The approximation problem for compact operators

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The following sufficient condition is obtained for the uniform approximability of compact operators on a reflexive Banach space by operators of finite rank: if \( S \) is the unit ball of \( X \) and \( \phi : X^* \to C(S) \) is the imbedding \( \phi(x^*)x = x^*(x) \) then we require \( \phi(x^*) \) to be complemented in \( C(S) \).

Let \( X \) denote a Banach space, \( B(X) \) the space of continuous endomorphisms of \( X \), \( C \) the set of compact operators in \( B(X) \) and \( F \) the set of finite dimensional operators in \( B(X) \). By the approximation problem, we shall mean: is \( C \) equal to the uniform closure of \( F \)? Only sufficient conditions on \( X \) are known for an affirmative solution to this problem. Most important among these is the condition d'approximation due to Grothendieck ([2] p. 165): every operator in \( B(X) \) can be uniformly approximated on compact subsets by operators in \( F \). It is, of course, not known whether this condition holds for all \( X \). However the conjecture that it is true for all \( X \) is known ([2] pp. 170-175) to be equivalent to a large number of other conjectures, some apparently more tractable.

At least we know that the approximation problem has an affirmative solution for a large number of the common Banach spaces. This can be deduced from the following ideas.

(1) A Banach space \( Y \) is called a \( P_\lambda \) space, if, given any Banach space \( \tilde{Y} \supset Y \), there exists a projection of norm not exceeding \( \lambda \) from \( \tilde{Y} \) to \( Y \).

(2) In [3], Lindenstrauss introduced the concept of an \( N_\lambda \) space: a Banach space \( Y \) is an \( N_\lambda \) space if there exists a set \( \{Y_1\} \) of finite
dimensional subspaces of $Y$, directed by inclusion, such that $UY_\tau$ is dense in $Y$ and such that each $Y_\tau$ is a $P_\lambda$ space.

(3) The concept of the $\lambda$ projection approximation property ($\lambda$-P.A.P.) was also introduced in [3]: a Banach space $Y$ has the $\lambda$-P.A.P. if there exists a set $\{Y_\tau\}$ of finite dimensional subspaces of $Y$, directed by inclusion, such that $UY_\tau$ is dense in $Y$ and such that, for each $\tau$, there exists a projection of $Y$ onto $Y_\tau$ with norm not exceeding $\lambda$.

It is evident that every $N_\lambda$ space has the $\lambda$-P.A.P. Moreover, it is not difficult to deduce that the $\lambda$-P.A.P. implies the condition $d'$approximation. For suppose $K$ is compact and $\varepsilon > 0$. Then we can find $x_1, x_2, \ldots, x_n \in K$ such that for every $x \in K$ there exists $x_\tau$ with $\|x-x_\tau\| < \varepsilon$. Then we can choose $y_1, y_2, \ldots, y_n \in UY_\tau$ such that $\|y_\tau-x_\tau\| < \varepsilon$. Thus there exists $\tau_0$ such that $y_1, y_2, \ldots, y_n \in Y_{\tau_0}$ and a projection $P_0$ of $Y$ onto $Y_{\tau_0}$ such that $\|P_0\| \leq \lambda$. If, given $x \in K$, $\|x-x_\tau\| < \varepsilon$ then we can write

$$\|Tx-TP_0 x\| \leq \|Tx-Tx_\tau\| + \|Tx_\tau-Ty_\tau\| + \|TP_0 y_\tau-TP_0 x_\tau\| + \|TP_0 x_\tau-TP_0 x\|$$

(i)

$$\leq 2\|T\|(\lambda+\lambda)\varepsilon.$$

Hence the result follows.

Now the separable Banach spaces with a basis, the $L_p(\mu)$ spaces ($1 \leq p \leq \infty$, $\mu$ arbitrary) and $C(\Omega)$ ($\Omega$ any topological space) are known to be spaces with the $\lambda$-P.A.P. for some $\lambda$ (see [3] pp. 25, 29). Hence for these spaces the approximation problem has an affirmative solution.

In that which follows, another sufficient condition is obtained for the case where $X$ is reflexive. For any reflexive Banach space $X$, let $S$ denote the unit ball with the weak topology and let $\phi : X^* \to C(S)$ denote the isometric imbedding $\phi(x^*)x = x^*(x)$.

