

1 Description of the Flow

The most complete mathematical formulation for the dynamics of a Newtonian fluid is unquestionably the Navier–Stokes equations, which account for all possible effects, i.e., compressibility, viscosity, and vorticity. Indeed, the Navier–Stokes equations are the most complete tool for describing the flow, and a vast literature exists dealing with and employing them in fluid mechanics, while several emblematic personalities dedicated their careers to this task (e.g., Batchelor, 1967). Navier–Stokes equations can be simplified under certain assumptions; i.e., neglecting viscosity, we arrive at the Euler equations. Further, eliminating terms that govern vorticity, a flow pattern governed by the celebrated Laplace domain may be obtained, in particular by the Laplace equation that assumes the flow of an ideal fluid, which in addition allows the engagement of the potential flow theory. A question that should be answered regarding the employment of the Laplace domain in marine hydrodynamics is, simply: Why potential flow theory? The answer is similarly simple: we do not need to complicate things when the flow field can indeed be described by the dynamics of an ideal fluid such as in the open ocean, where the liquid can be accurately assumed to be incompressible and inviscid and the flow irrotational. In addition, the Laplace equation is a relatively simple formula while potential flow theory is the most robust tool for formulating the dynamics of the free surface. Admittedly, when using more mathematically complete formulations one faces enormous difficulties in attempting to describe accurately the behavior of the free surface. That eventually impacts the way the free surface is formulated (by panels in boundary element methods, or by finite volumes in Navier–Stokes solvers) and accordingly the time required for computations.

The final goal of the text flow in this chapter is to derive the generic forms of most common boundary value problems in hydrodynamics, which are actually fluid–structure interaction problems. In other words, we are seeking the mathematical system that fully describes the flow field assuming a solid within it. The basic three-dimensional hydrodynamic problem, which differentiates it from similar problems in fluid mechanics, should account for given facts, i.e., the existence of the free surface, the existence of the impermeable bottom in shallow water cases, the behavior of the flow at infinity, and apparently the existence of the solid(s).

Toward the final goal one should start from the very beginning, namely from the very basic and complete mathematical formulation. To this end, the Eulerian

approach is followed to derive the Navier–Stokes and Euler equations, which are accordingly simplified, and under the assumptions of an ideal fluid and potential flow theory the Laplace domain is established. The basic three-dimensional hydrodynamic boundary value problem is accordingly defined and elaborated by the Stokes perturbations and Taylor expansion for the kinematics and the dynamics of the free surface. The main task in hydrodynamics is the calculation of the forces and the motions of the solid that exists within the reference volume of liquid. Motions are not an issue of concern here but the hydrodynamic parameters that affect them, added mass and hydrodynamic damping coefficients, are indeed considered. The same holds for the loading components. In this chapter only the general formulae are provided, which are accordingly used in the following chapters for more analytic elaboration. Potential flow problems are typically governed by the Green's functions. To this end the last section of this chapter is dedicated to the famous Green's theorem.

1.1 The Laplace Domain

The Laplace equation is derived from the equation of continuity provided that (1) the fluid is incompressible and (2) inviscid, and (3) the flow is irrotational. Incompressibility refers to the property of the fluid to maintain its density constant within an infinitesimal volume that moves with flow velocity. An inviscid fluid, on the other hand, is a fluid with zero viscosity, namely the term (property) of the fluid that quantifies the resistance of the fluid to its gradual deformation by shear stresses. Finally, irrotational flow assumes that individual parcels of infinitesimal volume of an incompressible fluid cannot be caused to rotate. In other words, the fluid is assumed to be frictionless with no shear stresses applied on the mutual surfaces of adjacent parcels of the fluid.

Properties (1)–(3) are precisely the assumptions taken to formulate large-scale flows of liquid, namely water flows in vast volumes, just like in the open ocean, or in other words, in a marine environment. The Laplace domain has numerous applications in various disciplines of modern science, such as electromagnetics, acoustics, optics, electrostatics, imaging, hydrodynamics, and others. In hydrodynamics, however, the situation is complicated to a major extent by the existence of the free surface and subsequently by the associated boundary conditions that must be satisfied. In fact, the free surface, namely the boundary surface where the water meets the air, is what makes hydrodynamics difficult and challenging, but at the same time, a fascinating discipline.

The Laplace equation in hydrodynamics is associated with the so-called velocity potential, a scalar function, the gradient of which provides the velocity field as well as the pressure at any point of the volume of reference. Having defined the velocity field by the gradient of the velocity potential, the Laplace equation immediately follows from the mass conservation law (or the equation of continuity). Knowing the velocity potential, one may derive all dynamic parameters associated with the

concerned hydrodynamic problem, namely, velocities, pressures, and accordingly hydrodynamic loads. All these, however, require the solution of relevant boundary value problems in which the boundary conditions on the free surface are of paramount importance for free-surface flows.

Describing the flow of a fluid through the Laplace equation is admittedly a simplification that leads inevitably to the assumption of an *ideal fluid*. This simplification, however, is a valid approach for large-scale flows in a marine environment, as has been proven beyond any doubt over the past decades. The reason is simple: in the open ocean the fluid (liquid) is water, and water is practically incompressible, nearly inviscid, while with a large degree of accuracy the flow is irrotational. In addition, use of potential theory (even nonlinear potential theory) allows the employment of analytical approaches that accordingly enable the derivation of robust analytical (or semianalytical) closed-form solutions. That advantage is provided by the separable solutions of the Laplace equation using the method of the separation of variables in various coordinate systems (Moon and Spencer, 1971), yielding expressions in the form of eigenfunction expansions.

It should be acknowledged, however, that all Newtonian fluids, such as water, are literally viscous. Therefore, for the completeness of the presentation, we derive in the following sections the more general formulations of the flow for Newtonian fluids, namely the Euler and the Navier–Stokes equations and eventually we focus on the flow of the ideal fluid. The analysis starts with the transport theorem, which is discussed in the section immediately following.

1.2 The Transport Theorem

We consider the following volume integral:

$$I(t) = \int_{V(t)} f(\mathbf{x}, t) dV \quad (1.1)$$

where $f(\mathbf{x}, t)$ is an arbitrary differentiable scalar function of position \mathbf{x} and time t . The integral is taken over the given volume $V(t)$, which changes with time. The surface S that contains $V(t)$ is also a function of time and we assume that it has a normal velocity equal to U_n . Let us further take the same volume integral at the time instant $t + \Delta t$. The change in integral ΔI is thus written as

$$\Delta I = I(t + \Delta t) - I(t) = \int_{V(t+\Delta t)} f(\mathbf{x}, t + \Delta t) dV - \int_{V(t)} f(\mathbf{x}, t) dV \quad (1.2)$$

Also, we expand $f(\mathbf{x}, t + \Delta t)$ in a Taylor series in t to obtain

$$f(\mathbf{x}, t + \Delta t) = f(\mathbf{x}, t) + \Delta t \frac{\partial f(\mathbf{x}, t)}{\partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 f(\mathbf{x}, t)}{\partial t^2} + \dots \quad (1.3)$$

In the following, the function $f(\mathbf{x}, t)$ is denoted simply by f . Retaining terms up to $O[(\Delta t)^2]$, (1.2) becomes

$$\Delta I = \int_{V(t+\Delta t)} \left(f + \Delta t \frac{\partial f}{\partial t} \right) dV - \int_{V(t)} f dV \quad (1.4)$$

In order to simplify further the change of the volume integral ΔI , we assume that $V(t+\Delta t)$ is composed of the volume $V(t)$ and an infinitesimal volume ΔV that is contained between the adjacent surfaces $S(t+\Delta t)$ and $S(t)$, i.e., $V(t+\Delta t) = V(t) + \Delta V(t)$. Hence, (1.4) is expanded in the following manner:

$$\Delta I = \int_{V(t)+\Delta V(t)} \left(f + \Delta t \frac{\partial f}{\partial t} \right) dV - \int_{V(t)} f dV = \int_{V(t)} f dV + \Delta t \int_{V(t)} \frac{\partial f}{\partial t} dV + O[(\Delta t)^2] \quad (1.5)$$

and it is understood that the $O[(\Delta t)^2]$ terms also contain terms of $O[\Delta t \Delta V(t)]$.

To evaluate the integral over $\Delta V(t)$ we need to consider the infinitesimal distance between $S(t+\Delta t)$ and $S(t)$ that is equal to the thickness of the small region $\Delta V(t)$. This thickness is the normal component of the distance traveled by $S(t)$ in time Δt , and is equal to the product $U_n \Delta t$. Hence, it is immediately deduced that the first integral in (1.5) is proportional to Δt . In addition, given the infinitesimal thickness of volume $\Delta V(t)$, we may assume that the integrand f is constant across that thin region in the direction normal to $S(t)$. Integrating in this direction only, we obtain

$$\Delta I = \int_{S(t)} (U_n \Delta t) f dS + \Delta t \int_{V(t)} \frac{\partial f}{\partial t} dV + O[(\Delta t)^2] \quad (1.6)$$

Taking the limit of $\Delta I / \Delta t$ as $\Delta t \rightarrow 0$ yields the differential form of the transport theorem, namely

$$\frac{dI}{dt} = \int_{S(t)} f U_n dS + \int_{V(t)} \frac{\partial f}{\partial t} dV \quad (1.7)$$

The surface integral in (1.7) represents the transport of the quantity f out of the volume $V(t)$ as a result of the motion of the boundary.

