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# On Some Topological Properties of Fourier Transforms of Regular Holonomic $\mathcal{D}$-Modules

Yohei Ito and Kiyoshi Takeuchi

*Abstract.* We study Fourier transforms of regular holonomic $\mathcal{D}$-modules. In particular, we show that their solution complexes are monodromic. An application to direct images of some irregular holonomic $\mathcal{D}$-modules will be given. Moreover, we give a new proof of the classical theorem of Brylinski and improve it by showing its converse.

## 1 Introduction

First, we recall Fourier transforms of algebraic $\mathcal{D}$-modules. Let $X = \mathbb{C}_z^N$ be a complex vector space and let $Y = \mathbb{C}_w^N$ be its dual. We regard them as algebraic varieties and use the notations $\mathcal{D}_X$ and $\mathcal{D}_Y$ for the rings of "algebraic" differential operators on them. Denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$) the category of coherent (resp. holonomic) $\mathcal{D}_X$-modules. Let $W_N := \mathbb{C}[z, \partial_z] \simeq \Gamma(X; \mathcal{D}_X)$ and $W_N^* := \mathbb{C}[w, \partial_w] \simeq \Gamma(Y; \mathcal{D}_Y)$ be the Weyl algebras over $X$ and $Y$, respectively. Then by the ring isomorphism

$$W_N \xrightarrow{\sim} W_N^* \qquad (z_i \longmapsto -\partial_{w_i},\ \partial_{z_i} \longmapsto w_i)$$

we can endow a left $W_N$-module $M$ with a structure of a left $W_N^*$-module. We call it the Fourier transform of $M$ and denote it by $M^\wedge$. For a ring $R$ we denote by $\mathrm{Mod}_f(R)$ the category of finitely generated $R$-modules. Recall that for the affine algebraic varieties $X$ and $Y$, we have the equivalences of categories

$$\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X) \simeq \mathrm{Mod}_f(\Gamma(X; \mathcal{D}_X)) = \mathrm{Mod}_f(W_N),$$
$$\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Y) \simeq \mathrm{Mod}_f(\Gamma(Y; \mathcal{D}_Y)) = \mathrm{Mod}_f(W_N^*)$$

(see *e.g.,* [HTT08, Propositions 1.4.4 and 1.4.13]). For a coherent $\mathcal{D}_X$-module $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$, we can thus define its Fourier transform $\mathcal{M}^\wedge \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Y)$. It follows that we obtain an equivalence of categories

$$(\cdot)^\wedge \colon \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Y)$$

between the categories of holonomic $\mathcal{D}$-modules. However, the Fourier transform $\mathcal{M}^\wedge$ of a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is not necessarily regular. For the regularity of $\mathcal{M}^\wedge$, we need some strong condition on $\mathcal{M}$. Recall that a constructible sheaf $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)$ on $X = \mathbb{C}^N$ is called *monodromic* if its cohomology sheaves are locally

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constant on each $\mathbb{C}^*$-orbit in $X = \mathbb{C}^N$. Note that this condition was introduced by Verdier [Ver83]. Then the following beautiful theorem is due to Brylinski [Bry86].

***Theorem 1.1*** (Brylinski [Bry86]) *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Assume that its solution complex $\mathrm{Sol}_X(\mathcal{M})$ is monodromic. Then its Fourier transform $\mathcal{M}^\wedge$ is regular, and $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.*

Recently in [IT18], the authors studied the Fourier transforms of general regular holonomic $\mathcal{D}$-modules very precisely by using the Riemann–Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules established by D'Agnolo and Kashiwara [DK16] and the Fourier–Sato transforms for enhanced ind-sheaves developed by Kashiwara and Schapira [KS16a]. In this process we found a new proof of Theorem 1.1 (see the proof of Theorem 3.2). Recall that Brylinski proved it by reducing the problem to the case $N = 1$ and using some deep results on nearby cycle $\mathcal{D}$-modules. Our new proof is purely geometric and relies on the Riemann–Hilbert correspondence of D'Agnolo and Kashiwara [DK16]. See the proof of Theorem 3.2 for the details. In our study of Fourier transforms of regular holonomic $\mathcal{D}$-modules, we also found that for a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, the enhanced solution complex $\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)$ of its Fourier transform $\mathcal{M}^\wedge$ satisfies a special condition. More precisely, for a $\mathbb{R}_+$-conic sheaf $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times\mathbb{R}})$ on $Y \times \mathbb{R} \simeq \mathbb{R}^{2N+1}$, we found an isomorphism

$$\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge) \simeq \mathbb{C}_Y^{\mathrm{E}} \overset{+}{\otimes} \mathcal{G},$$

where we regard $\mathcal{G}$ as an ind-sheaf on the bordered space $Y \times \mathbb{R}_\infty$ by the natural embedding

$$\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times\mathbb{R}}) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{Y\times\mathbb{R}_\infty});$$

see Corollary 3.6. From this, we obtain the following result.

***Theorem 1.2*** *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.*

It seems that this result is already implicit in the main theorem of Daia [Dai00]. Indeed, for regular holonomic $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$, it implies that $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ is $\mathbb{R}_+$-conic. Note that the recent result in [DHMS17, Lemma 6.1.3] of D'Agnolo, Hien, Morando, and Sabbah also implies also the same property of $Sol_Y(\mathcal{M}^\wedge)$ (see also [DHMS17, Lemma 1.5.2]). For a general theory of conic ind-sheaves, see [Pre11]. The monodromicity of $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ in Theorem 1.2 follows from its $\mathbb{C}$-constructibility and the $\mathbb{R}_+$-conicness (see Lemma 2.1). In this paper, we prove Theorem 1.2 by using the theory of enhanced ind-sheaves and our results in [IT18]. In this way, we can also improve Brylinski's Theorem 1.1 as follows.

***Corollary 1.3*** *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\mathcal{M}^\wedge$ is regular if and only if $\mathcal{M}$ is monodromic.*

Namely, we prove the converse of Brylinski's theorem. Moreover, as a simple application of Theorem 1.2, we obtain the following result, which may be of independent interest.

***Theorem 1.4*** *Let $\rho\colon X = \mathbb{C}^N \twoheadrightarrow Z = \mathbb{C}^{N-1}$ be a surjective linear map of codimension one and let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. For the dual $L \simeq \mathbb{C}^{N-1}$ of $Z$ let $\iota\colon L \hookrightarrow Y = \mathbb{C}^N$ be the injective linear map induced by $\rho$. Then for any point $a \in Y \backslash \iota(L)$ the direct image*

$$\mathbf{D}\rho_*(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Z)$$

*is concentrated in degree* 0.

