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On Analytic Functions of Bergman BMO in the Ball

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Abstract. Let $B = B_n$ be the open unit ball of C^n with volume measure ν, $U = B_1$ and $\mathcal{B}$ be the Bloch space on U. $\mathcal{A}^{2,\alpha}(B)$, $1 \le \alpha < \infty$, is defined as the set of holomorphic $f : B \to \mathrm{C}$ for which

$$\int_B |f(z)|^2 \left(\frac{1}{|z|} \log \frac{1}{1-|z|}\right)^{-\alpha} \frac{d\nu(z)}{1-|z|} < \infty$$

if $0 < \alpha < \infty$ and $\mathcal{A}^{2,1}(B) = H^2(B)$, the Hardy space. Our objective of this note is to characterize, in terms of the Bergman distance, those holomorphic $f : B \to U$ for which the composition operator $C_f : \mathcal{B} \to \mathcal{A}^{2,\alpha}(B)$ defined by $C_f(g) = g \circ f$, $g \in \mathcal{B}$, is bounded. Our result has a corollary that characterize the set of analytic functions of bounded mean oscillation with respect to the Bergman metric.

1 Introduction

Let B be the open unit ball of C^n and S be the boundary of B. Let ν and σ denote the normalized (Euclidean) volume and surface measures on B and S respectively. Let U and T stand for B and S respectively when $n = 1$. The Hermitian inner product and the associated Euclidean norm in C^n are denoted by $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$ and $|z| = \sqrt{\langle z, z\rangle}$. The Möbius group, *i.e.*, the group of biholomorphic self maps of B, will be denoted by $\mathcal{M}$. Let φ_z, $z \in B$, denote the self map of B defined by

$$\varphi_z(w) = \frac{z - \frac{\langle w,z\rangle}{\langle z,z\rangle} z - \sqrt{1-|z|^2}\left(w - \frac{\langle w,z\rangle}{\langle z,z\rangle} z\right)}{1 - \langle w, z\rangle}, \quad w \in B,$$

if $z \neq 0$ and $\varphi_0(w) = -w$, $w \in B$. It is known [R] that $\mathcal{M}$ consists of functions of the form $\mathcal{U}\varphi_z$, where $\mathcal{U}$ denotes a unitary operator on B.

The space $\mathrm{BMOA}(B)$, analytic functions of bounded mean oscillation, consists of holomorphic f defined on B for which

$$\sup_{z\in B} \lim_{r\to 1} \int_S |f \circ \varphi_z(r\zeta) - f \circ \varphi_z(0)|^2 \, d\sigma(\zeta) < \infty$$

(see [G] and [CRW]).

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The Bloch space is denoted by $\mathcal{B} = \mathcal{B}(U)$. It consists of holomorphic f defined on U for which

$$\|f\|_{\mathcal{B}} = \sup_{z\in U}(1-|z|^2)|f'(z)| < \infty.$$

We define $\mathcal{A}^{2,\alpha}(B)$, $1 \le \alpha < \infty$, as the set of holomorphic $f\colon B \to \mathrm{C}$ for which

$$\int_B |f(z)|^2 \left(\frac{1}{|z|}\log\frac{1}{1-|z|}\right)^{-\alpha} \frac{d\nu(z)}{1-|z|} < \infty$$

if $0 < \alpha < \infty$ and $\mathcal{A}^{2,1}(B) = H^2(B)$, the Hardy space.

Our objective of this note is to characterize, in terms of the Bergman distance, those holomorphic $f\colon B \to U$ for which the composition operator $C_f\colon \mathcal{B} \to \mathcal{A}^{2,\alpha}(B)$ defined by $C_f(g) = g \circ f$, $g \in \mathcal{B}$, is bounded.

We denote by ϱ the Bergman distance in U:

$$\varrho(z,w) = \frac{1}{2}\log\frac{|1-\bar{z}w| + |w-z|}{|1-\bar{z}w| - |w-z|}, \quad z,w \in U,$$

and we make use of the convention $\varrho(z) = \varrho(z,0)$, $z \in U$.

Using the Bergman distance, we define some new function classes.