1.3 Shear Stresses in Fluid Particles: The Eulerian Approach

Let us now assume a flow field in which the velocity of the fluid particles is defined by the velocity vector $\mathbf{U} = (u, v, w) = (u_1, u_2, u_3)$, where u, v , and w (or u_1, u_2, u_3 , respectively) are the velocity components in the three directions of the Cartesian frame of reference $(x, y, z) = (x_1, x_2, x_3)$. The density of the fluid is assumed to be constant and equal to ρ , while the fluid particles are subjected to the external force vector \mathbf{F}

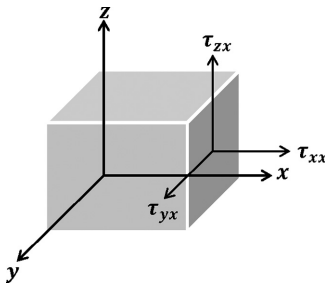


Figure 1.1 Stresses acting on a cubical fluid particle.

that is composed of the gravitational force and the surface stresses, or force per unit area, which act on adjacent surfaces of the fluid (Figure 1.1).

We note that each surface stress should be defined by both its direction and the orientation of the surface of the fluid particle on which it is acting. For the cubical parcel of Figure 1.1 in particular, a total of $3 \times 3 = 9$ stress components must be defined. The origin of the Cartesian system was chosen to coincide with one of the corners of the cube only for display purposes. The reasoning behind the tensor notation is that τ_{ij} acts in the direction i and is applied on the surface of constant j , with $i, j = x, y, z$. Clearly, τ_{ij} , $i \neq j$ denote actual shear stresses while τ_{ii} are normal stresses (often denoted by σ_i).

The definition of the stress components based on the cubical configuration of the fluid particles is in fact too restrictive. A more general configuration arises by assuming an infinitesimal volume of fluid (a fluid particle) in the shape of a tetrahedron with three orthogonal faces normal to the Cartesian coordinates as shown in Figure 1.2. The fourth face of the tetrahedron has an arbitrary oblique orientation. It is assumed the tetrahedron is sufficiently small and that the stresses are effectively constant along each face, and since the volume will be negligible compared with the surface area, the surface forces will dominate the body forces. To achieve an equilibrium state, the forces exerted on the four faces of the tetrahedron, which are induced by the surface stresses, must balance. By analogy with the stress theory in structural mechanics, the equilibrium state is secured by Cauchy's law according to which there exists a Cauchy stress tensor $\boldsymbol{\tau}$ that maps the normal to a surface to the traction vector acting on that surface. In particular, if $\mathbf{n} = (n_x, n_y, n_z)^T = (n_1, n_2, n_3)^T$ denotes the unit normal vector on the oblique face (see Figure 1.2), Cauchy's law implies that

$$\mathbf{T} = \boldsymbol{\tau} \cdot \mathbf{n} \quad (1.8)$$

or, in Cartesian representation,

$$T_x = \tau_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z \quad (1.9)$$

$$T_y = \tau_{yx}n_x + \tau_{yy}n_y + \tau_{yz}n_z$$

$$T_z = \tau_{zx}n_x + \tau_{zy}n_y + \tau_{zz}n_z$$

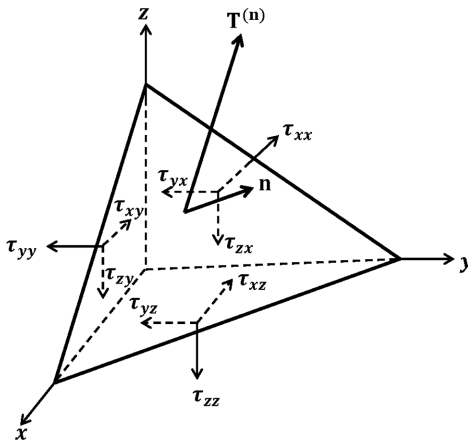


Figure 1.2 Stresses acting on the surfaces of a volume element in the shape of a tetrahedron with three orthogonal faces.

The stress tensor τ , which is also denoted by τ_{ij} , is written as

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_2 & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_3 \end{bmatrix} \quad (1.10)$$

Accordingly, the stress component in the i th direction, on a surface element with unit normal \mathbf{n} , is given by $\sum_{j=1}^3 \tau_{ij}n_j$, which for convenience will be denoted by the repeating indices product $\tau_{ij}n_j, i = 1, 2, 3$.

The stress tensor τ_{ij} is symmetric, i.e., $\tau_{ij} = \tau_{ji}$. To demonstrate that statement, we assume that the cube shown in Figure 1.1 is infinitesimal, which allows assuming that the surface moments dominate the body moments. In addition, changes in the magnitude of the stresses may be neglected across the cube and a positive value of τ_{zx} (Figure 1.1) will cause a counterclockwise moment about the centroid. The moment on the opposite face will have the same sign, since the normal vector has the opposite sense and hence a positive τ_{zx} acts downward. Accordingly, the counterclockwise moment is balanced by a clockwise moment that is developed only by the stress τ_{xz} that acts in the x -direction on the top and the bottom faces of the cube. Hence, it follows that $\tau_{zx} = \tau_{xz}$ and accordingly $\tau_{xy} = \tau_{yx}$ and $\tau_{yz} = \tau_{zy}$.

1.4 Mass Conservation and Momentum Conservation

Let us now assume that $V(t)$, defined originally in Section 1.2, is a material volume of fluid that is assembled by a group of material fluid particles. Accordingly, the total mass of the fluid in $V(t)$ is given by the volume integral of density ρ in $V(t)$,

while the mass conservation law requires that the total mass must be constant and independent of time t . Hence

$$\frac{d}{dt} \int_{V(t)} \rho dV = 0 \quad (1.11)$$

Further, the momentum conservation law is written as

$$\frac{d}{dt} \int_{V(t)} \rho u_i dV = \int_{S(t)} \tau_{ij} n_j dS + \int_{V(t)} F_i dV \quad (1.12)$$

which implies that the sum of all forces exerted on a volume of fluid must be equal to the rate of change of its momentum with respect to the Newtonian frame of reference. The momentum rate of change on the left-hand side of (1.12) refers to the i th component of the velocity, the surface integral on the right-hand side denotes the sum of the stress forces in the i th direction, while the last term is the total external force exerted on the concerned volume of fluid, again in the i th direction. Clearly, F_i is the i th component of the aforementioned vector \mathbf{F} .

The momentum conservation law can be effectively transformed in order to be expressed by volume integrals only. This can be achieved using the Gauss theorem (also known as the divergence theorem). In particular, for a vector \mathbf{W} that is continuous and differentiable in the volume $V(t)$ and the unit normal \mathbf{n} that is fixed on the surface S surrounding $V(t)$ and pointing to the exterior of $V(t)$, the Gauss theorem reads in vectorial and tensorial forms:

$$\int_{V(t)} \nabla \cdot \mathbf{W} dV = \int_{S(t)} \mathbf{W} \cdot \mathbf{n} dS, \quad \int_{V(t)} \frac{\partial \mathbf{W}_i}{\partial x_i} dV = \int_{S(t)} \mathbf{W}_i n_i dS \quad (1.13)$$

Hence, using (1.13) the momentum conservation law (1.12) becomes

$$\frac{d}{dt} \int_{V(t)} \rho u_i dV = \int_{V(t)} \left(\frac{\partial \tau_{ij}}{\partial x_j} + F_i \right) dV \quad (1.14)$$

1.5 The Equation of Continuity

Let us now consider the special case in which $V(t)$ is a material volume always composed of the same fluid particles, while the surface $S(t)$ in which the volume is contained moves with the same normal velocity as the fluid and $U_n = \mathbf{U} \cdot \mathbf{n} = u_i n_i$. Accordingly, using (1.1), (1.7), and the Gauss theorem (1.13) it follows that

$$\frac{d}{dt} \int_{V(t)} f dV = \int_{V(t)} \frac{\partial f}{\partial t} dV + \int_{S(t)} f u_i n_i dV = \int_{V(t)} \left[\frac{\partial f}{\partial t} + \frac{\partial (f u_i)}{\partial x_i} \right] dV \quad (1.15)$$

Returning to the mass conservation (1.11), (1.15) for $f = \rho$ yields

$$\frac{d}{dt} \int_{V(t)} \rho dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} \right] dV = 0 \quad (1.16)$$

Clearly, (1.16) should hold for an arbitrary group of fluid particles and for any instant of time t . Therefore the integrand itself should be zero, resulting in

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \quad (1.17)$$

Hence for an incompressible fluid and constant density, (1.17) yields readily the so-called equation of continuity, which reads

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \nabla \cdot \mathbf{U} = 0 \quad (1.18)$$

1.6 Euler Equations

If the transport theorem (1.7) is applied to the momentum conservation law (1.12), namely using ρu_i instead of ρ , it follows that

$$\int_{V(t)} \left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} \right] dV = \int_{V(t)} \left(\frac{\partial \tau_{ij}}{\partial x_j} + F_i \right) dV \quad (1.19)$$

or

$$\int_{V(t)} \left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} - F_i \right] dV = 0 \quad (1.20)$$

Again, for an arbitrary group of fluid particles and for any instant of time, (1.20) is satisfied if the integrand is zero. Thus

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} + F_i \quad (1.21)$$

Finally, for a constant density ρ and using the equation of continuity (1.18), (1.21) yields the Euler equations, i.e.,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{\rho} F_i \quad (1.22)$$

1.7 Stress Relations in a Newtonian Fluid

Inevitably, the stress tensor and the kinematical properties of the fluid are tightly correlated. Analogous correlations hold for the stress–strain relations in solid mechanics. Assuming that the fluid is at rest, then no shear stresses are developed between the fluid particles and only a normal pressure component (pressure stress) will exist within the fluid. With reference to Figure 1.2, it is deduced that in order to balance the forces acting across a small tetrahedron, the normal pressure must be isotropic. Given that no viscous shear was considered, the stress tensor will be composed only of the pressure stress, i.e.,

$$\tau_{ij} = -p\delta_{ij} \quad (1.23)$$

By hypothesis, there are no viscous forces if the fluid moves as a rigid mass without deformation, or with the velocity field

$$\mathbf{U} = \mathbf{A} + \mathbf{B} \times \mathbf{r} \quad (1.24)$$

where \mathbf{A} and \mathbf{B} are constant vectors equal to the translational and rotational velocities and \mathbf{r} is the position vector from the origin of the rotation. Viscous stresses will occur when the fluid velocity differs from (1.24) with relative motion between adjacent fluid particles. The simplest example is a uniform shear flow. Here, and for more general velocity fields, the fundamental assumption of a Newtonian fluid is that the stress tensor is a linear function of the nine gradients $\partial u_k / \partial x_j$. This ensures vanishing of the viscous stress components for uniform translation of the fluid. The rotational term in (1.24) will be stress-free provided the gradients occur only in the form of sums $\partial u_k / \partial x_i + \partial u_i / \partial x_k$. For an isotropic fluid, the values of the stress components must be independent of the choice of coordinates, and for flow in one plane there can be no shear stress in the direction normal to this plane. The most general linear function of the velocity gradients, consistent with these conditions and the requirement that τ_{ij} is symmetric, is of the form

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i \neq j \quad (1.25)$$

where μ is the viscous shear coefficient, also known as the coefficient of viscosity.

