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BIPROJECTIVITY AND BIFLATNESS OF LAU PRODUCT OF BANACH ALGEBRAS DEFINED BY A BANACH ALGEBRA MORPHISM

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Abstract

Let φ be a homomorphism from a Banach algebra $\mathcal{B}$ to a Banach algebra $\mathcal{A}$. We define a multiplication on the Cartesian product space $\mathcal{A} \times \mathcal{B}$ and obtain a new Banach algebra $\mathcal{A} \times_\varphi \mathcal{B}$. We show that biprojectivity as well as biflatness of $\mathcal{A} \times_\varphi \mathcal{B}$ are stable with respect to φ.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and let $\varphi \in \mathrm{hom}(\mathcal{B}, \mathcal{A})$, the space consisting of all Banach algebra homomorphisms from $\mathcal{B}$ into $\mathcal{A}$. Moreover, suppose that $\|\varphi\| \leq 1$. Following [2], the Cartesian product space $\mathcal{A} \times \mathcal{B}$ equipped with the algebra multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 + a_1 \varphi(b_2) + \varphi(b_1) a_2, b_1 b_2), \quad (a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}),$$

and the norm

$$\|(a, b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}},$$

is a Banach algebra which is denoted by $\mathcal{A} \times_\varphi \mathcal{B}$. Note that our definition of the multiplication $\times_\varphi$, is presented with a slight difference from that given by Bhatt and Dabhi [2]. In fact they give the definition with the assumption of commutativity of $\mathcal{A}$ and use $a_1\varphi(b_2)$ instead of $\varphi(b_2)a_1$. However this condition is redundant, and the definition can be provided for an arbitrary Banach algebra $\mathcal{A}$.

As in [2], when $\varphi = 0$, this multiplication is the usual coordinatewise product and so $\times_\varphi$ is in fact the perturbation of the coordinatewise product induced by φ. Furthermore, let $\mathcal{A}$ be unital with the unit element e and let $\theta : \mathcal{B} \to \mathbb{C}$ be a multiplicative linear

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functional. Define $\varphi_0 : \mathcal{B} \to \mathcal{A}$ as $\varphi_0(b) = \theta(b)e$, for each $b \in \mathcal{B}$. Then the above product with respect to φ_0 coincides with the product investigated by Lau [10], for certain classes of Banach algebras. This definition was extended by Sangani Monfared [13], for the general case.

The main motivation for the present work stems from [2], which gives some results related to Arens regularity and some notions of amenability of $\mathcal{A} \times_\varphi \mathcal{B}$. In fact, Arens regularity as well as amenability (together with its various avatars) of $\mathcal{A} \times_\varphi \mathcal{B}$ are shown to be stable with respect to φ; see [2, Theorem 3.1] and [2, Theorem 4.1].

Homological properties of Banach algebras have been studied by several authors. We refer to [5] as a standard reference in this field. Moreover, we refer to recent work, such as [1, 4, 9, 12, 14], closely related to the present work. The main purpose of this paper is to study biprojectivity and biflatness of $\mathcal{A} \times_\varphi \mathcal{B}$. We begin with some primary results about $\mathcal{A} \times_\varphi \mathcal{B}$. Indeed, we show that $\mathcal{A} \times_\varphi \mathcal{B}$ is commutative if and only if $\mathcal{A}$ and $\mathcal{B}$ are commutative. We also prove that $\mathcal{A} \times_\varphi \mathcal{B}$ has an (approximate) identity if and only if $\mathcal{A}$ and $\mathcal{B}$ also do. Then we study biprojectivity and biflatness of $\mathcal{A} \times_\varphi \mathcal{B}$ and show that these properties are stable with respect to φ. Finally, as an application of these results, we obtain related results about amenability and contractibility of $\mathcal{A} \times_\varphi \mathcal{B}$.