Recently, many mathematicians studied direct images of irregular holonomic $\mathcal{D}$-modules and obtained precise results. See, for example, Heizinger [Hei15], Hien–Roucairol [HR08], and Roucairol [Rou06, Rou07]. Note also that in the case $N = 1$, Fourier transforms of general holonomic $\mathcal{D}$-modules were precisely studied by many authors such as Bloch–Esnault [BE04], D'Agnolo–Kashiwara [DK17], Mochizuki [Moc10, Moc18], and Sabbah [Sab08] to name a few.

## 2 Preliminary Notions and Results

In this section, we briefly recall some basic notions and results that will be used in this paper. We assume here that the reader is familiar with the theory of sheaves and functors in the framework of derived categories, for which we follow the terminology in [KS90]. For a topological space $M$ denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ the derived category consisting of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on it. The following lemma will be used in the proofs of Theorems 3.2 and 3.9.

***Lemma 2.1*** *Assume that a $\mathbb{C}$-constructible sheaf $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{C}^N})$ on $\mathbb{C}^N$ is $\mathbb{R}_+$-conic. Then it is monodromic.*

**Proof** By restrictions, we can assume that $N = 1$. By the $\mathbb{C}$-constructibility of $\mathcal{G}$, there exists a finite subset $\{P_1, P_2, \dots, P_k\} \subset \mathbb{C}$ of $\mathbb{C} \simeq \mathbb{R}^2$ such that $(H^j\mathcal{G})|_{\mathbb{C}\backslash\{P_1,P_2,\dots,P_k\}}$ is a local system for any $j \in \mathbb{Z}$. For $1 \le i \le k$ such that $P_i \neq 0$ let $\ell_i = \mathbb{R}_+ P_i \simeq \mathbb{R}_+$ be the real half line in $\mathbb{C} \simeq \mathbb{R}^2$ passing through the point $P_i$. Then by our assumption, $(H^j\mathcal{G})|_{\ell_i}$ is a constant sheaf for any $j \in \mathbb{Z}$. This implies that for the function $h_i\colon \mathbb{C} \to \mathbb{C}$, $h_i(x) = x - P_i$ such that $h_i^{-1}(0) = \{P_i\} \subset \mathbb{C}$, we have $\phi_{h_i}(\mathcal{G}) \simeq 0$, where

$$\phi_{h_i}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{C}}) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{h_i^{-1}(0)})$$

is Deligne's vanishing cycle functor. From now on, we use an argument in Sabbah [Sab06, §8]. Let

$${}^p\phi_{h_i} = \phi_{h_i}[-1]\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{C}}) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{h_i^{-1}(0)})$$

be the perverse (or shifted) vanishing cycle functor. Recall that it preserves the perversity. For $j \in \mathbb{Z}$, let ${}^pH^j(\mathcal{G}) \in \mathrm{Perv}(\mathbb{C})$ be the $j$-th perverse cohomology sheaf of $\mathcal{G}$. Then ${}^p\phi_{h_i}({}^pH^j(\mathcal{G}))$ is concentrated in degree 0 for any $j \in \mathbb{Z}$. Hence, there exists an isomorphism $H^j({}^p\phi_{h_i}(\mathcal{G})) \simeq H^0({}^p\phi_{h_i}({}^pH^j(\mathcal{G})))$ for any $j \in \mathbb{Z}$. We thus obtain ${}^p\phi_{h_i}({}^pH^j(\mathcal{G})) \simeq 0$ for any $1 \le i \le k$ and $j \in \mathbb{Z}$. This shows that the perverse sheaves ${}^pH^j(\mathcal{G})$'s are smooth on $\mathbb{C}^*$; *i.e.*, $H^l({}^pH^j(\mathcal{G}))|_{\mathbb{C}^*}$ is a local system on $\mathbb{C}^*$ for any $j, l \in \mathbb{Z}$. Then the assertion immediately follows. ∎

## 2.1 Ind-sheaves

We recall some basic notions and results on ind-sheaves. References are made to Kashiwara–Schapira [KS01] and [KS06]. Let $M$ be a good topological space (which is locally compact, Hausdorff, countable at infinity, and has finite soft dimension). We denote by $\mathrm{Mod}(\mathbb{C}_M)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on it and by $\mathrm{I}\mathbb{C}_M$ that of ind-sheaves. Then there exists a natural exact embedding $\iota_M\colon \mathrm{Mod}(\mathbb{C}_M) \to \mathrm{I}\mathbb{C}_M$ of categories. We sometimes omit it. It has an exact left adjoint $\alpha_M$ that has in turn an exact fully faithful left adjoint functor $\beta_M$:

$$\mathrm{Mod}(\mathbb{C}_M) \underset{\beta_M}{\overset{\iota_M}{\underset{\longleftarrow\,\alpha_M\,}{\rightrightarrows}}} \mathrm{I}\mathbb{C}_M.$$

The category $\mathrm{I}\mathbb{C}_M$ does not have enough injectives. Nevertheless, we can construct the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$ for ind-sheaves and the six Grothendieck operations among them. We denote by $\otimes$ and $\mathrm{R}\mathcal{I}hom$ the operations of tensor products and internal homs, respectively. If $f\colon M \to N$ is a continuous map, we denote by $f^{-1}, \mathrm{R}f_*, f^!$, and $\mathrm{R}f_{!!}$ the operations of inverse images, direct images, proper inverse images, and proper direct images, respectively. We set also $\mathrm{R}\mathcal{H}om := \alpha_M \circ \mathrm{R}\mathcal{I}hom$. Note that $(f^{-1}, \mathrm{R}f_*)$ and $(\mathrm{R}f_{!!}, f^!)$ are pairs of adjoint functors.

## 2.2 Ind-sheaves on Bordered Spaces

For the results in this subsection, we refer the reader to D'Agnolo–Kashiwara [DK16]. A bordered space is a pair $M_\infty = (M, \check{M})$ of a good topological space $\check{M}$ and an open subset $M \subset \check{M}$. A morphism $f\colon (M, \check{M}) \to (N, \check{N})$ of bordered spaces is a continuous map $f\colon M \to N$ such that the first projection $\check{M} \times \check{N} \to \check{M}$ is proper on the closure $\overline{\Gamma}_f$ of the graph $\Gamma_f$ of $f$ in $\check{M} \times \check{N}$. The category of good topological spaces embeds into that of bordered spaces by the identification $M = (M, M)$. We define the triangulated category of ind-sheaves on $M_\infty = (M, \check{M})$ by

$$\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}})/\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}\setminus M}).$$