In order for τ_{ij} be a tensor, the only possible addition to (1.25) for $i = j$ is a second constant times the divergence $\partial u_i / \partial x_i$, which vanishes for an incompressible fluid. Thus, the total stress tensor for an incompressible fluid is given by

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1.26)$$

or in Cartesian representation

$$[\tau_{ij}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \mu \begin{bmatrix} 2 \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2 \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2 \frac{\partial w}{\partial z} \end{bmatrix} \quad (1.27)$$

The diagonal of the first matrix in (1.27) is occupied by the normal pressure stress. The second matrix is the viscous stress tensor and is proportional to the viscous shear coefficient μ . The diagonal elements of the second matrix are associated with the elongations of fluid elements. The off-diagonal elements are associated with shear deformations.

1.8 The Navier–Stokes Equations

Having derived the Euler equations and expressed the stress–strain relations via (1.25) the Navier–Stokes equations are obtained by direct substitution of (1.25) into (1.22). The Navier–Stokes equations express the conservation of momentum of a Newtonian fluid. Given that the equation of continuity requires that $\frac{\partial^2 u_j}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} = 0$, the derivatives of the stress tensor will read

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (1.28)$$

Hence, the Navier–Stokes equations are written as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\rho} F_i \quad (1.29)$$

where $\nu = \mu / \rho$ is the coefficient of the kinematic viscosity. The Navier–Stokes equations are often written in a vectorial form according to

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{U} + \frac{1}{\rho} \mathbf{F} \quad (1.30)$$

Finally, the Cartesian representation of the Navier–Stokes equations is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \frac{1}{\rho} F_x \quad (1.31)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v + \frac{1}{\rho} F_y \quad (1.32)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w + \frac{1}{\rho} F_z \quad (1.33)$$

The motion of a viscous fluid with constant density and a Newtonian stress–strain relation is fully described by the system of partial differential equations (1.31)–(1.33) and the equation of continuity (1.18). Most common fluids, including water and air, appear to comply with the requirement of a Newtonian stress–strain relation. Exceptions occur for nonisotropic fluids, in which case the stress–strain relation is said to be “non-Newtonian.” Even in the former case, however, the solution of the Navier–Stokes equations raises major difficulties. Major challenges originate from the fact that those equations literally form a coupled nonlinear system that has been solved analytically only for some very simple geometrical configurations, principally those in which the nonlinear convective acceleration terms $(\mathbf{U} \cdot \nabla)\mathbf{U}$ can be assumed to vanish (Newman, 1977).

1.9 Inviscid, Incompressible Fluid and Irrotational Flow: The Velocity Potential

1.9.1 Inviscid, Incompressible

Those assumptions lead to the concept of an ideal fluid and eventually to potential flow. Incompressibility is secured by the constant density of the fluid $\rho = \text{const}$. In addition, for a nonviscous fluid, the Navier–Stokes equations are reduced to

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} F_i \quad (1.34)$$

Further, we assume that the external force F_i is represented by only the gravitational force ρg (g is the gravitational acceleration), which is directed vertically downward along the $z \equiv x_3$ coordinate. Letting the $z \equiv x_3$ coordinate pointing upward, it follows that $F_i = (0, 0, -\rho g)$. Hence (1.34) may be rewritten as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial}{\partial x_i} (p + \rho g x_3) \quad (1.35)$$

1.9.2 Irrotational Flow: The Velocity Potential

The correlation of the velocity potential with the flow field is realized through its gradient, which provides the velocity vector of the fluid particles. The gain from using the velocity potential is that instead of three components (namely the velocities in all directions of the three-dimensional space) only one is needed, which apparently depends on the three spatial coordinates and the time. Further, although the velocities have an obvious physical meaning, the potential is somehow a vague term that is materialized physically through its correlation with the velocities. Nevertheless, it

can be proved that the potential is indeed an effective representation of the velocity field. To this end, only the assumption of the irrotational flow is required.

In particular let us consider the definite integral

$$\Phi(\mathbf{x}, t) = \int_{\mathbf{x}_0}^{\mathbf{x}} u_i dx_i \quad (1.36)$$

where the lower limit is some arbitrary constant position \mathbf{x}_0 and the upper limit is the point $\mathbf{x} = (x_1, x_2, x_3)$. This integral is independent of the particular path of integration between the points \mathbf{x}_0 and \mathbf{x} , since the difference in value of any two integrals, between the same two points, is equal to circulation around a closed path from \mathbf{x}_0 to \mathbf{x} , along one path and back to \mathbf{x}_0 along the other path, which is equal to zero if the fluid motion is irrotational. Thus, the integration in (1.36) can be performed along any desired path. If we choose a path that approaches the point \mathbf{x} , along a straight line parallel to the x_1 -axis, then along the final portion of the path of integration $u_i dx_i = u_1 dx_1$, so that

$$\frac{\partial \Phi}{\partial x_1} = \frac{\partial}{\partial x_1} \int_{\mathbf{x}_0}^{\mathbf{x}} u_1 dx_1 = u_1 \quad (1.37)$$

The remaining portion of the integral being a constant, does not contribute to the derivative. Applying a similar argument for the other two coordinates, we have in general that

$$u_i = \frac{\partial \Phi}{\partial x_i}, \quad \mathbf{U} = \nabla \Phi \quad (1.38)$$

The existence of the velocity potential therefore is univocally justified by the assumption of an irrotational flow.

Substituting (1.38) into the equation of continuity (1.18) immediately yields

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_i} = 0, \quad \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (1.39)$$

Equation (1.39) is the Laplace equation given in tensorial and Cartesian representations and $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$ is the Laplace operator.

1.9.3 The Unsteady Bernoulli Equation

The Laplace equation is the governing equation of the flow field under the basic assumptions made in this section. It will also be referred to as the *field equation*, which we recall is the outcome of the equation of continuity. To complete the problem's setup within the realm of potential theory we must exploit (1.35), which eventually provides the pressure field that can be also univocally obtained by the potential function only. We are particularly interested for the unsteady case,

including the time derivative term of (1.35). The steady case is a special case in which this term is absent. Using (1.38), (1.35) yields

$$\frac{\partial}{\partial t} \frac{\partial \Phi}{\partial x_i} + \frac{\partial \Phi}{\partial x_j} \frac{\partial}{\partial x_j} \frac{\partial \Phi}{\partial x_i} = -\frac{1}{\rho} \frac{\partial}{\partial x_i} (p + \rho g x_3) \quad (1.40)$$

It can be easily shown that

$$\frac{1}{2} \frac{\partial}{\partial x_i} \frac{\partial \Phi}{\partial x_j} \frac{\partial \Phi}{\partial x_j} = \frac{\partial \Phi}{\partial x_j} \frac{\partial}{\partial x_j} \frac{\partial \Phi}{\partial x_i} \quad (1.41)$$

Substituting (1.41) into (1.40) and rearranging terms yields

$$\frac{\partial}{\partial x_i} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\partial \Phi}{\partial x_j} \frac{\partial \Phi}{\partial x_j} + \frac{1}{\rho} (p + \rho g x_3) \right] = 0 \quad (1.42)$$

Finally, integrating with respect to x_i for $i = 1, 2, 3$ one gets

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\partial \Phi}{\partial x_j} \frac{\partial \Phi}{\partial x_j} = -\frac{1}{\rho} (p + \rho g x_3) + C(t) \quad (1.43)$$

The integration with respect to x_i , $i = 1, 2, 3$ will produce an integration factor that is a function of time $C(t)$. This can be incorporated into the velocity potential, writing

$$\Phi' = \Phi - \int_0^t C(\xi) d\xi \quad (1.44)$$

Finally, using (1.44) into (1.43) yields the celebrated unsteady Bernoulli equation

$$\frac{\partial \Phi'}{\partial t} + \frac{1}{2} \frac{\partial \Phi'}{\partial x_j} \frac{\partial \Phi'}{\partial x_j} = -\frac{1}{\rho} (p + \rho g x_3) \quad (1.45)$$

The prime will be dropped in the sequel. In the Cartesian frame of reference, (1.45) is written as

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla \Phi|^2 + \frac{p}{\rho} + g z = 0 \quad (1.46)$$

Hence, the flow field for incompressible and inviscid fluid and irrotational flow is fully described by potential flow theory via the Laplace equation (1.39) and the unsteady Bernoulli equation (1.46). Those equations describe explicitly the flow in a boundless medium. Practical hydrodynamic problems, however, involve always boundary conditions that are represented by mathematical constraints, i.e., differential or algebraic equations that must be satisfied together with the Laplace and the Bernoulli equations. Even the conditions at infinity can be regarded as boundary conditions. Thus, in hydrodynamics we are invited to consider boundary value

problems formed by several equations accompanying the fundamental equations (1.39) and (1.46).