2. Preliminaries

Let $\mathcal{A}$ be a Banach algebra. Then the dual space $\mathcal{A}^*$ of $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the module operations

$$\langle b, f \cdot a\rangle = \langle ab, f\rangle \quad \text{and} \quad \langle b, a \cdot f\rangle = \langle ba, f\rangle,$$

for all $a, b \in \mathcal{A}$ and $f \in \mathcal{A}^*$. We remark that the dual space $(\mathcal{A} \times_\varphi \mathcal{B})^*$ can be identified with $\mathcal{A}^* \times \mathcal{B}^*$, when we consider $\mathcal{A}^* \times \mathcal{B}^*$ under the norm

$$\|(f, g)\| = \|f\| + \|g\| \quad (f \in \mathcal{A}^*, g \in \mathcal{B}^*).$$

In fact, define $\theta : \mathcal{A}^* \times \mathcal{B}^* \to (\mathcal{A} \times_\varphi \mathcal{B})^*$ by

$$\langle (a, b), \theta((f, g))\rangle = \langle a, f\rangle + \langle b, g\rangle,$$

for $a \in \mathcal{A}$, $f \in \mathcal{A}^*, b \in \mathcal{B}$ and $g \in \mathcal{B}^*$. Some easy calculations imply that θ is the desired bounded linear map. Moreover, $(\mathcal{A} \times_\varphi \mathcal{B})^*$ can be turned into an $(\mathcal{A} \times_\varphi \mathcal{B})$-bimodule with the module operations defined by

$$(f, g) \cdot (a, b) := (f \cdot a + f \cdot \varphi(b), f \circ (L_a\varphi) + g \cdot b),$$
$$(a, b) \cdot (f, g) := (a \cdot f + \varphi(b) \cdot f, f \circ (R_a\varphi) + b \cdot g),$$

where $a \in \mathcal{A}, b \in \mathcal{B}, f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$. In fact, the module operations defined on $(A \times_\varphi B)*$ are the natural module actions of $A \times_\varphi B$ on its dual. In addition, $L_a\varphi : \mathcal{B} \to \mathcal{A}$ and $R_a\varphi : \mathcal{B} \to \mathcal{A}$ are defined as $L_a\varphi(y) = a\varphi(y)$ and $R_a\varphi(y) = \varphi(y)a$, for each $y \in \mathcal{B}$. Furthermore, $\mathcal{A} \times_\varphi \mathcal{B}$ is a Banach $\mathcal{A}$-bimodule under the module actions

$$c \cdot (a, b) := (c, 0) \cdot (a, b) \quad \text{and} \quad (a, b) \cdot c := (a, b) \cdot (c, 0),$$

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for all $a, c \in \mathcal{A}$ and $b \in \mathcal{B}$. Also $\mathcal{A} \times_\varphi \mathcal{B}$ can also be made into a Banach $\mathcal{B}$-bimodule in a similar fashion.

For completeness, we also recall the definitions and basic relationships of the standard homological properties. Following [5], we say that $\mathcal{A}$ is biprojective if there is a bounded $\mathcal{A}$-bimodule map ξ from $\mathcal{A}$ into the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \xi = \mathrm{id}_{\mathcal{A}}$. Here and in the sequel, $\pi_{\mathcal{A}}$ always denotes the product morphism from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ into $\mathcal{A}$, specified by $\pi_{\mathcal{A}}(a \otimes b) = ab$. Also $\mathcal{A}$ is biflat if there is a bounded $\mathcal{A}$-bimodule map $\lambda : (\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}^*$ such that $\lambda \circ {\pi_{\mathcal{A}}}^* = \mathrm{id}_{\mathcal{A}^*}$. We refer to [3, 5, 11] for the basic properties of biprojectivity and biflatness of Banach algebras.

