-algebra defined by \( L_n \). Then the identity in \( A_n \) cannot be written in the form \( \sum_{i=1}^{m} \xi_i \otimes \eta_i \) for \( \xi_i, \eta_i \in \Gamma(L_n) \) unless \( m > n \). (In fact \( A_n \equiv C(\mathbb{C}P^n) \), but this is not important here.)

**Proof of Proposition 5(2).** Let \( L_n \) be as in Corollary 10, and define a field of Hilbert spaces \( K \) over the disjoint union \( X = \bigcup_{n=1}^{\infty} \mathbb{C}P^n \) by taking \( K = L_n \) on \( \mathbb{C}P^n \). We now define a field \( H \) over the compactification \( T = X \cup \{ \infty \} \) by
\[
H(x) = C \oplus K(x) \cong C^2, \quad H(\infty) = C, \quad \Gamma(H) = C(T) \oplus \Gamma_0(K).
\]
Let \( A \) be the C*-algebra \( \Gamma(\mathbb{H}(H)) \) defined by \( H \); we claim that \( A \) is not just \( \Gamma(H) \otimes_{C(T)} \Gamma(H) \). For \( A \) contains the closure of \( \Gamma_0(K) \otimes_{C(T)} \Gamma_0(K) \), which is
the $c_0$-direct sum of the $C^*$-algebras $A_n$. If we define $f \in A_n$ by

$$f(x) = \frac{1}{n} 1_{K(x)} \text{ if } x \in CP^n,$$

then $f$ cannot be written in the form $\sum_{i=1}^{m} \xi_i \otimes \eta_i$ for any finite $m$, and so does not belong to the algebraic tensor product $T_0(K) \otimes T_0(\overline{K})$; this justifies the claim. The algebra $A$ contains the idempotent $p = 1_{C(T)} \otimes 1_{C(T)}$, and it is easy to see that $pAp = C(T)$, so $p$ is rank one, and if $A$ were central separable we would have $A \cong Ap \otimes C(T) pA$ by Lemma 6. However, $A' = \Gamma(H)$ so we have just shown this is not the case.

Finally, we observe that, although continuous trace $C^*$-algebras with compact spectrum are not in general central separable, they do always have a dense ideal which is. For any continuous trace $C^*$-algebra with spectrum $T$ can be constructed from a cover $\{N_i\}$ of $T$, fields of Hilbert spaces $H_i$ over $N_i$, and isomorphisms $h_{ij}$: $H_j|_{N_{ij}} \to H_i|_{N_{ij}}$ which satisfy

$$\text{Ad } h_{ij}(t) \circ \text{Ad } h_{jk}(t) = \text{Ad } h_{jk}(t) \text{ for } t \in N_{ijk}.$$ 

[3, 10.7.11]. For convenience we suppose $N_i$ is compact. Then the algebraic tensor product $A_i = \Gamma(H_i) \otimes_{C(N_i)} \Gamma(\overline{H}_i)$ is a central separable $C(N_i)$-algebra, and the isomorphisms $\text{Ad } h_{ij}$ map $A_j|_{N_{ij}}$ onto $A_i|_{N_{ij}}$. We can therefore use them to piece together a central separable algebra (cf. the proof of Proposition 4) which is clearly dense in $A$. Conversely, if $A$ is a $C^*$-algebra with Hausdorff spectrum $T$ and $A$ contains a dense central separable $C(T)$-subalgebra then as in the proof of Proposition 5 it is not hard to see that $A$ satisfies Fell's condition. Thus this property characterises continuous trace $C^*$-algebras.

References

Continuous trace C*-algebras


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