**THEOREM.** If $\phi(X^*)$ has a closed complement in $C(S)$, then the approximation problem in $X$ has an affirmative solution.
Proof. Let $P_0$ denote a projection $C(S) + \phi(X^*)$. By a result of Lindenstrauss ([3] p. 29), we know that $C(S)$ is a $N_1$ space for each $\lambda > 1$. Moreover ([3] p. 22), every finite dimensional subspace of a $N_1$ space is contained in a finite dimensional $P_1$ space for any $\lambda' > \lambda$. We shall fix $\lambda$ and $\lambda'$, $\lambda' > \lambda > 1$.

Now suppose $T$ is a compact operator in $B(X)$. Then by a theorem of Lacey ([1] p. 85), given $\varepsilon > 0$, there exists a closed subspace $N_\varepsilon$ of $X$ with finite codimension such that \[ \|T|N_\varepsilon\| < \varepsilon. \] Let $N_\varepsilon^\perp$ denote the functionals in $X^*$ which vanish on $N_\varepsilon$. Then $N_\varepsilon^\perp$ is a finite dimensional subspace of $X^*$ and $\phi(N_\varepsilon^\perp)$ is finite dimensional in $C(S)$. By the remark about $N_1$ spaces, there exists a finite dimensional subspace $X_\varepsilon$ in $C(S)$ with $X_\varepsilon \supset \phi(N_\varepsilon^\perp)$ and a projection $P_\varepsilon : C(S) + X_\varepsilon$ with $\|P_\varepsilon\| < \lambda'$. Consider the product $P_0 P_\varepsilon$. Evidently this is a finite dimensional operator whose range $R$ satisfies $\phi(N_\varepsilon^\perp) \subseteq R \subseteq \phi(X^*)$. Hence we can define $\tilde{P}_\varepsilon = \phi^{-1} P_0 P_\varepsilon \phi$ which will be a finite dimensional operator in $X^*$.

Now $\phi(N_\varepsilon^\perp) \subseteq X_\varepsilon$ so that

\[ N_\varepsilon^\perp \subseteq \phi^{-1}(X_\varepsilon) = \phi^{-1} R(P_\varepsilon) = \left\{ x^* \in X^* : \phi x^* \in R(P_\varepsilon) \right\} \]
\[ = \left\{ x^* \in X^* : P_\varepsilon \phi x^* = \phi x^* \right\} \]
\[ \subseteq \left\{ x^* \in X^* : P_0 P_\varepsilon \phi x^* = \phi x^* \right\} \]
\[ = \left\{ x^* \in X^* : \tilde{P}_\varepsilon x^* = x^* \right\} \]
\[ = N(I-\tilde{P}_\varepsilon) \]
\[ = R(I-\tilde{P}_\varepsilon) \]

where we are identifying $\tilde{P}_\varepsilon$ with the corresponding operator in $B(X)$.

Hence $N_\varepsilon \supseteq R(I-\tilde{P}_\varepsilon)$.

We can now write
Since $\lambda'$ and $\|P_0\|$ are fixed and $\varepsilon$ arbitrary, and since $\tilde{P}_\varepsilon^*$ has finite dimensional range, the proof is complete.

**REMARK.** Examination of the above proof reveals that the hypotheses can be weakened considerably. Suppose we define the following property $(P)$: a Banach space $X$ will be said to have property $(P)$ if there exists $M > 0$ such that, given any finite dimensional subspace $F$ of $X$, there exists a finite dimensional projection $P$ in $\mathcal{B}(X)$ such that $\|P\| \leq M$ and $R(P) \supseteq F$. An examination of the proof of Lemma 3.1 of [3] shows that any space with the $\lambda$-P.A.P. also has property $(P)$; moreover, using simple calculations similar to (i), we can deduce that property $(P)$ implies the condition d'approximation of Grothendieck. Hence we can state

**COROLLARY.** Let $X$ be a reflexive Banach space such that there exists a Banach space $Y$ with property $(P)$ and a linear homeomorphism $\phi : X^* \rightarrow Y$ with $\phi(X^*)$ complemented. Then the approximation problem for $X$ has an affirmative solution.

**References**


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