We also introduce some required mappings similar to the mappings defined in [9], which will be used several times in this paper. Let $p_{\mathcal{A}} : \mathcal{A} \times_\varphi \mathcal{B} \to \mathcal{A}$ and $p_{\mathcal{B}} : \mathcal{A} \times_\varphi \mathcal{B} \to \mathcal{B}$ be the usual projections which are defined by $p_{\mathcal{A}}((a, b)) = a$ and $p_{\mathcal{B}}((a, b)) = b$, respectively $(a \in \mathcal{A}, b \in \mathcal{B})$. Also $q_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \times_\varphi \mathcal{B}$ and $q_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \times_\varphi \mathcal{B}$ are the usual injections, defined by $q_{\mathcal{A}}(a) = (a, 0)$ and $q_{\mathcal{B}}(b) = (0, b)$, respectively. Moreover, we define the mappings $r_{\mathcal{A}} : \mathcal{A} \times_\varphi \mathcal{B} \to \mathcal{A}$ and $s_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \times_\varphi \mathcal{B}$ by $r_{\mathcal{A}}((a, b)) := a + \varphi(b)$ and $s_{\mathcal{B}}(b) := (-\varphi(b), b)$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. One can easily check that $q_{\mathcal{A}}$, $r_{\mathcal{A}}$ are Banach $\mathcal{A}$-bimodule maps and $p_{\mathcal{B}}$, $q_{\mathcal{B}}$ and $s_{\mathcal{B}}$ are Banach $\mathcal{B}$-bimodule maps.

3. Results on identity and approximate identity

We commence with the following proposition. Recall that $\mathcal{A}$ is said to be faithful if $\{a \in \mathcal{A} : a\mathcal{A} = \mathcal{A}a = \{0\}\} = \{0\}$, where

$$\mathcal{A}a = \{ba : b \in \mathcal{A}\} \quad \text{and} \quad a\mathcal{A} = \{ab : b \in \mathcal{A}\}.$$

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi, \psi \in \mathrm{hom}(\mathcal{B}, \mathcal{A})$. Then the following statements hold.*

(i) *$\mathcal{A} \times_\varphi \mathcal{B}$ is commutative if and only if both $\mathcal{A}$ and $\mathcal{B}$ are commutative.*
(ii) *Suppose that there exist Banach algebra isomorphisms $\Psi : \mathcal{A} \to \mathcal{A}$ and $\Phi : \mathcal{B} \to \mathcal{B}$ such that $\Psi \circ \psi = \varphi \circ \Phi$. Then the Banach algebras $\mathcal{A} \times_\varphi \mathcal{B}$ and $\mathcal{A} \times_\psi \mathcal{B}$ are isomorphic.*

Proof. (i) This is easy to prove and is left to the reader.

(ii) Define $F : \mathcal{A} \times_\psi \mathcal{B} \to \mathcal{A} \times_\varphi \mathcal{B}$ by

$$F(a, b) = (\Psi(a), \Phi(b)).$$

Then F is obviously a bounded bijective linear map. Moreover, for all (a_1, b_1), $(a_2, b_2) \in \mathcal{A} \times_\psi \mathcal{B}$, we have

$$\begin{aligned} F((a_1, b_1) \cdot (a_2, b_2)) &= F((a_1a_2 + a_1\psi(b_2) + \psi(b_1)a_2, b_1b_2)) \\ &= (\Psi(a_1)\Psi(a_2) + \Psi(a_1)\Psi(\psi(b_2)) + \Psi(\psi(b_1))\Psi(a_2), \Phi(b_1)\Phi(b_2)) \\ &= (\Psi(a_1)\Psi(a_2) + \Psi(a_1)\varphi(\Phi(b_2)) + \varphi(\Phi(b_1))\Psi(a_2), \Phi(b_1)\Phi(b_2)) \\ &= (\Psi(a_1), \Phi(b_1)) \cdot (\Psi(a_2), \Phi(b_2)) \\ &= F((a_1, b_1)) \cdot F((a_2, b_2)). \end{aligned}$$

Thus F is a Banach algebra isomorphism. □

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Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi \in \hom(\mathcal{B}, \mathcal{A})$. Then the following statements hold.*

(i) *$\mathcal{A} \times_\varphi \mathcal{B}$ has an identity if and only if $\mathcal{A}$ and $\mathcal{B}$ have identities.*

(ii) *$\mathcal{A} \times_\varphi \mathcal{B}$ has a bounded left (right, or two-sided) approximate identity if and only if $\mathcal{A}$ and $\mathcal{B}$ have bounded left (right, or two-sided) approximate identities.*