1.10 Free-Surface Flow in the Laplace Domain

The hydrodynamical applications associated with the marine environment are in most of the cases fluid–structure interaction problems. In relevant situations, the factor that makes the difference is the existence of the free surface, i.e., the mutual surface that separates water and air. The effect of the free surface is of paramount importance in wave–structure interaction problems and also in hydrodynamical problems without incident waves. As an example, which will be analyzed extensively in Chapter 5, we refer to the case of a solid moving under an undisturbed free surface with no waves, namely without periodic or nonperiodic deformation of the free surface. This is the so-called “wave resistance” problem. Clearly, the motion of the solid affects the kinematical condition of the liquid up to the free surface especially in cases where the solid is in close proximity with the free surface. It is evident therefore that the free surface will affect the flow field, i.e., the velocities of the liquid particles and the pressure field exerted on the solid.

There are two main challenges associated with the existence of the free surface: (1) it imposes constraints, or in other words boundary conditions, which are in fact nonlinear and (2) the boundary where those conditions should be satisfied is time dependent and is one of the unknowns to be determined. Both issues are discussed in the present section.

In marine hydrodynamics the fluid is liquid. The domain of interest is the material volume of liquid below the free surface and the alterations of its kinematical condition(s) due to the kinematics of both the liquid (e.g., water waves, current) and the body(ies) that exist within the liquid. We start the analysis with the most typical hydrodynamical problem of a body with arbitrary geometry that floats on the free surface of a liquid field that extends to infinity both in x and y directions as shown in Figure 1.3 (in several cases it is most convenient to assume that the surface at infinity S_∞ is a cylindrical one, situated at $r \rightarrow \infty$, where r denotes the radial coordinate of a polar frame of reference). The free surface itself is a function of time and apparently of the horizontal plane coordinates (x, y) . Therefore a fixed and explicitly defined plane of reference is required to define the free-surface elevation H as a function of (x, y, t) . Given the fact that in the general case the topography of the bottom, which is also a fixed surface, can be arbitrary, the only alternative is to assume as a plane of reference the undisturbed free surface and define it on $z = 0$. It is evident that the schematic in Figure 1.3 assumes a plane horizontal bottom that is located at a distance h below the undisturbed free surface. The depth h can be finite or infinite.

The control volume Ω in Figure 1.3 is bounded from above by the free surface, coined FS , represented by the function $H(x, y, t)$, from below by the bottom surface, coined S_{BT} , and by the impermeable surface of the body S_0 . The basic condition is to

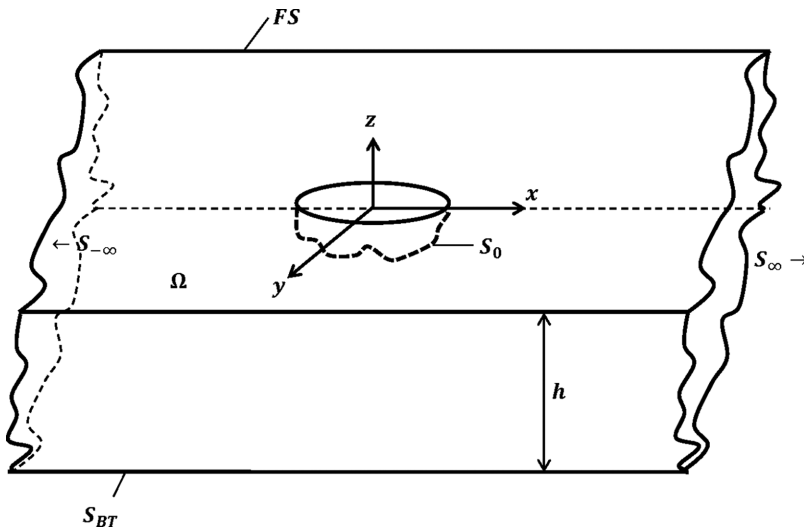


Figure 1.3 A solid that floats in a liquid field of infinite extent.

assume infinite extend for $(x, y) \in (-\infty, +\infty)$, while one can add more complications, e.g., by restricting the extent of the control volume, which is achieved by introducing additional boundary conditions, or putting more solids into the liquid field, floating, fully immersed or bottom seated.

According to the Laplace domain approach, namely inviscid, incompressible liquid and irrotational flow, the velocity field, described by the velocity components of the liquid particles (u, v, w) , in the three directions (x, y, z) respectively, is expressed in terms of the scalar velocity potential $\Phi(x, y, z, t)$ as

$$(u, v, w) = \nabla\Phi(x, y, z, t) = \frac{\partial u}{\partial x} \vec{i} + \frac{\partial v}{\partial y} \vec{j} + \frac{\partial w}{\partial z} \vec{k} \quad (1.47)$$

where $(\vec{i}, \vec{j}, \vec{k})$ are the unit vectors in (x, y, z) .

The basic hydrodynamic boundary value problem is thus defined by the following set of equations:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0, \quad (x, y, z) \in \Omega \quad (1.48)$$

$$\nabla\Phi \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} + \boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{n}), \quad (x, y, z) \in S_0 \quad (1.49)$$

$$\nabla\Phi \cdot \mathbf{n} = 0, \quad (x, y, z) \in S_{BT} \quad (1.50)$$

$$\frac{\partial^2\Phi}{\partial t^2} + g \frac{\partial\Phi}{\partial z} + 2\nabla\Phi \cdot \nabla \left(\frac{\partial\Phi}{\partial t} \right) + \frac{1}{2} \nabla\Phi \cdot \nabla (\nabla\Phi \cdot \nabla\Phi) = 0, \quad z = H(x, y, t) \quad (1.51)$$

$$H = -\frac{1}{g} \left(\frac{\partial\Phi}{\partial t} + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right), \quad z = H(x, y, t) \quad (1.52)$$

Equation (1.48) is the Laplace equation, (1.49) is the boundary condition that must be satisfied on the surface of the solid, (1.50) is the kinematic boundary condition on the bottom, (1.51) denotes the combined dynamic and kinematic boundary condition of the free surface, and finally (1.52) provides the exact position of the free surface. In (1.50) \mathbf{x} is the position vector of a point on S_0 measured from the center of rotation, while $\mathbf{v} = (v_1, v_2, v_3)$ and $\boldsymbol{\omega} = (v_4, v_5, v_6)$ are the translational and rotational vectors of the solid in surge, sway, heave, roll, pitch, and yaw respectively. Clearly \mathbf{n} represents the unit normal vector upon the associated impermeable surface that is pointing out of the fluid domain.

More detailed discussion is required for the free-surface condition (1.51). The above boundary value problem should be completed by an appropriate radiation condition at infinity, i.e., as $r = \sqrt{x^2 + y^2} \rightarrow \infty$.

1.11 Free-Surface Kinematics

The conditions associated with the presence of the free surface are the so-called dynamic and kinematic boundary conditions. Omitting the effect of the surface tension, the former condition implies that the pressure on the free surface should be continuous. The pressure at any point in the liquid is obtained through the Bernoulli equation (1.46) in which p will denote the pressure of the liquid, relative to the atmospheric pressure. As the density of the air is very small compared to the density of the liquid, the motion of the air can be neglected, implying no dynamic fluctuations of the pressure on the free surface. Therefore, p can be taken equal to zero and accordingly the dynamic condition on the free surface, evaluated on $z = H(x, y, t)$, becomes

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla \Phi|^2 + gH = 0, \quad z = H(x, y, t) \quad (1.53)$$

Also, continuity of the interface (the surface between the air and the liquid) requires that the liquid particles on the interface remain on it during the motion of the free surface. That is expressed mathematically by

$$\frac{D(H-z)}{Dt} = 0, \quad \frac{\partial H}{\partial t} + \nabla \Phi \cdot \nabla H = \frac{\partial \Phi}{\partial z}, \quad z = H(x, y, t) \quad (1.54)$$

where D/Dt denotes the material derivative. Next, we take the gradient and the time derivative of (1.53) to yield

$$\nabla \left(\frac{\partial \Phi}{\partial t} \right) + \frac{1}{2} \nabla \cdot (\nabla \Phi \cdot \nabla \Phi) + g \nabla H = 0, \quad z = H(x, y, t) \quad (1.55)$$

and

$$\frac{\partial^2 \Phi}{\partial t^2} + \nabla \Phi \cdot \nabla \left(\frac{\partial \Phi}{\partial t} \right) + g \frac{\partial H}{\partial t} = 0, \quad z = H(x, y, t) \quad (1.56)$$

Substituting ∇H from (1.55) into (1.54) and the resulting expression into (1.56) establishes (1.51). The exact position of the free surface is obtained directly from (1.53), which yields (1.52). Equation (1.51) is the fundamental, combined dynamic and kinematic free surface boundary condition. The fact that it has been expressed solely in terms of the velocity potential allows some simplification of the intended hydrodynamic problems, which that way, could be considered only in terms of the potential.