The Bergman Hardy class on B, $\varrho H^p(B)$, $0 < p < \infty$, is defined to consist of holomorphic $f\colon B \to U$ for which

$$\lim_{r\to 1}\int_S \{\varrho\big(f(r\zeta)\big)\}^p \, d\sigma(\zeta) < \infty.$$

The class $\varrho\,\mathrm{BMOA}(B)$, analytic functions of bounded mean oscillation under the Bergman metric, consists of uniformly (with respect $\mathcal{M}$) $\varrho H^1(B)$ functions, that is, it consists of holomorphic $f\colon B \to U$ for which

$$\sup_{z\in B}\lim_{r\to 1}\int_S \varrho\big(f\circ\varphi_z(r\zeta), f(z)\big)\, d\sigma(\zeta) < \infty.$$

The class $\varrho\mathcal{A}^{1,\alpha}(B)$, $1 \le \alpha < \infty$, consists of holomorphic $f\colon B \to U$ for which

$$\int_B \varrho\big(f(z)\big)\left(\frac{1}{|z|}\log\frac{1}{1-|z|}\right)^{-\alpha}\frac{d\nu(z)}{1-|z|} < \infty$$

if $0 < \alpha < \infty$ and $\varrho\mathcal{A}^{1,1}(B) = \varrho H^1(B)$.

The classes $\varrho H^p(U)$ and $\varrho\,\mathrm{BMOA}(U)$, when $n = 1$, were defined and studied by S. Yamashita (see [Y1] and [Y2]). The class $\varrho\mathcal{A}^{1,\alpha}(U)$ was studied in connection with $\varrho H^p(U)$ (see [K2]).

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Our results in this note are as follows.

Theorem *Let $f\colon B \to U$ be holomorphic and $1 \le \alpha < \infty$. Then the following are equivalent.*

(1) $g \circ f \in \mathcal{A}^{2,\alpha}(B)$ *for all* $g \in \mathcal{B}$.
(2) $f \in \varrho\mathcal{A}^{1,\alpha}(B)$.

Corollary *Let $f\colon B \to U$ be holomorphic. Then*

$$g \circ f \in \mathrm{BMOA}(B) \quad \text{for all } g \in \mathcal{B} \iff f \in \varrho\,\mathrm{BMOA}(B).$$

Without proof and without using the terminology of $\varrho\,\mathrm{BMOA}(B)$, the Corollary actually appeared in [RU, p. 22]. See [K1] for the same vein.

2 Lemmas

∇, Δ and $\tilde{\Delta}$ denote respectively the complex gradient, Laplacian and $\mathcal{M}$-invariant Laplacian of B [R].

For notational convenience, we let $d\tau(z) = d\nu(z)/(1-|z|^2)^{n+1}$ and

$$d\mu_\alpha(z) = c(\alpha)(1-|z|)^{-1}\left(\frac{1}{|z|}\log\frac{1}{1-|z|}\right)^{-\alpha} d\nu(z),$$

where $c(\alpha)$ is a constant determined to satisfy $\int_B d\mu_\alpha(z) = 1$.

***Lemma 1* ([P, Lemma 2.5])** *If $f \in C^2(B)$, $0 < r < 1$ and $0 < \varepsilon < 1$, then*

$$\frac{d}{dr}\int_S f(r\zeta)\,d\sigma(\zeta) = \frac{1}{2n}\frac{(1-r^2)^{n-1}}{r^{2n-1}}\int_{rB}(\tilde{\Delta} f)\,d\tau$$

and

$$f(0) = \int_S f(\varepsilon\zeta)\,d\sigma(\zeta) - \int_{\varepsilon B}\tilde{\Delta} f(z) G(|z|,\varepsilon)\,d\tau(z), \tag{2.1}$$

where

$$G(t,\varepsilon) = \frac{1}{2n}\int_t^\varepsilon \frac{(1-r^2)^{n-1}}{r^{2n-1}}\,dr.$$

Lemma 2 *Let f be a positive function of $C^2(B)$ with $\Delta f \ge 0$ and $\tilde{\Delta} f \ge 0$. If the radial limit function of f, which is also denoted by f, is in $L^1(S, d\sigma)$, then*

$$\int_B G(z)\tilde{\Delta} f(z)\,d\tau(z) = \int_S f(\zeta)\,d\sigma(\zeta) - f(0), \tag{2.2}$$

where

$$G(z) = \frac{1}{2n}\int_{|z|}^1 \frac{(1-r^2)^{n-1}}{r^{2n-1}}\,dr.$$

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If $1 < \alpha < \infty$ *and if* $f \in L^1(B, d\mu_\alpha)$*, then*

$$\int_B \mathcal{G}(z)\tilde{\Delta} f(z)\, d\tau(z) = \int_B f(z)\, d\mu_\alpha(z) - f(0), \tag{2.3}$$

where

$$\mathcal{G}(z) = c(\alpha)\int_{|z|}^1 \int_r^1 (1-\varepsilon)^{-1}\Big(\frac{1}{\varepsilon}\log\frac{1}{1-\varepsilon}\Big)^{-\alpha}(1-r)^{n-1}\Big(\frac{\varepsilon}{r}\Big)^{2n-1} d\varepsilon\, dr.$$

Proof Apply (2.1) and let $\varepsilon \to 1$. Then the positivity of f and $\tilde{\Delta} f$ with the increasing property of $\int_S f(\varepsilon\zeta)\, d\sigma(\zeta)$ (because $\Delta f \geq 0$) and that of $G(|z|, \varepsilon)$ makes the monotone convergence theorem legitimate, whence follows (2.2).