Proof. (i) Let (a_0, b_0) be the identity of $\mathcal{A} \times_\varphi \mathcal{B}$. Then, for each $b \in \mathcal{B}$, we have

$$\begin{aligned}(0, b) \cdot (a_0, b_0) &= (\varphi(b)a_0, bb_0)\\ &= (0, b),\end{aligned}$$

and so $bb_0 = b$. Similarly $b_0 b = b$, for each $b \in \mathcal{B}$. It follows that b_0 is the identity element of $\mathcal{B}$. Similarly, for each $a \in \mathcal{A}$, we have

$$\begin{aligned}(a, 0)(a_0, b_0) &= (aa_0 + a\varphi(b_0), 0)\\ &= (a, 0),\end{aligned}$$

which implies that $a_0 + \varphi(b_0)$ is the identity element of $\mathcal{A}$. Conversely suppose that $\mathcal{A}$ and $\mathcal{B}$ have identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Then $(e_{\mathcal{A}} - \varphi(e_{\mathcal{B}}), e_{\mathcal{B}})$ is the identity element of $\mathcal{A} \times_\varphi \mathcal{B}$. In fact, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have

$$\begin{aligned}(a, b)(e_{\mathcal{A}} - \varphi(e_{\mathcal{B}}), e_{\mathcal{B}}) &= (a - a\varphi(e_{\mathcal{B}}) + a\varphi(e_{\mathcal{B}}) + \varphi(b)e_{\mathcal{A}} - \varphi(b)\varphi(e_{\mathcal{B}}), be_{\mathcal{B}})\\ &= (a + \varphi(b) - \varphi(b), b) = (a, b).\end{aligned}$$

Similarly, $(e_{\mathcal{A}} - \varphi(e_{\mathcal{B}}), e_{\mathcal{B}})(a, b) = (a, b)$. Thus the result is obtained.

(ii) Suppose that $((a_\alpha, b_\alpha))_{\alpha \in I}$ is a bounded left approximate identity for $\mathcal{A} \times_\varphi \mathcal{B}$. Then, for each $b \in \mathcal{B}$, we have

$$\|(0, b) - (a_\alpha, b_\alpha) \cdot (0, b)\| \to 0$$

and so $\|b - b_\alpha b\|_{\mathcal{B}} \to 0$. It follows that $(b_\alpha)_{\alpha \in I}$ is a bounded left approximate identity for $\mathcal{B}$. Also, for each $a \in \mathcal{A}$,

$$\|(a, 0) - (a_\alpha, b_\alpha) \cdot (a, 0)\| = \|(a, 0) - (a_\alpha a + \varphi(b_\alpha)a, 0)\| \to 0.$$

Thus $\|a - (a_\alpha + \varphi(b_\alpha))a\|_{\mathcal{A}} \to 0$, which implies that $(a_\alpha + \varphi(b_\alpha))_{\alpha \in I}$ is a bounded left approximate identity for $\mathcal{A}$. For the converse, assume that $(a_\alpha)_{\alpha \in I}$ and $(b_\beta)_{\beta \in J}$ are bounded left approximate identities for $\mathcal{A}$ and $\mathcal{B}$, respectively. We show that $(a_\alpha - \varphi(b_\beta), b_\beta)_{(\alpha,\beta) \in I \times J}$ is a bounded left approximate identity for $\mathcal{A} \times_\varphi \mathcal{B}$. Indeed, for each $(a, b) \in \mathcal{A} \times_\varphi \mathcal{B}$, we have

$$\begin{aligned}\lim_{\alpha,\beta} \|(a, b) - (a_\alpha - \varphi(b_\beta), b_\beta)(a, b)\| &= \lim_{\alpha,\beta} \|(a, b) - (a_\alpha a - \varphi(b_\beta)a + a_\alpha\varphi(b)\\ &\qquad - \varphi(b_\beta)\varphi(b) + \varphi(b_\beta)a, b_\beta b)\|\\ &= \lim_\alpha \lim_\beta \|a - (a_\alpha a + a_\alpha\varphi(b) - \varphi(b_\beta)\varphi(b))\|_{\mathcal{A}}\\ &\qquad + \|b - b_\beta b\|_{\mathcal{B}}\\ &\le \lim_\alpha \lim_\beta \|a - a_\alpha a\|_{\mathcal{A}} + \|\varphi(b) - a_\alpha\varphi(b)\|_{\mathcal{A}}\\ &\qquad + \|\varphi(b_\beta b) - \varphi(b)\|_{\mathcal{A}} + \|b - b_\beta b\|_{\mathcal{B}}\\ &= 0,\end{aligned}$$

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where we have used the iterated limit theorem from of [8, page 69]. This completes the proof. □

4. Results on biprojectivity

In this section, we investigate biprojectivity of $\mathcal{A} \times_\varphi \mathcal{B}$. We use a slightly modified version of the technique used in the main theorem of [9].