1.12 The Taylor Expansion of the Free Surface

The main problem associated with the free surface is that its conditions should be valid in a time-varying boundary, namely on $z = H(x, y, t)$. To overcome this difficulty, Stokes suggested the expansion of the boundary conditions (assuming small displacements of the free surface) around a mean position that coincides with the undisturbed free surface at $z = 0$. Therefore, any function $f(z)$ evaluated on $z = H(x, y, t)$ is written as

$$f(z)|_{z=H} = f(z)|_{z=0} + H \left. \frac{\partial f(z)}{\partial z} \right|_{z=0} + \frac{1}{2} H^2 \left. \frac{\partial^2 f(z)}{\partial z^2} \right|_{z=0} + O(H^3) \quad (1.57)$$

Accordingly, (1.52) gives for the free surface

$$H = -\frac{1}{g} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi - \frac{1}{g} \frac{\partial \Phi}{\partial t} \frac{\partial^2 \Phi}{\partial t \partial z} - \frac{1}{2g} \frac{\partial^2 \Phi}{\partial t \partial z} \nabla \Phi \cdot \nabla \Phi - \frac{1}{g} \frac{\partial \Phi}{\partial t} \nabla \Phi \cdot \nabla \left(\frac{\partial \Phi}{\partial z} \right) + \frac{1}{2g^2} \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial t} \frac{\partial^3 \Phi}{\partial t \partial z^2} \right] + O(\Phi^4) \quad (1.58)$$

while the boundary condition (1.51) becomes

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = & -2 \nabla \Phi \cdot \nabla \left(\frac{\partial \Phi}{\partial t} \right) - \frac{1}{2} \nabla \Phi \cdot \nabla (\nabla \Phi \cdot \nabla \Phi) + \frac{1}{g} \frac{\partial \Phi}{\partial t} \left(\frac{\partial^3 \Phi}{\partial t^2 \partial z} + g \frac{\partial^2 \Phi}{\partial z^2} \right) \\ & + \frac{2}{g} \frac{\partial \Phi}{\partial t} \left[\nabla \left(\frac{\partial \Phi}{\partial z} \right) \cdot \nabla \left(\frac{\partial \Phi}{\partial t} \right) + \nabla \Phi \cdot \nabla \left(\frac{\partial^2 \Phi}{\partial t \partial z} \right) \right] \\ & - \frac{1}{g} \left(\frac{1}{g} \frac{\partial \Phi}{\partial t} \frac{\partial^2 \Phi}{\partial t \partial z} - \frac{1}{2} \nabla \Phi \cdot \nabla \Phi \right) \left(\frac{\partial^3 \Phi}{\partial t^2 \partial z} + g \frac{\partial^2 \Phi}{\partial z^2} \right) \\ & - \frac{1}{2g^2} \left(\frac{\partial \Phi}{\partial t} \right)^2 \left(\frac{\partial^4 \Phi}{\partial t^2 \partial z^2} + g \frac{\partial^3 \Phi}{\partial z^3} \right) + O(\Phi^4) \end{aligned} \quad (1.59)$$

Both (1.58) and (1.59) have been expressed on the known boundary of the undisturbed free surface on $z = 0$.

1.13 Expansion in Perturbations

Although the Taylor series expansion tackles the problem of the time-varying boundary in which the free-surface conditions should apply, it is unable to simplify these conditions, which remain strongly nonlinear. Admittedly, processing of non-linear systems is a very challenging task. Consideration of (1.59) in its complete form is feasible only in the time domain. The frequency domain approximation requires further simplifications, which in hydrodynamics (and other scientific disciplines) are employed using expansions in series of perturbations (e.g., Kevorkian and Cole, 1981). The expansion of the velocity potential (and the other hydrodynamic components) in perturbation series was first employed by Stokes and accordingly the perturbation scheme is widely referred as *Stokes perturbations* corresponding to the sequence of Stokes waves. Use of expansions in perturbations requires a small parameter (the scaling factor) that is the problem's property and preferentially known. In free-surface hydrodynamics, this role is assumed by the wave steepness $\epsilon = k_0 A \ll 1$, where k_0 denotes the wavenumber (being equal to $2\pi/\lambda$) of the regular waves propagating in a liquid field of specified water depth (λ is the wave length) and A denotes the linear amplitude of the regular waves. Thus, using the scaling factor ϵ , the velocity potential is expressed as an infinite sum according to

$$\Phi(x, y, z, t) = \sum_{n=1}^{\infty} \epsilon^n \Phi^{(n)}(x, y, z, t) \quad (1.60)$$

There is a direct correlation between the wavenumber k_0 and the (circular) frequency ω of the propagating waves expressed through the so-called “dispersion” relation, which will be discussed in the sequel. This correlation involves the water depth as well, which modifies the wave length for the same wave frequency. More details will be given in the sequel. Therefore, the actual reasoning behind the expansion (1.60) in terms of the wave steepness is that the total velocity potential involves terms (powers of ϵ) depending on the pairs (A, ω) , $(A^2, 2\omega)$, $(A^3, 3\omega)$, and so on.

Further, assuming that at the leading (first-) order, the potential varies periodically with circular frequency ω , we can write

$$\epsilon \Phi^{(1)}(x, y, z, t) = \text{Re} \left[\phi^{(1)}(x, y, z) e^{-i\omega t} \right] \quad (1.61)$$

where Re denotes the real part of a complex argument. The nonlinear products of the inhomogeneous term in (1.59) suggest taking the following forms for the second- and the third-order potentials (phenomena higher than third-order will not be considered):

$$\epsilon^2 \Phi^{(2)}(x, y, z, t) = \bar{\phi}^{(2)}(x, y, z) + \text{Re} \left[\phi^{(2)}(x, y, z) e^{-2i\omega t} \right] \quad (1.62)$$

$$\epsilon^3 \Phi^{(3)}(x, y, z, t) = \text{Re} \left[\bar{\phi}^{(3)}(x, y, z) e^{-i\omega t} \right] + \text{Re} \left[\phi^{(3)}(x, y, z) e^{-3i\omega t} \right] \quad (1.63)$$

Working the same way, the free-surface elevation is expanded in a series of perturbations according to

$$H(x, y, t) = \sum_{n=1}^{\infty} \epsilon^n H^{(n)}(x, y, t) = \operatorname{Re} \left[\eta^{(1)}(x, y) e^{-i\omega t} \right] + \bar{\eta}^{(2)}(x, y) + \operatorname{Re} \left[\eta^{(2)}(x, y) e^{-2i\omega t} \right] + \operatorname{Re} \left[\bar{\eta}^{(3)}(x, y) e^{-i\omega t} \right] + \operatorname{Re} \left[\eta^{(3)}(x, y) e^{-3i\omega t} \right] + O(\epsilon^4) \quad (1.64)$$

Clearly, the second-order (double frequency) problem involves a steady component, while the third-order problem embraces also a term that varies with the leading order frequency ω .

The next step is to introduce (1.61)–(1.63) into the free-surface boundary condition (1.59), and (1.64) into the free-surface elevation (1.58). Doing so and equating like powers of ϵ (up to ϵ^3) will result in the following boundary conditions and free surface elevations at the various orders.

Leading (first-) order problem $O(\epsilon)$

$$-K\phi^{(1)} + \frac{\partial\phi^{(1)}}{\partial z} = 0 \quad (1.65)$$

$$\eta^{(1)} = \frac{i\omega}{g} \phi^{(1)} \quad (1.66)$$

Second-order problem $O(\epsilon^2)$

$$-4K\phi^{(2)} + \frac{\partial\phi^{(2)}}{\partial z} = \frac{i\omega}{g} \left[\nabla\phi^{(1)} \cdot \nabla\phi^{(1)} - \frac{1}{2}\phi^{(1)} \left(\frac{\partial^2\phi^{(1)}}{\partial z^2} - K \frac{\partial\phi^{(1)}}{\partial z} \right) \right] \quad (1.67)$$

$$\eta^{(2)} = \frac{2i\omega}{g} \phi^{(2)} - \frac{1}{4g} \nabla\phi^{(1)} \cdot \nabla\phi^{(1)} - \frac{K^2}{2g} \phi^{(1)}\phi^{(1)} \quad (1.68)$$

$$\bar{\eta}^{(2)} = -\frac{1}{4g} \nabla\phi^{(1)} \cdot \nabla\phi^{(1)*} + \frac{K^2}{2g} \phi^{(1)}\phi^{(1)*} \quad (1.69)$$

where the asterisk denotes the equivalent complex conjugate (*cc*) of the associate component.

Third-order problem $O(\epsilon^3)$

$$\begin{aligned} -9K\phi^{(3)} + \frac{\partial\phi^{(3)}}{\partial z} &= \frac{3i\omega}{g} \nabla\phi^{(2)} \cdot \nabla\phi^{(1)} \\ &- \frac{i\omega}{2g} \left[\phi^{(1)} \left(\frac{\partial^2\phi^{(2)}}{\partial z^2} - 4K \frac{\partial\phi^{(2)}}{\partial z} \right) + 2\phi^{(2)} \left(\frac{\partial^2\phi^{(1)}}{\partial z^2} - K \frac{\partial\phi^{(1)}}{\partial z} \right) \right] \\ &- \frac{1}{8g} \nabla\phi^{(1)} \cdot \nabla(\nabla\phi^{(1)} \cdot \nabla\phi^{(1)}) - \frac{K}{g} \phi^{(1)} \nabla\phi^{(1)} \cdot \nabla \left(\frac{\partial\phi^{(1)}}{\partial z} \right) \\ &+ \frac{1}{4g} \left(K\phi^{(1)} \frac{\partial\phi^{(1)}}{\partial z} + \frac{1}{2} \nabla\phi^{(1)} \cdot \nabla\phi^{(1)} \right) \left(\frac{\partial^2\phi^{(1)}}{\partial z^2} - K \frac{\partial\phi^{(1)}}{\partial z} \right) \end{aligned} \quad (1.70)$$

$$\eta^{(3)} = \frac{3i\omega}{g} \phi^{(3)} - \frac{1}{2g} \nabla \phi^{(1)} \cdot \nabla \phi^{(2)} - \frac{K}{g} \phi^{(1)} \frac{\partial \phi^{(2)}}{\partial z} - \frac{K}{g} \phi^{(2)} \frac{\partial \phi^{(1)}}{\partial z} - \frac{i\omega}{8g^2} \frac{\partial \phi^{(1)}}{\partial z} \nabla \phi^{(1)} \cdot \nabla \phi^{(1)} - \frac{i\omega}{4g^2} \phi^{(1)} \nabla \phi^{(1)} \cdot \nabla \left(\frac{\partial \phi^{(1)}}{\partial z} \right) - \frac{i\omega^3}{8g^3} \phi^{(1)} \phi^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial z^2} \quad (1.71)$$