Let

$$\mathbf{q}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{M}}) \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$$

be the quotient functor. For a morphism $f\colon M_\infty \to N_\infty$ of bordered spaces, we have the Grothendieck operations,

$$\begin{aligned}
\otimes\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \times \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) &\longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}),\\
\mathrm{R}\mathcal{I}hom\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) &\longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}),\\
\mathrm{R}f_*\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) &\longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty}),\\
f^{-1}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty}) &\longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}),\\
\mathrm{R}f_{!!}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) &\longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty}),\\
f^!\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_\infty}) &\longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})
\end{aligned}$$

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(see [DK16, Definitions 3.3.1 and 3.3.4]). Moreover, there exists a natural embedding

$$\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}).$$

### 2.3 Enhanced Sheaves

For the results in this subsection, see Tamarkin [Tam08], Kashiwara–Schapira [KS16a], and D'Agnolo–Kashiwara [DK17]. Let $M$ be a good topological space. We consider the maps

$$M \times \mathbb{R}^2 \xrightarrow{p_1, p_2, \mu} M \times \mathbb{R} \xrightarrow{\pi} M,$$

where $p_1, p_2$ are the first and the second projections, and we set $\pi(x,t) := x$ and $\mu(x, t_1, t_2) := (x, t_1 + t_2)$. Then the convolution functors for sheaves on $M \times \mathbb{R}$ are defined by

$$F_1 \overset{+}{\otimes} F_2 := \mathrm{R}\mu_!(p_1^{-1}F_1 \otimes p_2^{-1}F_2),$$

$$\mathrm{R}\mathcal{H}om^+(F_1, F_2) := \mathrm{R}p_{1*}\mathrm{R}\mathcal{H}om(p_2^{-1}F_1, \mu^! F_2).$$

We define the triangulated category of enhanced sheaves on $M$ by

$$\mathbf{E}^{\mathrm{b}}(\mathbb{C}_M) := \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{M\times\mathbb{R}})/\pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M).$$

Let

$$\mathbf{Q}\colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{M\times\mathbb{R}}) \longrightarrow \mathbf{E}^{\mathrm{b}}(\mathbb{C}_M)$$

be the quotient functor. The convolution functors are also defined for enhanced sheaves. We denote them by the same symbols $\overset{+}{\otimes}$, $\mathrm{R}\mathcal{H}om^+$. For a continuous map $f\colon M \to N$, we can define naturally the operations $\mathbf{E}f^{-1}$, $\mathbf{E}f_*$, $\mathbf{E}f^!$, $\mathbf{E}f_!$ for enhanced sheaves. We also have a natural embedding $\varepsilon\colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M) \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}_M)$, defined by

$$\varepsilon(F) := \mathbf{Q}(\mathbb{C}_{\{t\geq 0\}} \otimes \pi^{-1}F).$$

For a continuous function $\varphi\colon U \to \mathbb{R}$ defined on an open subset $U \subset M$ of $M$, we define the exponential enhanced sheaf by

$$\mathrm{E}^{\varphi}_{U|M} := \mathbf{Q}(\mathbb{C}_{\{t+\varphi\geq 0\}}),$$

where $\{t + \varphi \geq 0\}$ stands for $\{(x,t) \in M \times \mathbb{R} \mid x \in U, t + \varphi(x) \geq 0\}$.

### 2.4 Enhanced Ind-sheaves

We recall some basic notions and results on enhanced ind-sheaves. References are made to D'Agnolo–Kashiwara [DK16] and Kashiwara–Schapira [KS16b]. Let $M$ be a good topological space. Set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ for $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$, and let $t \in \mathbb{R}$ be the affine coordinate. Then we define the triangulated category of enhanced ind-sheaves on $M$ by

$$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M) := \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})/\pi^{-1}\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M),$$

where $\pi\colon M \times \mathbb{R}_\infty \to M$ is a morphism of bordered spaces induced by the first projection $M \times \mathbb{R} \to M$. The quotient functor

$$\mathbf{Q}\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty}) \longrightarrow \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$$

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has fully faithful left and right adjoints $\mathbf{L}^{\mathrm{E}}, \mathbf{R}^{\mathrm{E}}$ defined by

$$\mathbf{L}^{\mathrm{E}}(\mathbf{Q}K) := (\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}) \overset{+}{\otimes} K,$$
$$\mathbf{R}^{\mathrm{E}}(\mathbf{Q}K) := \mathrm{R}\mathcal{I}hom^{+}(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, K),$$

where $\{t \geq 0\}$ stands for $\{(x,t) \in M \times \overline{\mathbb{R}} \mid t \in \mathbb{R}, t \geq 0\}$ and $\{t \leq 0\}$ is defined similarly.

We consider the maps

$$M \times \mathbb{R}^2_{\infty} \xrightarrow{p_1, p_2, \mu} M \times \mathbb{R}_{\infty},$$

where $p_1$ and $p_2$ are morphisms of bordered spaces induced by the projections. And $\mu$ is a morphism of bordered spaces induced by the map $M \times \mathbb{R}^2 \ni (x, t_1, t_2) \mapsto (x, t_1 + t_2) \in M \times \mathbb{R}$. Then the convolution functors for ind-sheaves on $M \times \mathbb{R}_{\infty}$ are defined by

$$F_1 \overset{+}{\otimes} F_2 := \mathrm{R}\mu_{!!}(p_1^{-1}F_1 \otimes p_2^{-1}F_2),$$
$$\mathrm{R}\mathcal{I}hom^{+}(F_1, F_2) := \mathrm{R}p_{1*}\mathrm{R}\mathcal{I}hom(p_2^{-1}F_1, \mu^{!}F_2).$$

The convolution functors are also defined for enhanced ind-sheaves. We denote them by the same symbols, $\overset{+}{\otimes}$ and $\mathrm{R}\mathcal{I}hom^{+}$. For a continuous map $f\colon M \to N$, we can define also the operations $\mathbf{E}f^{-1}, \mathbf{E}f_*, \mathbf{E}f^{!}, \mathbf{E}f_{!!}$ for enhanced ind-sheaves. For example, by the natural morphism $\widetilde{f}\colon M \times \mathbb{R}_{\infty} \to N \times \mathbb{R}_{\infty}$ of bordered spaces associated to $f$, we set $\mathbf{E}f_*(\mathbf{Q}K) = \mathbf{Q}\big(\mathrm{R}\widetilde{f}_*(K)\big)$. The other operations are defined similarly. We thus obtain the six operations $\overset{+}{\otimes}$, $\mathrm{R}\mathcal{I}hom^{+}$, $\mathbf{E}f^{-1}, \mathbf{E}f_*, \mathbf{E}f^{!}, \mathbf{E}f_{!!}$ for enhanced ind-sheaves. Set

$$\mathbb{C}^{\mathrm{E}}_M := \mathbf{Q}\Big(\text{“}\varinjlim_{a\to+\infty}\text{''}\, \mathbb{C}_{\{t\geq a\}}\Big) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M).$$