THEOREM 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\varphi \in \hom(\mathcal{B}, \mathcal{A})$, such that $\mathcal{A} \times_\varphi \mathcal{B}$ is biprojective. Then $\mathcal{A}$ and $\mathcal{B}$ are biprojective.*

PROOF. By the hypothesis, there exists a bounded $(\mathcal{A} \times_\varphi \mathcal{B})$-bimodule map

$$\eta : \mathcal{A} \times_\varphi \mathcal{B} \to (\mathcal{A} \times_\varphi \mathcal{B}) \hat{\otimes} (\mathcal{A} \times_\varphi \mathcal{B})$$

such that $\pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ \eta = \mathrm{id}_{\mathcal{A}\times_\varphi\mathcal{B}}$. A direct verification shows that the identities

$$\pi_{\mathcal{A}} \circ (r_{\mathcal{A}} \otimes r_{\mathcal{A}}) = r_{\mathcal{A}} \circ \pi_{\mathcal{A}\times_\varphi\mathcal{B}} \quad \text{and} \quad \pi_{\mathcal{B}} \circ (p_{\mathcal{B}} \otimes p_{\mathcal{B}}) = p_{\mathcal{B}} \circ \pi_{\mathcal{A}\times_\varphi\mathcal{B}}$$

hold. Define $\xi : \mathcal{A} \to \mathcal{A} \hat{\otimes} \mathcal{A}$ and $\rho : \mathcal{B} \to \mathcal{B} \hat{\otimes} \mathcal{B}$ by

$$\xi := (r_{\mathcal{A}} \otimes r_{\mathcal{A}}) \circ \eta \circ q_{\mathcal{A}} \quad \text{and} \quad \rho := (p_{\mathcal{B}} \otimes p_{\mathcal{B}}) \circ \eta \circ q_{\mathcal{B}}.$$

Some easy calculations show that ξ and ρ are $\mathcal{A}$-bimodule and $\mathcal{B}$-bimodule maps, respectively. Moreover,

$$\begin{aligned} \pi_{\mathcal{A}} \circ \xi &= \pi_{\mathcal{A}} \circ ((r_{\mathcal{A}} \otimes r_{\mathcal{A}}) \circ \eta \circ q_{\mathcal{A}}) \\ &= r_{\mathcal{A}} \circ (\pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ \eta) \circ q_{\mathcal{A}} \\ &= \mathrm{id}_{\mathcal{A}}. \end{aligned}$$

Also one can easily obtain that $\pi_{\mathcal{B}} \circ \rho = \mathrm{id}_{\mathcal{B}}$. These observations show that $\mathcal{A}$ and $\mathcal{B}$ are biprojective, as claimed. □

The converse of Theorem 4.1 holds in the case where $\mathcal{A}$ is unital. It is provided in the next result.

THEOREM 4.2. *Let $\mathcal{A}$ be a unital Banach algebra, let $\mathcal{B}$ be a Banach algebra and let $\varphi \in \hom(\mathcal{B}, \mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are biprojective, then $\mathcal{A} \times_\varphi \mathcal{B}$ is biprojective.*

PROOF. By the hypothesis, there exist an $\mathcal{A}$-bimodule map $\xi : \mathcal{A} \to \mathcal{A} \hat{\otimes} \mathcal{A}$ and a $\mathcal{B}$-bimodule map $\rho : \mathcal{B} \to \mathcal{B} \hat{\otimes} \mathcal{B}$, such that $\pi_{\mathcal{A}} \circ \xi = \mathrm{id}_{\mathcal{A}}$ and $\pi_{\mathcal{B}} \circ \rho = \mathrm{id}_{\mathcal{B}}$. We immediately have the identities

$$q_{\mathcal{A}} \circ \pi_{\mathcal{A}} = \pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}}) \quad \text{and} \quad s_{\mathcal{B}} \circ \pi_{\mathcal{B}} = \pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}}).$$