$$\begin{aligned} \bar{\eta}^{(3)} = & -\frac{1}{g} \nabla \phi^{(1)} \cdot \nabla \bar{\phi}^{(2)} - \frac{1}{2g} \nabla \phi^{(1)*} \cdot \nabla \phi^{(2)} + \\ & \frac{K}{g} \phi^{(1)*} \frac{\partial \phi^{(2)}}{\partial z} + \frac{K}{g} \phi^{(2)} \frac{\partial \phi^{(1)*}}{\partial z} - \frac{i\omega}{8g^2} \left[2 \frac{\partial \phi^{(1)}}{\partial z} \nabla \phi^{(1)} \cdot \nabla \phi^{(1)*} - \frac{\partial \phi^{(1)*}}{\partial z} \nabla \phi^{(1)} \cdot \nabla \phi^{(1)} \right] \\ & - \frac{i\omega}{4g^2} \left[\phi^{(1)} \nabla \phi^{(1)} \cdot \nabla \left(\frac{\partial \phi^{(1)*}}{\partial z} \right) + \phi^{(1)} \nabla \phi^{(1)*} \cdot \nabla \left(\frac{\partial \phi^{(1)}}{\partial z} \right) - \phi^{(1)*} \nabla \phi^{(1)} \cdot \nabla \left(\frac{\partial \phi^{(1)}}{\partial z} \right) \right] \\ & + \frac{i\omega^3}{8g^3} \left[\phi^{(1)} \phi^{(1)} \frac{\partial^2 \phi^{(1)*}}{\partial z^2} + 2\phi^{(1)} \phi^{(1)*} \frac{\partial^2 \phi^{(1)}}{\partial z^2} \right] \end{aligned} \quad (1.72)$$

where $K = \omega^2 / g$. Higher-order problems are governed by associated relations of the form

$$-n^2 K \phi^{(n)} + \frac{\partial \phi^{(n)}}{\partial z} = C^{(n-1)}, \quad (1.73)$$

$$\eta^{(n)} = \frac{ni\omega}{g} \phi^{(n)} + \mathcal{D}^{(n-1)}$$

where $C^{(n-1)}$ and $\mathcal{D}^{(n-1)}$ denote expressions that involve terms up to $(n - 1)$ order.

Equations (1.65), (1.67), and (1.70) are boundary conditions that should hold on $z = 0$, while (1.66), (1.68), (1.69), (1.71), and (1.72) should be evaluated on $z = 0$.

We recall that all potentials, at various orders, must satisfy the Laplace equation and appropriate far-field radiation conditions. Finally, the total velocity potentials should account for the boundary conditions on the fixed boundaries of the liquid field (1.49) and (1.50).

1.14 Diffraction and Radiation Potentials at the Leading Order

In the realm of the linear theory it is common to decompose the total velocity potential at the leading order into three components: the incident wave component, the diffraction component, and finally the radiation component. These are denoted by Φ_I , Φ_D and Φ_R , respectively. The sum $\Phi_s = \Phi_D + \Phi_R$ is usually called the *Scattered* velocity potential. The suggested decomposition has the meaning that any structure that is subjected to incident propagating waves causes the scattering phenomenon, which is realized by waves diffracted and radiated to infinity.

The physical acceptance of the decomposition is justified by the sequence of events that originate from the interaction of waves with the solid that exists in the

wave field. Let us assume a regular wave train that is generated far away from the body, say at $-\infty$, after some time interacts with it and accordingly propagates far away from the body, say to $+\infty$. The interaction of the regular waves with the body causes the distortion of the wave field and rationally thinking the disturbance of the flow is more profound in the vicinity of the body. Note that the surface of the body is considered rigid and non-deformable. The details of the disturbance depend on the volume of the body, its geometry and shape. The disturbance caused by slender bodies, for example, is practically negligible.

The physical intuition is that any disturbance caused by the interaction should vanish far away from the body and accordingly the velocity field and the flow should be identical in $\pm\infty$. The interaction of the waves with the solid, which is assumed to move freely, will induce oscillatory motions in all six degrees of freedom. Those motions will produce additional disturbances that are superposed to those generated by the impact of waves on the solid. All disturbances should disappear far away from the body. Therefore, the complete phenomenon can be decomposed into three subphenomena: (1) the regular wave train, represented by the incident wave potential; (2) the waves (disturbance) generated assuming that the solid is fixed and immovable (diffracted waves); and (3) the disturbance caused only by the oscillatory motions of the rigid body (radiated waves). By the linear superposition principle, the complete phenomenon is composed by a superposition of the three discrete phenomena (1), (2), and (3). As a result, mathematical synthesis requires the linear superposition (simple summation) of the factors that describe the discrete flow fields, namely the individual velocity potentials. Hence, we may write

$$\Phi = \Phi_I + \Phi_D + \Phi_R = \text{Re}(\phi_I e^{-i\omega t}) + \text{Re}(\phi_D e^{-i\omega t}) + \text{Re}(\phi_R e^{-i\omega t}) \quad (1.74)$$

It has been assumed that the diffracted waves are generated assuming that the structure is fixed while the radiated waves originate due to the motions of the structure in the six degrees of freedom. Hence the body condition (1.49) is accordingly decomposed into

$$\frac{\partial \Phi_I}{\partial n} + \frac{\partial \Phi_D}{\partial n} = 0, \quad \frac{\partial \phi_I}{\partial n} + \frac{\partial \phi_D}{\partial n} = 0, \quad (x, y, z) \in S_0 \quad (1.75)$$

$$\frac{\partial \Phi_R}{\partial n} = V_n, \quad (x, y, z) \in S_0 \quad (1.76)$$

where

$$\frac{\partial \Phi}{\partial n} = \nabla \Phi \cdot \mathbf{n} \quad (1.77)$$

$$V_n = \mathbf{v} \cdot \mathbf{n} + \boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{n}) \quad (1.78)$$

Here S_0 denotes the submerged surface of the solid below the undisturbed free surface, assumed on $z = 0$. Further we let

$$V_n = \sum_{j=1}^6 v_j n_j \quad (1.79)$$

where n_j $j = 1, 2, \dots, 6$ are the surge, sway, heave, roll, pitch, and yaw components of the unit normal. For harmonic oscillations one may assume that

$$v_j = \text{Re}(U_j e^{-i\omega t}) \quad (1.80)$$

while the radiation potential can be decomposed in terms associated with the six degrees of freedom, i.e.,

$$\phi_R = \sum_{j=1}^6 U_j \phi_j \quad (1.81)$$

Finally, on combining together (1.76) and (1.79)–(1.81) immediately yields the boundary conditions of the six radiation (Kirchhoff) potentials in the six modes of motion, as

$$\frac{\partial \phi_j}{\partial n} = n_j, \quad (x, y, z) \in S_0, \quad j = 1, 2, \dots, 6 \quad (1.82)$$

1.15 The Incident Wave Potential

1.15.1 Leading Order Problem

The incident wave potential constitutes the major ambient, time-varying excitation in hydrodynamical free-surface flows. The associated velocity potential at leading order $\phi_I \equiv \phi_I^{(1)}$ has the following properties:

1. It satisfies the Laplace equation.
2. It is regular (periodic) with circular frequency ω in the entire liquid domain (should be an entire function of $r = \sqrt{x^2 + y^2}$).
3. It satisfies the kinematic condition on the bottom.
4. It satisfies the linearized (combined kinematic and dynamic) boundary condition on the free surface (1.65).
5. When the water depth is infinite, the velocity potential should be zero for $z \rightarrow \pm\infty$; the plus or minus sign depends on the orientation of the z -axis, which is assumed fixed on the undisturbed free surface.

At the leading order the incident wave potential for infinite water depth is given by (z is assumed pointing upward)

$$\phi_I = -i \frac{gA}{\omega} e^{Kz} e^{iK(x\cos\beta + y\sin\beta)} = -i \frac{gA}{\omega} e^{Kz} e^{iKr \cos(\theta - \beta)} \quad (1.83)$$

where β is the angle of propagation with respect to the horizontal x -axis while $x = r \cos \theta$ and $y = r \sin \theta$. In the general case of finite water depth h , we can derive

an analytic expression if the bottom is assumed flat and horizontal in the entire control volume. The bottom boundary condition for the incident wave potential thus becomes $\partial\phi_I/\partial z = 0$ on $z = -h$. The potential ϕ_I is expressed as

$$\phi_I = -i \frac{gA}{\omega} \frac{\cosh k_0(z+h)}{\cosh k_0 h} e^{ik_0(x\cos\beta+y\sin\beta)} = -i \frac{gA}{\omega} \frac{\cosh k_0(z+h)}{\cosh k_0 h} e^{ik_0 r \cos(\theta-\beta)} \quad (1.84)$$

where the wavenumber k_0 is given by the celebrated dispersion relation

$$k_0 \tanh(k_0 h) = K = \omega^2 / g \quad (1.85)$$

the derivation of which will be discussed in Chapter 2.

1.15.2 Second-Order Problem

The double-frequency incident wave component $\phi_I^{(2)}$ should comply with the aforementioned properties (1) and (3). Also, it should be regular with circular frequency 2ω in the entire fluid domain and finally should satisfy the free-surface boundary condition at second order (1.67). The right-hand-side term of (1.67) is constructed by terms involving products of incident ϕ_I and diffraction potentials ϕ_D at the leading order. The inhomogeneous (right-hand side) term that is considered for determining $\phi_I^{(2)}$ should involve products of ϕ_I only. This term, denoted by $Q_I^{(2)}$ and expressed in polar (r, θ) coordinates, is given by (Mei, 1983; Malenica and Molin, 1995; Malenica et al., 1999)

$$Q_I^{(2)}(r, \theta) = \frac{i\omega}{2g}(3K^2 - k_0^2)(\phi_I)^2 + \frac{i\omega}{g}(\nabla_0 \phi_I)^2 = \frac{3i\omega K A^2}{2\sinh^2 k_0 h} e^{2ik_0 r \cos(\theta-\beta)}, \quad z = 0 \quad (1.86)$$

Equation (1.86) utilizes the plane gradient convention $\nabla_0 = (\partial/\partial r, 1/r \partial/\partial \theta, 0)$.