Then we have natural embeddings $\varepsilon, e\colon \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$ defined by

$$\varepsilon(F) := \mathbf{Q}(\mathbb{C}_{\{t\geq 0\}} \otimes \pi^{-1}F)$$
$$e(F) := \mathbb{C}^{\mathrm{E}}_M \otimes \pi^{-1}F \simeq \mathbb{C}^{\mathrm{E}}_M \overset{+}{\otimes} \varepsilon(F).$$

For a continuous function $\varphi\colon U \to \mathbb{R}$ defined on an open subset $U \subset M$ of $M$ we define the exponential enhanced ind-sheaf by

$$\mathbb{E}^{\varphi}_{U|M} := \mathbb{C}^{\mathrm{E}}_M \overset{+}{\otimes} \mathrm{E}^{\varphi}_{U|M} = \mathbb{C}^{\mathrm{E}}_M \overset{+}{\otimes} \mathbf{Q}\mathbb{C}_{\{t+\varphi\geq 0\}},$$

where $\{t + \varphi \geq 0\}$ stands for $\{(x,t) \in M \times \overline{\mathbb{R}} \mid t \in \mathbb{R}, x \in U, t + \varphi(x) \geq 0\}$.

## 2.5 $\mathcal{D}$-Modules

In this subsection we recall some basic notions and results on $\mathcal{D}$-modules. References are made to [HTT08], [KS01, §7], [DK16, §8, 9], and [KS16b, §3, 4, 7]. For a complex manifold $X$ we denote by $d_X$ its complex dimension. Denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the sheaves of holomorphic functions and holomorphic differential operators on $X$, respectively. Let $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ be the bounded derived category of left $\mathcal{D}_X$-modules. Moreover, we denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, and $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ the full triangulated subcategories of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with coherent, holonomic and regular holonomic cohomologies, respectively. For a morphism $f\colon X \to Y$ of complex manifolds, denote by

$$\overset{D}{\otimes}, \quad \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}, \quad \mathbf{D}f_*, \quad \mathbf{D}f^*, \quad \mathbb{D}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$$

the standard operations for $\mathcal{D}$-modules. The classical solution functor is defined by

$$\mathrm{Sol}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)^{\mathrm{op}} \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X), \qquad \mathcal{M} \longmapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

One defines the ind-sheaf $\mathcal{O}^{\mathrm{t}}_X$ of tempered holomorphic functions as the Dolbeault complex with coefficients in the ind-sheaf of tempered distributions. More precisely, denoting by $X^c$ the complex conjugate manifold to $X$ and by $X_{\mathbb{R}}$ the underlying real analytic manifold of $X$, we set

$$\mathcal{O}^{\mathrm{t}}_X := \mathrm{R}\mathcal{I}hom_{\mathcal{D}_{X^c}}(\mathcal{O}_{X^c}, \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}),$$

where $\mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}$ is the ind-sheaf of tempered distributions on $X_{\mathbb{R}}$ (for the definition, see [KS01, Definition 7.2.5]). Then the tempered solution functor is defined by

$$\mathrm{Sol}^{\mathrm{t}}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)^{\mathrm{op}} \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), \quad \mathcal{M} \longmapsto \mathrm{R}\mathcal{I}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{t}}_X).$$

Note that we have isomorphisms

$$\mathrm{Sol}_X(\mathcal{M}) \simeq \alpha_X \, \mathrm{Sol}^{\mathrm{t}}_X(\mathcal{M}).$$

Let $i\colon X \times \mathbb{R}_\infty \to X \times \mathbb{P}$ be the natural morphism of bordered spaces and $\tau \in \mathbb{C} \subset \mathbb{P}$ the affine coordinate such that $\tau|_{\mathbb{R}}$ is that of $\mathbb{R}$. We then define an object $\mathcal{O}^{\mathrm{E}}_X \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathcal{D}_X)$ by

$$\mathcal{O}^{\mathrm{E}}_X := \mathrm{R}\mathcal{I}hom_{\mathcal{D}_{X^c}}(\mathcal{O}_{X^c}, \mathcal{D}b^{\mathrm{T}}_{X_{\mathbb{R}}}) \simeq i^! \mathrm{R}\mathcal{I}hom_{\mathcal{D}_{\mathbb{P}}}(\mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{O}^{\mathrm{t}}_{X\times\mathbb{P}})[2],$$

where $\mathcal{D}b^{\mathrm{T}}_{X_{\mathbb{R}}}$ stands for the enhanced ind-sheaf of tempered distributions on $X_{\mathbb{R}}$ (for the definition see [DK16, Definition 8.1.1]). We call $\mathcal{O}^{\mathrm{E}}_X$ the enhanced ind-sheaf of tempered holomorphic functions. Note that there exists an isomorphism

$$i_0^! \mathbf{R}^{\mathrm{E}} \mathcal{O}^{\mathrm{E}}_X \simeq \mathcal{O}^{\mathrm{t}}_X,$$

where $i_0\colon X \to X \times \mathbb{R}_\infty$ is the inclusion map of bordered spaces induced by $x \mapsto (x, 0)$. The enhanced solution functor is defined by

$$\mathrm{Sol}^{\mathrm{E}}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), \quad \mathcal{M} \longmapsto \mathrm{R}\mathcal{I}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_X).$$

Then for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, we have an isomorphism

$$\mathrm{Sol}^{\mathrm{t}}_X(\mathcal{M}) \simeq i_0^! \mathbf{R}^{\mathrm{E}} \, \mathrm{Sol}^{\mathrm{E}}_X(\mathcal{M}).$$

Finally, we recall the following theorem of [DK16].