Let $e_{\mathcal{A}}$ be the identity of $\mathcal{A}$ and define

$$\eta : \mathcal{A} \times_\varphi \mathcal{B} \to (\mathcal{A} \times_\varphi \mathcal{B}) \hat{\otimes} (\mathcal{A} \times_\varphi \mathcal{B})$$

by

$$\eta((a, b)) := (a, b) \cdot ((q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e_{\mathcal{A}}))) + ((s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho \circ p_{\mathcal{B}})(a, b),$$

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for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. We first show that η is a bounded $(\mathcal{A} \times_\varphi \mathcal{B})$-bimodule map. Clearly η is bounded. For all $(a, b), (c, d) \in \mathcal{A} \times_\varphi \mathcal{B}$, we have

$$\begin{aligned}
\eta((a, b) \cdot (c, d)) &= (a, b) \cdot (c, d) \cdot ((q_{\mathcal{A}} \otimes q_{\mathcal{A}}) \circ \xi(e)) + ((s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho)(bd) \\
&= (a, b) \cdot (c, d) \cdot (q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e)) + (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(b \cdot \rho(d)) \\
&= (a, b) \cdot (c, d) \cdot (q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e)) + (0, b) \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(d)) \\
&= (a, b) \cdot [(c, d) \cdot (q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e)) + (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(d))] \\
&\qquad - (a, 0) \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(d)) \\
&= (a, b) \cdot \eta((c, d)) - (a, 0) \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(d)).
\end{aligned}$$

But $(a, 0) \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(d)) = 0$. In fact, let $\rho(d) = \sum_{i=1}^{\infty} b_i \otimes d_i$, for some sequences (b_i) and (d_i) in $\mathcal{B}$ with $\sum_{i=1}^{\infty} \|b_i\|_{\mathcal{B}} \|d_i\|_{\mathcal{B}} < \infty$. Then

$$\begin{aligned}
(a, 0) \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(d)) &= (a, 0) \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \Big(\sum_{i=1}^{\infty} b_i \otimes d_i \Big) \\
&= (a, 0) \cdot \Big(\sum_{i=1}^{\infty} s_{\mathcal{B}}(b_i) \otimes s_{\mathcal{B}}(d_i) \Big) \\
&= \sum_{i=1}^{\infty} (a, 0) \cdot [(-\varphi(b_i), b_i) \otimes (-\varphi(d_i), d_i)] = 0.
\end{aligned}$$

Thus $\eta((a, b) \cdot (c, d)) = (a, b) \cdot \eta((c, d))$, and so η is a left $(\mathcal{A} \times_\varphi \mathcal{B})$-module map. To see that η is a right $(\mathcal{A} \times_\varphi \mathcal{B})$-module map, we note that, for all $x, y \in \mathcal{A}$,

$$(a, b) \cdot ((q_{\mathcal{A}} \otimes q_{\mathcal{A}})(x \otimes y)) = (q_{\mathcal{A}} \otimes q_{\mathcal{A}})((a + \varphi(b)) \cdot (x \otimes y)).$$

Similarly, $((q_{\mathcal{A}} \otimes q_{\mathcal{A}})(x \otimes y)) \cdot (a, b) = (q_{\mathcal{A}} \otimes q_{\mathcal{A}})((x \otimes y) \cdot (a + \varphi(b)))$. Hence

$$\begin{aligned}
(a, b) \cdot (q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e_{\mathcal{A}})) &= (q_{\mathcal{A}} \otimes q_{\mathcal{A}})((a + \varphi(b)) \cdot \xi(e_{\mathcal{A}})) \\
&= (q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e_{\mathcal{A}}) \cdot (a + \varphi(b))) \\
&= (q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e_{\mathcal{A}})) \cdot (a, b).
\end{aligned}$$