This yields the following form for the double-frequency incident wave component:

$$\phi_I^{(2)} = -i \frac{3\omega A^2}{8} \frac{\cosh 2k_0(z+h)}{\sinh^4 k_0 h} e^{2ik_0 r \cos(\theta-\beta)} \quad (1.87)$$

1.15.3 Third-Order Problem

In a similar manner, the triple-frequency incident wave component should satisfy conditions (1) and (3) of Section 1.15.1 and should be an entire function of $r = \sqrt{x^2 + y^2}$ in the entire fluid domain with circular frequency 3ω . The free-surface boundary condition associated with the third-order problem has been given in (1.70). For the incident wave problem the inhomogeneous term of (1.70) will be constructed by products of ϕ_I and $\phi_I^{(2)}$ only. Hence assuming finite water depth and introducing (1.84) and (1.87) into the right-hand-side term of (1.70) yields the inhomogeneous term of the free-surface boundary condition for the triple frequency incident wave potential. This is (Malenica and Molin, 1995)

$$Q_I^{(3)}(r, \theta) = \frac{3i\omega k_0^2 A^3}{8 \sinh^4 k_0 h} (11 - 2 \cosh 2k_0 h) e^{3ik_0 r \cos(\theta - \beta)}, \quad z = 0 \quad (1.88)$$

while the solution for the component $\phi_I^{(3)}$ is

$$\phi_I^{(3)} = -i \frac{\omega k_0 A^3}{64} (11 - 2 \cosh 2k_0 h) \frac{\cosh 3k_0(z+h)}{\sinh^7 k_0 h} e^{3ik_0 r \cos(\theta - \beta)} \quad (1.89)$$

1.16 The Far-Field Radiation Condition

Also referred as the Sommerfeld radiation condition (Sommerfeld, 1949) is the condition that must be satisfied by the diffraction and the radiation components of the potential, ϕ_D , ϕ_R . The scattered wave field $\phi_s = \phi_D + \phi_R$ must decay at infinity, so that the flow in the far field is described only by the incident wave component. The far-field condition applies directly only to the first-order terms and is valid unconditionally for arbitrary geometries. In two dimensions, with the horizontal x -axis extending to infinity is expressed by

$$\lim_{kx \rightarrow \pm\infty} \left(\frac{\partial}{\partial x} \mp ik \right) G = 0 \quad (1.90)$$

where G denotes either ϕ_D or ϕ_R and k is the wavenumber, which is equal to either k_0 or K depending on the water depth. In three dimensions the associated radiation condition is

$$\lim_{kr \rightarrow \infty} r^{1/2} \left(\frac{\partial}{\partial r} - ik \right) G = 0 \quad (1.91)$$

The radiation conditions (1.90) and (1.91) require the waves at infinity to be progressing outwards and imposes a uniqueness that otherwise would not be present. Equation (1.91) describes circular waves of decreasing amplitude as $r \rightarrow \infty$ while the factor $r^{1/2}$ is required by energy conservation arguments.

The radiation condition for the second-order problem (diffraction potential in particular) has long been a controversial issue (Malenica and Molin, 1995). In fact, there is no generally accepted asymptotic formula although there have been studies that propose explicit formulations of (1.91) for the double-frequency diffraction problem (Liu and Miao, 1991). Instead, the existing theory focuses on the asymptotic behavior of the double-frequency diffraction component, here coined $\phi_D^{(2)}$. In Molin's original analysis (Molin, 1979) it was tentatively shown that the second-order diffraction potential, to the leading order $[O(A^2/r^{1/2})]$, consists of two components: waves "locked" to the first-order wave field (also known as the "locked" wave component, or the particular solution), and "free" waves (also known as the "free" wave component or homogeneous solution) traveling in the radial direction with wavenumber k_2 obtained by the second-order dispersion relation

$$k_2 \tanh(k_2 h) = 4K = 4\omega^2 / g \quad (1.92)$$

Even though the asymptotic behaviors of the particular and the homogeneous solutions were not established on purely rigorous mathematical foundations, they have now been accepted as being correct (Kim and Yue, 1989). The corresponding far-field asymptotic behaviors of the homogeneous and the particular solutions are

$$\phi_H \sim r^{-1/2} e^{ik_2 r} + O(r^{-3/2}), \quad r \gg 1 \quad (1.93)$$

$$\phi_P \sim r^{-1/2} P(\theta, z) e^{ik_0 r(1+\cos\theta)} + O(r^{-1}), \quad r \gg 1 \quad (1.94)$$

where

$$P(\theta, z) = p(\theta) \cosh \left\{ k_0 \left[2(1 + \cos \theta) \right]^{1/2} (z + h) \right\} + O(r^{-1/2}) \quad (1.95)$$

Equations (1.93) and (1.94) assume that the field equation and the condition on the flat horizontal bottom have been satisfied. Analogous expressions have also been found for the third-order diffraction component and its constituent terms (see Malenica and Molin, 1995). A detailed analysis regarding the decomposition of the second-order diffraction potential into “locked” and “free” wave components is provided in Chapter 3.

1.17 Hydrodynamic Loading

The hydrodynamic loading on the structure is taken by integrating the hydrodynamic pressure distribution over the wetted surface. The pressure is expressed through the Bernoulli equation given in (1.46). Exciting forces and moments require that the structure is considered to be fixed. Integration of pressure over the surface S_0 yields

$$F_j = \int_{S_0} p n_j dS \quad (1.96)$$

where, as usual, $j = 1, 2, \dots, 6$ denotes the generalized direction being considered. In order to collect the terms at the different orders of wave steepness ϵ , the integral on the wetted surface is decomposed according to

$$\int_{S_0} = \int_{S_{B0}} + \int_{\Delta S} = \int_{S_{B0}} + \int_{C_{B0}} \int_0^H \quad (1.97)$$

where S_{B0} is the mean wetted surface and C_{B0} the mean waterline. In (1.97), S_0 , S_{B0} , and ΔS denote surface integrals while C_{B0} denotes a line integral. Equation (1.97) has the meaning that the pressure is integrated over the mean wetted surface plus an additional part to account for the free-surface elevation above the undisturbed

free surface. The expansion in perturbations (1.60) and (1.64) suggests taking the following series expansion for the hydrodynamic loading as well:

$$F_j = \epsilon F_j^{(1)} + \epsilon^2 F_j^{(2)} + \epsilon^3 F_j^{(3)} + O(\epsilon^4) = \text{Re} \left\{ \mathcal{F}_j^{(1)} e^{-i\omega t} \right\} + \bar{\mathcal{F}}_j^{(2)} + \text{Re} \left\{ \mathcal{F}_j^{(2)} e^{-2i\omega t} \right\} + \text{Re} \left\{ \bar{\mathcal{F}}_j^{(3)} e^{-i\omega t} \right\} + \text{Re} \left\{ \mathcal{F}_j^{(3)} e^{-3i\omega t} \right\} + O(\epsilon^4) \tag{1.98}$$

Hence the following expressions for the time varying hydrodynamic loading up to the first three orders are derived:

$$O(\epsilon) \mathcal{F}_j^{(1)} = \int_{S_{B0}} i\omega\rho\phi^{(1)}n_j dS \tag{1.99}$$

$$O(\epsilon^2) \mathcal{F}_j^{(2)} = \int_{S_{B0}} \left[2i\omega\rho\phi^{(2)} - \frac{1}{4}\rho\nabla\phi^{(1)} \cdot \nabla\phi^{(1)} \right] n_j dS + \frac{1}{4}\rho g \int_{C_{B0}} \eta^{(1)}\eta^{(1)}n_j dC \tag{1.100}$$

$$O(\epsilon^3) \mathcal{F}_j^{(3)} = \int_{S_{B0}} \left[3i\omega\rho\phi^{(3)} - \frac{1}{2}\rho\nabla\phi^{(1)} \cdot \nabla\phi^{(2)} \right] n_j dS + \frac{1}{2}\rho g \int_{C_{B0}} \eta^{(1)} \left[\eta^{(2)} - \frac{1}{4}K\eta^{(1)}\eta^{(1)} \right] n_j dC \tag{1.101}$$

1.18 Added Mass and Hydrodynamic Damping Coefficients

The component of the hydrodynamic loading that arises from the pressure integration associated with the forced oscillations with frequency ω and the radiation component can be written as (see also Newman, 1977, Section 6.17; Linton and McIver, 2001, Section 1.3.4)

$$f_{ij} = i\omega\rho \int_{S_{B0}} \phi_i n_j dS \tag{1.102}$$

The factor f_{ij} , being by default a complex number, is convenient to be decomposed into a real and an imaginary part as

$$f_{ij} = i\omega \left(\mu_{ij} + i \frac{\lambda_{ij}}{\omega} \right) \tag{1.103}$$

yielding

$$\mu_{ij} + i \frac{\lambda_{ij}}{\omega} = \rho \int_{S_{B0}} \phi_i n_j dS \tag{1.104}$$

The term μ_{ij} is known as the added mass coefficient given that it represents a force that is proportional to acceleration. In contrast, λ_{ij} represents a force that

is proportional to the velocity of the body and thus it is called the hydrodynamic damping coefficient. In fact they represent the associated hydrodynamic parameter in direction i due to the oscillatory motion of the body with frequency ω and small amplitude in the direction j .