***Theorem 2.2*** ([DK16, Theorem 9.5.3 (Irregular Riemann–Hilbert Correspondence)]) *The enhanced solution functor induces a fully faithful one:*

$$\mathrm{Sol}^{\mathrm{E}}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \longrightarrow \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X).$$

## 3 Fourier Transforms of Regular Holonomic $\mathcal{D}$-modules

In this section, we inherit the situation and the notations in Section 1. Let

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y$$

be the projections. Then by Katz–Laumon [KL85], for an algebraic holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$, we have an isomorphism

$$\mathcal{M}^{\wedge} \simeq \mathbf{D}q_*(\mathbf{D}p^*\mathcal{M} \overset{D}{\otimes} \mathcal{O}_{X\times Y}e^{-\langle z,w\rangle}),$$

where $\mathbf{D}p^*, \mathbf{D}q_*, \overset{D}{\otimes}$ are the operations for algebraic $\mathcal{D}$-modules and $\mathcal{O}_{X\times Y}e^{-\langle z,w\rangle}$ is the integral connection of rank one on $X\times Y$ associated with the canonical paring $\langle , \rangle \colon X\times Y \to \mathbb{C}$. In particular, the right-hand side is concentrated in degree zero. Let $\overline{X} \simeq \mathbb{P}^N$ (resp. $\overline{Y} \simeq \mathbb{P}^N$) be the projective compactification of $X$ (resp. $Y$). By the inclusion map $i_X \colon X = \mathbb{C}^N \hookrightarrow \overline{X} = \mathbb{P}^N$, we extend a holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ on $X$ to the one $\widetilde{\mathcal{M}} := i_{X*}\mathcal{M} \simeq \mathbf{D}i_{X*}\mathcal{M}$ on $\overline{X}$. Denote by $\overline{X}^{\mathrm{an}}$ the underlying complex manifold of $\overline{X}$ and define the analytification $\widetilde{\mathcal{M}}^{\mathrm{an}} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{\overline{X}^{\mathrm{an}}})$ of $\widetilde{\mathcal{M}}$ by $\widetilde{\mathcal{M}}^{\mathrm{an}} := \mathcal{O}_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}}} \widetilde{\mathcal{M}}$. Then we set

$$\mathrm{Sol}^{\mathrm{E}}_{\overline{X}}(\widetilde{\mathcal{M}}) := \mathrm{Sol}^{\mathrm{E}}_{\overline{X}^{\mathrm{an}}}(\widetilde{\mathcal{M}}^{\mathrm{an}}) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\overline{X}^{\mathrm{an}}}).$$

Similarly, for the Fourier transform $\mathcal{M}^{\wedge} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Y)$, by the inclusion map $i_Y \colon Y = \mathbb{C}^N \hookrightarrow \overline{Y} = \mathbb{P}^N$, we define $\mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\widetilde{\mathcal{M}^{\wedge}}) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\overline{Y}^{\mathrm{an}}})$. Let

$$\overline{X}^{\mathrm{an}} \xleftarrow{\overline{p}} \overline{X}^{\mathrm{an}} \times \overline{Y}^{\mathrm{an}} \xrightarrow{\overline{q}} \overline{Y}^{\mathrm{an}}$$

be the projections. Then the following theorem is essentially due to Kashiwara–Schapira [KS16a] and D'Agnolo–Kashiwara [DK17]. For $F \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\overline{X}^{\mathrm{an}}})$ we set

$${}^{\mathrm{L}}F := \mathbf{E}\overline{q}_*(\mathbf{E}\overline{p}^{-1}F \overset{+}{\otimes} \mathbb{E}^{-\mathrm{Re}\langle z,w\rangle}_{X\times Y|\overline{X}\times\overline{Y}}[N]) \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\overline{Y}^{\mathrm{an}}})$$

(here we denote $X^{\mathrm{an}} \times Y^{\mathrm{an}}$ etc. by $X \times Y$ etc. for short) and call it the Fourier–Sato (Fourier–Laplace) transform of $F$.

***Theorem 3.1*** *For $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$, there exists an isomorphism*

$$\mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\widetilde{\mathcal{M}^{\wedge}}) \simeq {}^{\mathrm{L}}\,\mathrm{Sol}^{\mathrm{E}}_{\overline{X}}(\widetilde{\mathcal{M}}).$$

From now on, we focus our attention on Fourier transforms of regular holonomic $\mathcal{D}_X$-modules. For such a $\mathcal{D}_X$-module $\mathcal{M}$, by [HTT08, Theorem 7.1.1] we have an isomorphism $\mathrm{Sol}_{\overline{X}}(\widetilde{\mathcal{M}}) \simeq i_{X!}\,\mathrm{Sol}_X(\mathcal{M})$, where the right-hand side $i_{X!}\,\mathrm{Sol}_X(\mathcal{M}) \in D^b(\mathbb{C}_{\overline{X}^{\mathrm{an}}})$ is the extension by zero of the classical solution complex of $\mathcal{M}$ to $\overline{X}^{\mathrm{an}}$. Moreover, by [DK16, Proposition 9.1.3 and Corollary 9.4.9], there exists an isomorphism

$$\mathrm{Sol}^{\mathrm{E}}_{\overline{X}}(\widetilde{\mathcal{M}}) \simeq \mathbb{C}^{\mathrm{E}}_{\overline{X}^{\mathrm{an}}} \overset{+}{\otimes} \varepsilon(i_{X!}\,\mathrm{Sol}_X(\mathcal{M})).$$

For an enhanced sheaf $F \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}_{\overline{X}^{\mathrm{an}}})$ on $\overline{X}^{\mathrm{an}}$ we define its Fourier–Sato (Fourier–Laplace) transform ${}^{\mathrm{L}}F \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}_{\overline{Y}^{\mathrm{an}}})$ by

$$ {}^{\mathrm{L}}F := \mathbf{E}\overline{q}_*(\mathbf{E}\overline{p}^{-1}F \overset{+}{\otimes} \mathrm{E}^{-\mathrm{Re}\langle z,w\rangle}_{X\times Y|\overline{X}\times\overline{Y}}[N]) \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}_{\overline{Y}^{\mathrm{an}}}). $$

Since we have

$$ {}^{\mathrm{L}}(\mathbb{C}^{\mathrm{E}}_{\overline{X}^{\mathrm{an}}} \overset{+}{\otimes} (\,\cdot\,)) \simeq \mathbb{C}^{\mathrm{E}}_{\overline{Y}^{\mathrm{an}}} \overset{+}{\otimes} {}^{\mathrm{L}}(\,\cdot\,), $$

for the calculation of $\mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\widetilde{\mathcal{M}^\wedge})$, it suffices to calculate the Fourier-Sato transform of the enhanced sheaf $\varepsilon(i_{X!}\,\mathrm{Sol}_X(\mathcal{M})) \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}_{\overline{X}^{\mathrm{an}}})$ on $\overline{X}^{\mathrm{an}}$. The following theorem is due to Brylinski [Bry86]. Here we give a new geometric proof for it.