To see (2.3), integrate both sides of (2.1) with respect to

$$2nc(\alpha)\varepsilon^{2n-1}(1-\varepsilon)^{-1}\Big(\frac{1}{r}\log\frac{1}{1-r}\Big)^{-\alpha} d\varepsilon$$

over $(0, 1)$ then the result becomes

$$f(0) = \int_B f(z)\, d\mu_\alpha(z) - \int_B \mathcal{G}(z)\tilde{\Delta} f(z)\, d\tau(z).$$

■

Lemma 3 *Let* $f : B \to U$ *be holomorphic and let* $\lambda(z) = -\log(1 - |z|^2)$*,* $z \in B$*. Let* $g \in \mathcal{B}$ *and let* $F = g \circ f$*. Then*

$$\tilde{\Delta}|F|^2(z) \leq \|g\|_{\mathcal{B}}^2 \quad \tilde{\Delta}(\lambda \circ f)(z), \quad z \in B. \tag{2.4}$$

Proof Since

$$\nabla(F \circ \varphi_z)(0) = g' \circ f(z)\nabla(f \circ \varphi_z)(0),$$

we have

$$|\nabla(F \circ \varphi_z)(0)| \leq \|g\|_{\mathcal{B}} \frac{|\nabla(f \circ \varphi_z)(0)|}{1 - |f(z)|^2}. \tag{2.5}$$

It is easy to see that

$$\Delta(\lambda \circ f) = 4\frac{|\nabla f|^2}{(1-|f|^2)^2},$$

so that

$$\tilde{\Delta}(\lambda \circ f)(z) = \Delta(\lambda \circ f \circ \varphi_z)(0) = 4\frac{|\nabla(f \circ \varphi_z)(0)|^2}{\big(1 - |f(z)|^2\big)^2}. \tag{2.6}$$

Noting that

$$\tilde{\Delta}|F|^2(z) = 4|\nabla(F \circ \varphi_z)(0)|^2,$$

(2.4) now follows from (2.5) and (2.6). ■

***Lemma 4* ([RU, Proposition 5.4])** *There exist Bloch functions* g_j*,* $j = 1, 2$*, such that*

$$\sum_{j=1}^2 |g_j'(z)| \geq \frac{1}{1 - |z|^2}$$

for all $z \in U$.

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3 Proof of the results

Proof of Theorem Note first that

$$\lambda \circ f \le 2\varrho(f) \le 2\log 2 + \lambda \circ f \tag{3.1}$$

for $f : B \to U$ holomorphic, so that

$$\int_S (\lambda \circ f)(\zeta)\, d\sigma(\zeta) < \infty \iff \int_S \varrho\big(f(\zeta)\big)\, d\sigma(\zeta) < \infty.$$

Suppose $f \in \varrho\mathcal{A}^{1,\alpha}(B)$. Let $g \in \mathcal{B}$ and $F = g \circ f$. Then $|F|^2$ and $\lambda \circ f$ are plurisubharmonic (see [R, 7.2]), so that $\Delta|F|^2 \ge 0$, $\Delta(\lambda \circ f) \ge 0$, $\tilde{\Delta}|F|^2 \ge$ and $\tilde{\Delta}(\lambda \circ f) \ge 0$. Hence we have

$$\int_B |F(\zeta)|^2\, d\mu_\alpha(z) - |F(0)|^2 \le \|g\|_{\mathcal{B}}^2 \left\{ \int_B (\lambda \circ f)(z)\, d\mu_\alpha(z) - (\lambda \circ f)(0) \right\} \tag{3.2}$$

if $1 < \alpha < \infty$. In fact,

$$\int_B \mathcal{G}(z) \tilde{\Delta}|F|^2(z)\, d\tau(z) \tag{3.3}$$

is bounded, by (2.4), by

$$\|g\|_{\mathcal{B}}^2 \int_B \mathcal{G}(z) \tilde{\Delta}(\lambda \circ f)(z)\, d\tau(z),$$

which is, by (2.3), equal to the right side of (3.2), whence (3.3) is finite and, by (2.3) again, equal to the left side of (3.2).