From the physical point of view the added mass is a virtual mass that depends on the geometry and the volume of the body, and apparently the density of the liquid, and expresses the added inertia introduced into the oscillatory system because of the volume of liquid that is displaced as the body is accelerating. In other words, in order for the body to move inside the liquid must first accelerate some volume of it. In the same manner, the hydrodynamic damping coefficient expresses a quantity of added damping introduced into the system. The hydrodynamic damping is defined in the form of a linear component and is directly proportional to the velocity.

1.19 The Green's Theorem

The Green's theorem follows from Green's second identity that is derived in the following manner. Let ϕ and G be continuously differentiable functions in a region Ω bounded by a surface S . Both ϕ and G have continuous partial derivatives of the second order in Ω . Then the pair of functions ϕ and G satisfy the following (Kellogg, 1967; p. 212)

$$\int_{\Omega} \phi \nabla^2 G \, dV + \int_{\Omega} (\nabla G \cdot \nabla \phi) \, dV = \int_S \phi \frac{\partial G}{\partial n} \, dS \quad (1.105)$$

If we interchange ϕ and G and the resulting equation is subtracted from (1.105) the result is Green's second identity, namely

$$\int_{\Omega} (\phi \nabla^2 G - G \nabla^2 \phi) \, dV = \int_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) \, dS \quad (1.106)$$

where n is a coordinate directed in the outward normal direction to the surface S . If ϕ and G are both solutions of the Laplace or Helmholtz equation, the latter being given by

$$\nabla^2 U + k^2 U = 0, \quad U = \phi, G \quad (1.107)$$

in the region Ω , for an arbitrary k [for $k = 0$, (1.107) yields the Laplace equation] then (1.106) is reduced to

$$\int_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) \, dS = 0 \quad (1.108)$$

which is widely referred as the Green's theorem.

Let us next consider the two-dimensional problem of the function ϕ at a particular point $P \in \Omega$, $\phi(P)$, so that the domain Ω is some surface bounded by the closed

curve S , as shown in Figure 1.4. We use capital letters to denote points within Ω and lowercase letters to denote points on the boundary S of Ω . The distance between P and an arbitrary point Q within Ω is denoted by R_{pQ} , while the Green's function is now denoted by $G(P, Q)$. The Green's function must be singular at Q , i.e., for $R_{pQ} \rightarrow 0$.

Our goal is to calculate the value of the function ϕ at point Q , $\phi(Q)$ and therefore we consider a point upon S , coined p . Given that $G(p, Q)$ is singular as $R_{pQ} \rightarrow 0$, the point Q must be excluded from region Ω . Therefore, we define a small circle S' embracing Q , with radius $\tilde{n} \rightarrow 0$. Further, use is made of the Green's theorem to ϕ and G over the region between S and S' . Thus the line of integration is $S \cup S'$ and we write

$$\begin{aligned} & \int_{S \cup S'} \left[\phi(p) \frac{\partial G(p, Q)}{\partial n_p} - G(p, Q) \frac{\partial \phi(p)}{\partial n_p} \right] dS \\ &= \int_S \left[\phi(p) \frac{\partial G(p, Q)}{\partial n_p} - G(p, Q) \frac{\partial \phi(p)}{\partial n_p} \right] dS \\ &+ \lim_{\tilde{n} \rightarrow 0} \int_{S'} \left[\phi(p) \frac{\partial G(p, Q)}{\partial n_p} - G(p, Q) \frac{\partial \phi(p)}{\partial n_p} \right] dS = 0 \quad (1.109) \end{aligned}$$

Here, n_p denotes the outward normal to the surface S located on the point p . Elaborating further (1.109) we obtain

$$\begin{aligned} & \int_S \left[\phi(p) \frac{\partial G(p, Q)}{\partial n_p} - G(p, Q) \frac{\partial \phi(p)}{\partial n_p} \right] dS \\ &+ \lim_{\tilde{n} \rightarrow 0} \int_0^{2\pi} \left\{ \phi(p) \left[-\frac{\partial G(p, Q)}{\partial R_{pQ}} \right] - G(p, Q) \left[-\frac{\partial \phi(p)}{\partial R_{pQ}} \right] \right\} \tilde{n} d\theta = 0 \quad (1.110) \end{aligned}$$

The second term in the left member of (1.110) results in the value $-\phi(Q)$ if $G(p, Q)$ behaves in such a way that

$$\lim_{\tilde{n} \rightarrow 0} \left[\tilde{n} G(p, Q) \right]_{R_{pQ}=\tilde{n}} = 0 \quad (1.111)$$

and

$$\lim_{\tilde{n} \rightarrow 0} \left[\tilde{n} \frac{\partial G(p, Q)}{\partial R_{pQ}} \right]_{R_{pQ}=\tilde{n}} \neq 0 \quad (1.112)$$

and yet is finite. In particular (1.112) should give 2π , while it is recalled that $G(p, Q)$ must be singular for $R_{pQ} \rightarrow 0$. All these requirements are satisfied by taking

$$G(P, Q) = \frac{1}{2\pi} \ln R_{pQ} \quad (1.113)$$

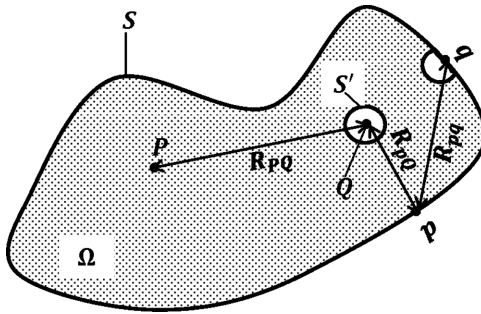


Figure 1.4 Definitions for integral representations using the Green's function.

which implies a logarithmic singularity at $R_{pQ} \rightarrow 0$. Finally, (1.110) yields the following integral representation for $\phi(Q)$:

$$\phi(Q) = \int_S \left[\phi(p) \frac{\partial G(p, Q)}{\partial n_p} - G(p, Q) \frac{\partial \phi(p)}{\partial n_p} \right] dS, \quad Q \in \Omega \quad (1.114)$$

If Q is taken outside Ω , then it holds that

$$\int_S \left[\phi(p) \frac{\partial G(p, Q)}{\partial n_p} - G(p, Q) \frac{\partial \phi(p)}{\partial n_p} \right] dS = 0 \quad (1.115)$$

Further, we discuss some properties of the integral representation (1.114). In hydrodynamics, the velocity potential typically satisfies Neumann or Dirichlet boundary conditions on a line contour S that bounds a region Ω , in the form $\partial \phi(p) / \partial n_p = 0$ or $\phi(p) = 0$, respectively, for all $p \in S$. Note the linear free-surface boundary condition (1.65) is of Robin type. In relevant cases, (1.114) is reduced to single-term integration. Further, if $\phi(p)$ is known on S and $G(p, Q)$ can be constructed in such a way so that $G(p, Q) = 0$ for all $p \in S$, then $\phi(Q)$ is obtained explicitly. Accordingly if $\partial \phi(p) / \partial n_p$ is known on S and $\partial G(p, Q) / \partial n_p = 0$ for all $p \in S$, then, again, $\phi(Q)$ is obtained explicitly. If the point (say point q) in which we want to calculate ϕ lies on the contour line S (see Figure 1.4), then to apply the Green's theorem we must exclude that point and we must assume that it is enclosed by a semicircle instead of a circle. Therefore, application of the Green's theorem over the domain Ω and the semicircle in S will yield the following integral representation for $\phi(q)$:

$$\frac{1}{2} \phi(q) = \int_S \left[\phi(p) \frac{\partial G(p, q)}{\partial n_p} - G(p, q) \frac{\partial \phi(p)}{\partial n_p} \right] dS, \quad q \in S \quad (1.116)$$

An integral equation for the boundary values of ϕ (points lying on S) arises from (1.115) if $\partial \phi(p) / \partial n_p = 0$ is known for all $p \in S$.

Equation (1.116) allows some simplifications if the Green's function can be constructed in such a way that satisfies the same boundary conditions as ϕ on a

subinterval $S_0 \subset S$. Provided that G satisfies Neumann or Dirichlet conditions on S_0 , the contribution of the integral over S_0 vanishes.

Equation (1.116) is used to calculate ϕ on all points lying on S . The next step, after solving the integral equation, is to employ (1.114) to calculate ϕ on all points inside the region Ω .

In particular problems the Green's function may be symmetrical, i.e., $G(P, Q) = G(Q, P)$. Hence from (1.114) and (1.116) we have

$$\phi(P) = \int_S \left[\phi(q) \frac{\partial G(P, q)}{\partial n_q} - G(P, q) \frac{\partial \phi(q)}{\partial n_q} \right] dS, \quad P \in \Omega \quad (1.117)$$

$$\frac{1}{2} \phi(p) = \int_S \left[\phi(q) \frac{\partial G(p, q)}{\partial n_q} - G(p, q) \frac{\partial \phi(q)}{\partial n_q} \right] dS, \quad p \in S \quad (1.118)$$

Integral equations (1.117)–(1.118) may now be interpreted in terms of distributions of sources and dipoles over the boundary surface S .

The same approach can be taken for the three-dimensional problem where Ω is a volume bounded by the surface S . The analysis and the equations for the three-dimensional problem are explicitly the same with those presented previously that referred to the two-dimensional space, provided that the integrals over S are regarded as surface integrals. To exclude the point Q from surface Ω , we should assume a sphere containing Q , while when the point lies on the surface S (denoted by q) it should be bounded by a hemisphere. The Green's function in the three-dimensional case will read

$$G(P, Q) = -\frac{1}{4\pi R_{PQ}} \quad (1.119)$$

which is evidently singular for $R_{PQ} \rightarrow 0$.