***Theorem 3.2*** *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Assume that $\mathrm{Sol}_X(\mathcal{M})$ is monodromic. Then $\mathcal{M}^\wedge$ is also a regular holonomic $\mathcal{D}_Y$-module and $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.*

**Proof** By the above argument we have isomorphisms

$$ \begin{aligned} \mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\widetilde{\mathcal{M}^\wedge}) &\simeq {}^{\mathrm{L}}\,\mathrm{Sol}^{\mathrm{E}}_{\overline{X}}(\widetilde{\mathcal{M}}) \\ &\simeq {}^{\mathrm{L}}\Big(\mathbb{C}^{\mathrm{E}}_{\overline{X}^{\mathrm{an}}} \overset{+}{\otimes} \varepsilon\big(i_{X!}\,\mathrm{Sol}_X(\mathcal{M})\big)\Big) \\ &\simeq \mathbb{C}^{\mathrm{E}}_{\overline{Y}^{\mathrm{an}}} \overset{+}{\otimes} {}^{\mathrm{L}}\Big(\varepsilon\big(i_{X!}\,\mathrm{Sol}_X(\mathcal{M})\big)\Big) \\ &\simeq \mathbb{C}^{\mathrm{E}}_{\overline{Y}^{\mathrm{an}}} \overset{+}{\otimes} \varepsilon\big(i_{Y!}\,\mathrm{Sol}_X(\mathcal{M})^\wedge\big), \end{aligned} $$

where $(\,\cdot\,)^\wedge$ stands for the Fourier–Sato transform for $\mathbb{R}_+$-conic sheaves (see [KS90]) and in the last isomorphism, we applied [KS16a, Theorem 5.7] to the $\mathbb{R}_+$-conic sheaf $\mathrm{Sol}_X(\mathcal{M})$. Note that $\mathrm{Sol}_X(\mathcal{M})^\wedge$ is not only $\mathbb{R}_+$-conic but also $\mathbb{C}$-constructible by [KS90, Proposition 10.3.18]. Hence, it is monodromic by Lemma 2.1. Moreover, by applying the functor $i_0^!\mathrm{R}^{\mathrm{E}}(\,\cdot\,)$ to the isomorphism $\mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\widetilde{\mathcal{M}^\wedge}) \simeq \mathbb{C}^{\mathrm{E}}_{\overline{Y}^{\mathrm{an}}} \overset{+}{\otimes} \varepsilon(i_{Y!}\,\mathrm{Sol}_X(\mathcal{M})^\wedge)$ we obtain an isomorphism

$$ \mathrm{Sol}_{\overline{Y}}(\widetilde{\mathcal{M}^\wedge}) \simeq i_{Y!}\,\mathrm{Sol}_X(\mathcal{M})^\wedge. $$

This implies that $i_{Y!}\,\mathrm{Sol}_X(\mathcal{M})^\wedge$ is an (algebraic) constructible sheaf on the algebraic variety $\overline{Y}$. By [HTT08, Corollary 7.2.4], we can take a regular holonomic $\mathcal{D}$-module $\mathcal{N} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_{\overline{Y}})$ on $\overline{Y}$ such that $\mathrm{Sol}_{\overline{Y}}(\mathcal{N}) \simeq i_{Y!}\,\mathrm{Sol}_X(\mathcal{M})^\wedge$. Then we have isomorphisms

$$ \begin{aligned} \mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\widetilde{\mathcal{M}^\wedge}) &\simeq \mathbb{C}^{\mathrm{E}}_{\overline{Y}^{\mathrm{an}}} \overset{+}{\otimes} \varepsilon\big(i_{Y!}\,\mathrm{Sol}_X(\mathcal{M})^\wedge\big) \\ &\simeq \mathbb{C}^{\mathrm{E}}_{\overline{Y}^{\mathrm{an}}} \overset{+}{\otimes} \varepsilon\big(\mathrm{Sol}_{\overline{Y}}(\mathcal{N})\big) \\ &\simeq \mathrm{Sol}^{\mathrm{E}}_{\overline{Y}}(\mathcal{N}). \end{aligned} $$

By Theorem 2.2, we thus obtain an isomorphism

$$ (\widetilde{\mathcal{M}^\wedge})^{\mathrm{an}} \simeq \mathcal{N}^{\mathrm{an}} \qquad \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_{\overline{Y}^{\mathrm{an}}}) $$

of analytic $\mathcal{D}$-modules on $\overline{Y}^{\mathrm{an}}$. Then the assertion follows from Lemma 3.3 of Brylinski [Bry86, Théorème 7.1] below. ∎

***Lemma 3.3*** (Brylinski [Bry86, Théorème 7.1]) *Let $Z$ be a smooth projective variety. Then the analytification functor*

$$(\,\cdot\,)^{\mathrm{an}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_Z) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{Z^{\mathrm{an}}})$$

*is an equivalence of categories.*

**Proof** This result is due to Brylinski [Bry86, Théorème 7.1]. We shall give a new proof to it. Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_Z)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Z^{\mathrm{an}}})$) be the derived category of $\mathbb{C}$-constructible sheaves on the algebraic variety $Z$ (resp. the complex manifold $Z^{\mathrm{an}}$). Then we have a commutative diagram of functors

$$\begin{tikzcd} \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_Z) \arrow[r, "\sim"] \arrow[d, "(\,\cdot\,)^{\mathrm{an}}"'] & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_Z) \arrow[d] \\ \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{Z^{\mathrm{an}}}) \arrow[r, "\sim"] & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Z^{\mathrm{an}}}), \end{tikzcd}$$

where the horizontal arrows are the Riemann–Hilbert correspondences of algebraic and analytic $\mathcal{D}$-modules respectively (see *e.g.*, [HTT08, Theorem 7.2.2]). By Chow's theorem, the right vertical arrow

$$\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_Z) \longrightarrow \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Z^{\mathrm{an}}})$$

is also an equivalence of categories. Then the assertion immediately follows. ∎

For $s \in \mathbb{R}_+$, let

$$m_s\colon Y = \mathbb{C}^N \xrightarrow{\sim} Y = \mathbb{C}^N, \quad w \longmapsto sw$$

be the multiplication by $s$. We shall use also the morphism $\ell_s\colon Y \times \mathbb{R}_\infty \to Y \times \mathbb{R}_\infty$ on the bordered space $Y \times \mathbb{R}_\infty$ induced by the diagonal action

$$\ell_s\colon Y \times \mathbb{R} \xrightarrow{\sim} Y \times \mathbb{R}, \quad (w,t) \longmapsto (sw, st).$$

Let $f\colon X \times Y \times \mathbb{R} \to X$, $g\colon X \times Y \times \mathbb{R} \to Y \times \mathbb{R}$ be the projections. Then the following lemma was obtained in (the proof) of Ito–Takeuchi [IT18, Theorem 4.4].