It follows that $(q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e_{\mathcal{A}}))$ commutes with the elements of $\mathcal{A} \times_\varphi \mathcal{B}$. Consequently,

$$\begin{aligned}
\eta((c, d) \cdot (a, b)) &= (c, d) \cdot (a, b) \cdot ((q_{\mathcal{A}} \otimes q_{\mathcal{A}}) \circ \xi(e_{\mathcal{A}})) + ((s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho)(db) \\
&= [(c, d) \cdot ((q_{\mathcal{A}} \otimes q_{\mathcal{A}}) \circ \xi(e_{\mathcal{A}}))] \cdot (a, b) + ((s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho(d)) \cdot (0, b) \\
&= \eta((c, d)) \cdot (a, b) - ((s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho(d)) \cdot (a, 0) \\
&= \eta((c, d)) \cdot (a, b),
\end{aligned}$$

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where we have used the fact that $((s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho(d)) \cdot (a, 0) = 0$. Finally, we show that $\pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ \eta = \mathrm{id}_{\mathcal{A}\times_\varphi\mathcal{B}}$. In fact, for each $(a, b) \in \mathcal{A} \times_\varphi \mathcal{B}$, we have

$$\begin{aligned}(\pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ \eta)((a, b)) &= \pi_{\mathcal{A}\times_\varphi\mathcal{B}}((a, b) \cdot ((q_{\mathcal{A}} \otimes q_{\mathcal{A}})(\xi(e_{\mathcal{A}}))) \\ &\qquad + (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(\rho(b))) \\ &= (a, b) \cdot ((\pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}}))(\xi(e_{\mathcal{A}}))) \\ &\qquad + (\pi_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}}))(\rho(b)) \\ &= (a, b) \cdot q_{\mathcal{A}} \circ (\pi_{\mathcal{A}} \circ \xi)(e_{\mathcal{A}}) \\ &\qquad + s_{\mathcal{B}} \circ (\pi_{\mathcal{B}} \circ \rho)(b) \\ &= (a, b) \cdot q_{\mathcal{A}}(e_{\mathcal{A}}) + s_{\mathcal{B}}(b) \\ &= (a, b) \cdot (e_{\mathcal{A}}, 0) + (-\varphi(b), b) \\ &= (a, b).\end{aligned}$$

Therefore $\mathcal{A} \times_\varphi \mathcal{B}$ is biprojective, as claimed. □

5. Results on biflatness

In this section we provide necessary and sufficient conditions for the biflatness of $\mathcal{A} \times_\varphi \mathcal{B}$. We note that our proof is inspired by the proof of the main theorem in [9], with some slight modifications.

Theorem 5.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\varphi \in \mathrm{hom}(\mathcal{B}, \mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are biflat, then $\mathcal{A} \times_\varphi \mathcal{B}$ is biflat.*

Proof. By the hypothesis, there exist an $\mathcal{A}$-bimodule map $\lambda_{\mathcal{A}} : (\mathcal{A} \hat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ and a $\mathcal{B}$-bimodule map $\lambda_{\mathcal{B}} : (\mathcal{B} \hat{\otimes} \mathcal{B})^* \to \mathcal{B}^*$, such that $\lambda_{\mathcal{A}} \circ \pi^*_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}^*}$ and $\lambda_{\mathcal{B}} \circ \pi^*_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}^*}$. Consider the identification $(\mathcal{A} \times_\varphi \mathcal{B})^* \simeq \mathcal{A}^* \times \mathcal{B}^*$ and define $\lambda : ((\mathcal{A} \times_\varphi \mathcal{B}) \hat{\otimes} (\mathcal{A} \times_\varphi \mathcal{B}))^* \to (\mathcal{A} \times_\varphi \mathcal{B})^*$ by

$$\begin{aligned}\lambda(h) := ((\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h), (\lambda_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*)(h) \\ + (\varphi^* \circ \lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h)),\end{aligned}$$

for each $h \in ((\mathcal{A} \times_\varphi \mathcal{B}) \hat{\otimes} (\mathcal{A} \times_\varphi \mathcal{B}))^*$. Since $\lambda_{\mathcal{A}}$ and $\lambda_{\mathcal{B}}$ are bounded, it follows that λ is bounded as well. For each $(a, b) \in \mathcal{A} \times_\varphi \mathcal{B}$ and $h \in ((\mathcal{A} \times_\varphi \mathcal{B}) \hat{\otimes} (\mathcal{A} \times_\varphi \mathcal{B}))^*$ the following identities immediately hold:

$$\begin{aligned}(q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h \cdot (a, b)) &= (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h) \cdot (a + \varphi(b)), \\ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*((a, b) \cdot h) &= (a + \varphi(b)) \cdot (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h), \\ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(h \cdot (a, b)) &= (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(h) \cdot b, \\ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*((a, b) \cdot h) &= b \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(h).\end{aligned}$$