When $\alpha = 1$, the above process, using (2.2) instead of (2.3) and $G(z)$ in place of $\mathcal{G}(z)$, gives

$$\int_S |g \circ f(\zeta)|^2\, d\sigma(\zeta) - |g \circ f(0)|^2 \le \|g\|_{\mathcal{B}}^2 \left\{ \int_S (\lambda \circ f)(\zeta)\, d\sigma(\zeta) - (\lambda \circ f)(0) \right\}. \tag{3.4}$$

By (3.2), (3.4) with (3.1), $g \circ f \in \mathcal{A}^{2,\alpha}(B)$.

Conversely, suppose that $g \circ f \in \mathcal{A}^{2,\alpha}(B)$ for all $g \in \mathcal{B}$. Then by Lemma 4, there are $g_j \in \mathcal{B}$, $j = 1, 2$, such that

$$\sum_{j=1}^{2} |\nabla(g_j \circ f \circ \varphi_z)(0)| \ge \frac{|\nabla(f \circ \varphi_z)(0)|}{1 - |f(z)|^2},$$

so that

$$\sum_{j=1}^{2} \tilde{\Delta}|g \circ f|^2(z) \ge \tilde{\Delta}(\lambda \circ f)(z), \quad z \in B,$$

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whence we have, using Lemma 2 with the same argument that we just applied in proving (3.2) and (3.4),

$$\sum_{j=1}^{2}\Big(\int_B |g_j\circ f(z)|^2\,d\mu_\alpha(z) - |g_j\circ f(0)|^2\Big) \geq \int_B (\lambda\circ f)(z)\,d\mu_\alpha(z) - (\lambda\circ f)(0) \tag{3.5}$$

if $1<\alpha<\infty$ or

$$\sum_{j=1}^{2}\Big(\int_S |g_j\circ f(\zeta)|^2\,d\sigma(\zeta) - |g_j\circ f(0)|^2\Big) \geq \int_S (\lambda\circ f)(\zeta)\,d\sigma(\zeta) - (\lambda\circ f)(0) \tag{3.6}$$

if $\alpha = 1$. By (3.5), (3.6) with (3.1), $f \in \varrho\mathcal{A}^{1,\alpha}(B)$ in any case. ■

Proof of the Corollary Let $f\colon B \to U$ be holomorphic. Since the function $\log|1-\bar{f}(z)f\circ\varphi_z(w)|^2$, $w\in B$, is harmonic,

$$\int_S \log|1-\bar{f}(z)f\circ\varphi_z(r\zeta)|^2\,d\sigma(\zeta) = \log\big(1-|f(z)|^2\big)^2,$$

whence

$$\begin{aligned}
\int_S \lambda\Big(\frac{f(z)-f\circ\varphi_z(r\zeta)}{1-f(z)f\circ\varphi_z(r\zeta)}\Big)\,d\sigma(\zeta) &= \int_S \log\frac{1}{1-|\frac{f(z)-f\circ\varphi_z(r\zeta)}{1-f(z)f\circ\varphi_z(r\zeta)}|^2}\,d\sigma(\zeta)\\
&= \int_S \log\frac{|1-\bar{f}(z)f\circ\varphi_z(r\zeta)|^2}{\big(1-|f(z)|^2\big)\big(1-|f\circ\varphi_z(r\zeta)|^2\big)}\,d\sigma(\zeta)\\
&= \int_S \log\frac{1-|f(z)|^2}{1-|f\circ\varphi_z(r\zeta)|^2}\,d\sigma(\zeta)\\
&= \int_S (\lambda\circ f\circ\varphi_z)(r\zeta)\,d\sigma(\zeta) - (\lambda\circ f)(z), \quad 0<r<1.
\end{aligned} \tag{3.7}$$

By the monotone convergence theorem, the last quantity of (3.7) tends to the right side quantity of (3.6) with $f\circ\varphi_z$ in place of f as $r\to 1$.

On the other hand, the Möbius invariance of ϱ (see [G]) says

$$\varrho\big(f\circ\varphi_z(r\zeta), f(z)\big) = \varrho\Big(\frac{f(z)-f\circ\varphi_z(r\zeta)}{1-f(z)f\circ\varphi_z(r\zeta)}\Big). \tag{3.8}$$

Now, by taking the supremum for all $z\in B$ on both sides of (3.4) and (3.6) after replacing f there by $f\circ\varphi_z$, the result follows by use of (3.1), (3.7) and (3.8). ■

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