***Lemma 3.4*** *Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. Then we have an isomorphism*

$${}^{\mathrm{L}}(\varepsilon(i_{X!}\mathcal{F})) \simeq \mathbf{Q}\big(\widetilde{i_{Y}}_{!}\mathrm{R}g_!(\mathbb{C}_{\{t-\mathrm{Re}\langle z,w\rangle \geq 0\}} \otimes f^{-1}\mathcal{F})[N]\big)$$

*of enhanced sheaves.*

For $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$, let us set

$$L(\mathcal{F}) = \mathrm{R}g_!\big(\mathbb{C}_{\{t-\mathrm{Re}\langle z,w\rangle \geq 0\}} \otimes f^{-1}\mathcal{F}\big)[N] \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times\mathbb{R}}).$$

***Lemma 3.5*** *Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. Then for any $s \in \mathbb{R}_+$, we have an isomorphism $\ell_s^{-1}(L(\mathcal{F})) \simeq L(\mathcal{F})$ in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times\mathbb{R}})$. In other words, $L(\mathcal{F})$ is a $\mathbb{R}_+$-conic sheaf on $Y \times \mathbb{R} \simeq \mathbb{R}^{2N+1}$.*

**Proof** Consider the Cartesian diagram

$$\begin{tikzcd} X \times Y \times \mathbb{R} \arrow[r, "\mathrm{id}_X \times \ell_s"] \arrow[d, "g"'] & X \times Y \times \mathbb{R} \arrow[d, "g"] \\ Y \times \mathbb{R} \arrow[r, "\ell_s"'] & Y \times \mathbb{R}. \end{tikzcd}$$

Then we have isomorphisms

$$\begin{aligned} \ell_s^{-1}\big(L(\mathcal{F})\big) &\simeq \ell_s^{-1}\mathrm{R}g_!(\mathbb{C}_{\{t-\mathrm{Re}\langle z,w\rangle\geq 0\}} \otimes f^{-1}\mathcal{F})[N] \\ &\simeq \mathrm{R}g_!(\mathrm{id}_X \times \ell_s)^{-1}(\mathbb{C}_{\{t-\mathrm{Re}\langle z,w\rangle\geq 0\}} \otimes f^{-1}\mathcal{F})[N] \\ &\simeq \mathrm{R}g_!(\mathbb{C}_{\{t-\mathrm{Re}\langle z,w\rangle\geq 0\}} \otimes f^{-1}\mathcal{F})[N] \simeq L(\mathcal{F}), \end{aligned}$$

where in the third isomorphism we used

$$st - \mathrm{Re}\langle z, sw\rangle \geq 0 \quad \Longleftrightarrow \quad t - \mathrm{Re}\langle z, w\rangle \geq 0$$

and $f \circ (\mathrm{id}_X \times \ell_s) = f$. ■

From now on, we consider the special case where $\mathcal{F} = \mathrm{Sol}_X(\mathcal{M}) \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ for $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$.

***Corollary 3.6*** *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ be an algebraic regular holonomic $\mathcal{D}_X$-module on $X$. Then there exists an $\mathbb{R}_+$-conic sheaf $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times\mathbb{R}})$ on $Y \times \mathbb{R} \simeq \mathbb{R}^{2N+1}$ such that*

$$\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge) \simeq \mathbb{C}_Y^{\mathrm{E}} \overset{+}{\otimes} \mathbf{Q}(\mathcal{G}).$$

**Proof** By Lemma 3.4 we have isomorphisms

$$\begin{aligned} \mathrm{Sol}_{\overline{Y}}^{\mathrm{E}}(\widetilde{\mathcal{M}^\wedge}) &\simeq \mathbb{C}_{\overline{Y}}^{\mathrm{E}} \overset{+}{\otimes} {}^{\mathrm{L}}\varepsilon(i_{X!}\,\mathrm{Sol}_X(\mathcal{M})) \\ &\simeq \mathbb{C}_{\overline{Y}}^{\mathrm{E}} \overset{+}{\otimes} \mathbf{Q}(\widetilde{i_{Y!}}L(\mathrm{Sol}_X(\mathcal{M}))). \end{aligned}$$

Then by the restriction to $Y \subset \overline{Y}$ and Lemma 3.5, we obtain the assertion. ■

***Proposition 3.7*** *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ be an algebraic regular holonomic $\mathcal{D}_X$-module on $X$. Then for any $s \in \mathbb{R}_+$, we have an isomorphism*

$$\ell_s^{-1}\mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge) \simeq \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge).$$

**Proof** There exist isomorphisms

$$\begin{aligned} \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_{\overline{Y}}^{\mathrm{E}}(\widetilde{\mathcal{M}^\wedge}) &\simeq \mathrm{L}^{\mathrm{E}}\big(\mathbb{C}_{\overline{Y}}^{\mathrm{E}} \overset{+}{\otimes} \mathbf{Q}(\widetilde{i_{Y!}}L(\mathrm{Sol}_X(\mathcal{M})))\big) \\ &\simeq \mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} \widetilde{i_{Y!}}L\big(\mathrm{Sol}_X(\mathcal{M})\big). \end{aligned}$$

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We extend $\ell_s$ to $\overline{Y} \times \mathbb{R}$ naturally and denote it by the same symbol. Then by Lemma 3.5 for $s \in \mathbb{R}_+$, we have isomorphisms

$$\begin{aligned}\ell_s^{-1}\mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_{\overline{Y}}^{\mathrm{E}}(\widetilde{\mathcal{M}^\wedge}) &\simeq \ell_s^{-1}\big(\mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} \widetilde{i_Y}_!L(\mathrm{Sol}_X(\mathcal{M}))\big)\\ &\simeq \mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} \ell_s^{-1}\big(\widetilde{i_Y}_!L(\mathrm{Sol}_X(\mathcal{M}))\big)\\ &\simeq \mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} \widetilde{i_Y}_!L(\mathrm{Sol}_X(\mathcal{M}))\\ &\simeq \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_{\overline{Y}}^{\mathrm{E}}(\widetilde{\mathcal{M}^\wedge}).\end{aligned}$$

We obtain the assertion by the restriction to $Y \subset \overline{Y}$. ∎

***Proposition 3.8*** *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then for any $s \in \mathbb{R}_+$, we have an isomorphism*

$$m_s^{-1}\,\mathrm{Sol}_Y(\mathcal{M}^\wedge) \simeq \mathrm{Sol}_Y(\mathcal{M}^\wedge).$$

**Proof** By (the proof) of Ito–Takeuchi [IT18, Lemma 3.13], there exist isomorphisms

$$\begin{aligned}\mathrm{Sol}_Y(\mathcal{M}^\wedge) &\simeq \alpha_Y i_0^!\mathrm{R}^{\mathrm{E}}\big(\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big)\\ &\simeq \alpha_Y \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom\big(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big).\end{aligned}$$

Consider the commutative diagram

$$\begin{tikzcd} Y \times \mathbb{R}_\infty \arrow[r, "\pi"] \arrow[d, "\ell_s"'] & Y \arrow[d, "m_s"] \\ Y \times \mathbb{R}_\infty \arrow[r, "\pi"'] & Y. \end{tikzcd}$$

It is easy to see that it is Cartesian. Then by Proposition 3.7, we have isomorphisms

$$\begin{aligned}m_s^{-1}\,\mathrm{Sol}_Y(\mathcal{M}^\wedge) &\simeq m_s^{-1}\alpha_Y \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom\big(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big)\\ &\simeq \alpha_Y m_s^! \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom\big(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big)\\ &\simeq \alpha_Y \mathrm{R}\pi_*\ell_s^!\mathrm{R}\mathcal{I}hom\big(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big)\\ &\simeq \alpha_Y \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom\big(\ell_s^{-1}(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}), \ell_s^!\mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big)\\ &\simeq \alpha_Y \mathrm{R}\pi_*\mathrm{R}\mathcal{I}hom\big(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, \mathrm{L}^{\mathrm{E}}\,\mathrm{Sol}_Y^{\mathrm{E}}(\mathcal{M}^\wedge)\big)\\ &\simeq \mathrm{Sol}_Y(\mathcal{M}^\wedge),\end{aligned}$$

where in the fifth isomorphism we used $\ell_s^! \simeq \ell_s^{-1}$ (see [DK16, Corollary 3.3.11]). ∎

***Theorem 3.9*** *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.*

**Proof** Since the Fourier transform $\mathcal{M}^\wedge$ of $\mathcal{M}$ is also holonomic, $\mathrm{Sol}_Y(\mathcal{M}^\wedge)$ is $\mathbb{C}$-constructible. Moreover, it is $\mathbb{R}_+$-conic by Proposition 3.8. Then the assertion follows from Lemma 2.1. ∎

By this theorem, we can improve Brylinski's Theorem 3.2 as follows.

***Corollary 3.10*** *Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\mathcal{M}^\wedge$ is regular if and only if $\mathcal{M}$ is monodromic.*

**Proof** By Theorem 3.2, the Fourier transform $\mathcal{M}^\wedge$ is regular if $\mathcal{M}$ is monodromic. It suffices to show the converse. Assume that $\mathcal{M}^\wedge$ is regular. Let

$$(\cdot)^\vee \colon \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Y) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$$

be the inverse Fourier transform. Then by Theorems 3.2 and 3.9, the original regular holonomic $\mathcal{D}_X$-module $\mathcal{M} \simeq (\mathcal{M}^\wedge)^\vee$ is monodromic. ∎

## 4 An Application to Direct Images of $\mathcal{D}$-Modules

In this section, we apply our results to direct images of some irregular holonomic $\mathcal{D}$-modules. We inherit the situation and the notations in Section 1. For a point $a \in Y = \mathbb{C}^N$, let $\tau_a \colon Y \xrightarrow{\sim} Y, w \mapsto w + a$ be the translation by it.

***Lemma 4.1*** *For $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ and $a \in Y = \mathbb{C}^N$, we have an isomorphism*

$$\mathbf{D}\tau_a^*(\mathcal{M}^\wedge) \simeq (\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle})^\wedge.$$

**Proof** By Katz–Laumon [KL85], there exist isomorphisms

$$\begin{aligned}(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle})^\wedge &\simeq \mathbf{D}q_*\big(\mathbf{D}p^*(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle}) \overset{D}{\otimes} \mathcal{O}_{X\times Y} e^{-\langle z,w\rangle}\big)\\ &\simeq \mathbf{D}q_*\big(\mathbf{D}p^*\mathcal{M} \overset{D}{\otimes} \mathcal{O}_{X\times Y} e^{-\langle z,w+a\rangle}\big)\\ &\simeq \mathbf{D}\tau_a^*(\mathcal{M}^\wedge).\end{aligned}$$ ∎

***Theorem 4.2*** *Let $\rho \colon X = \mathbb{C}^N \twoheadrightarrow Z = \mathbb{C}^n$ be a surjective linear map and $\mathcal{M}$ an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. For the dual $L \simeq \mathbb{C}^n$ of $Z$, let $\iota \colon L \hookrightarrow Y = \mathbb{C}^N$ be the injective linear map induced by $\rho$. Assume that for a point $a \in Y \setminus \iota(L)$ the affine linear subspace $K = \tau_a(\iota(L)) \subset Y = \mathbb{C}^N$ is non-characteristic for the Fourier transform $\mathcal{M}^\wedge \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Y)$ of $\mathcal{M}$. Then the direct image $\mathbf{D}\rho_*(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Z)$ is concentrated in degree 0.*

**Proof** Let $i_L = \iota \colon L \hookrightarrow Y = \mathbb{C}^N$ and $i_K \colon K \hookrightarrow Y = \mathbb{C}^N$ be the inclusion maps. Then via the identification $L \simeq K$ induced by the translation $\tau_a$, we have isomorphisms

$$\begin{aligned}\mathbf{D}i_K^*(\mathcal{M}^\wedge) &\simeq \mathbf{D}i_L^*\mathbf{D}\tau_a^*(\mathcal{M}^\wedge)\\ &\simeq \mathbf{D}i_L^*(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle})^\wedge\\ &\simeq \big(\mathbf{D}\rho_*(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle})\big)^\wedge,\end{aligned}$$

where in the second (resp. third) isomorphism we used Lemma 4.1 (resp. [HTT08, Proposition 3.2.6]). By our assumption, the left-hand side $\mathbf{D}i_K^*(\mathcal{M}^\wedge) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_K)$ is concentrated in degree 0. Then the assertion follows from the fact that the Fourier transform is an exact functor. ∎

***Corollary 4.3*** *In the situation of Theorem 4.2 assume also that $n = N - 1$ i.e., the surjective linear map $\rho\colon X = \mathbb{C}^N \twoheadrightarrow Z = \mathbb{C}^n$ is of codimension one. Then for any point $a \in Y \backslash \iota(L)$ the direct image $\mathbf{D}\rho_*(\mathcal{M} \overset{D}{\otimes} \mathcal{O}_X e^{-\langle z,a\rangle}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Z)$ is concentrated in degree 0.*

**Proof** By Theorem 3.9, the Fourier transform $\mathcal{M}^\wedge$ of $\mathcal{M}$ is monodromic. Since the affine linear subspace $K = \tau_a(\iota(L)) \subset Y = \mathbb{C}^N$ does not contain the origin $0 \in Y = \mathbb{C}^N$, this implies that $K$ is non-characteristic for $\mathcal{M}^\wedge$. Then the assertion follows from Theorem 4.2. ∎

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*Graduate School of Mathematical Science, The University of Tokyo, 3-8-1, Komaba, Meguro, Tokyo, 153-8914, Japan*
*e-mail:* yitoh@ms.u-tokyo.ac.jp

*Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan*
*e-mail:* takemicro@nifty.com