Moreover,

$$\varphi^* \circ \lambda_{\mathcal{A}}((q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h) \cdot \varphi(b)) = (\varphi^* \circ \lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h)) \cdot b$$

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and

$$\varphi^* \circ \lambda_{\mathcal{A}}((q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h) \cdot a) = (\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h))(L_a \varphi).$$

Using these identities, we show that λ is a $(\mathcal{A} \times_\varphi \mathcal{B})$-bimodule map. To this end, let $(a, b) \in \mathcal{A} \times_\varphi \mathcal{B}$ and $h \in ((\mathcal{A} \times_\varphi \mathcal{B}) \otimes (\mathcal{A} \times_\varphi \mathcal{B}))^*$. Then

$$\begin{aligned}
\lambda(h \cdot (a, b)) &= ((\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h \cdot (a, b)), (\lambda_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*)(h \cdot (a, b)) \\
&\qquad + (\varphi^* \circ \lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h \cdot (a, b))) \\
&= (\lambda_{\mathcal{A}}((q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h)) \cdot (a + \varphi(b)), \lambda_{\mathcal{B}}((s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(h) \cdot b) \\
&\qquad + \varphi^* \circ \lambda_{\mathcal{A}}(((q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h)) \cdot (a + \varphi(b)) \\
&= ([(\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h)] \cdot a + [(\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*)(h)] \cdot \varphi(b), \\
&\qquad [(\lambda_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*)(h)] \cdot b + \varphi^* \circ \lambda_{\mathcal{A}}[(q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h) \cdot a] \\
&\qquad\quad + \varphi^* \circ \lambda_{\mathcal{A}}[(q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h) \cdot \varphi(b)]) \\
&= ((\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h)) \cdot a + (\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h)) \cdot \varphi(b), \\
&\qquad (\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h))(L_a \varphi) \\
&\qquad\quad + [\lambda_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(h) + \varphi^* \circ \lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^*(h)] \cdot b) \\
&= \lambda(h) \cdot (a, b).
\end{aligned}$$

With similar arguments, we obtain that $\lambda((a, b) \cdot h) = (a, b) \cdot \lambda(h)$, and consequently λ is a bounded $(\mathcal{A} \times_\varphi \mathcal{B})$-bimodule. Finaly, we prove that

$$\lambda \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}} = \mathrm{id}_{(\mathcal{A} \times_\varphi \mathcal{B})^*}.$$

First, note that by using the identification $(\mathcal{A} \times_\varphi \mathcal{B}) \simeq (\mathcal{A}^* \times \mathcal{B}^*)$, one can easily obtain that

$$(s_{\mathcal{B}} \otimes s_{\mathcal{B}})^* \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}} = \pi^*_{\mathcal{B}} \circ (s_{\mathcal{B}})^* \quad \text{and} \quad (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^* \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}} = \pi^*_{\mathcal{A}} \circ (q_{\mathcal{A}})^*.$$

Moreover, for each $(f, g) \in (\mathcal{A}^* \times \mathcal{B}^*) \cong (\mathcal{A} \times_\varphi \mathcal{B})^*$ we have $(q_{\mathcal{A}})^*((f, g)) = f$, $(s_{\mathcal{B}})^*((f, g)) = -f \circ \varphi + g$, and $(\varphi^* \circ (q_{\mathcal{A}})^*)((f, g)) = f \circ \varphi$. Now suppose that $(a, b) \in \mathcal{A} \times_\varphi \mathcal{B}$ and $(f, g) \in (\mathcal{A}^* \times \mathcal{B}^*) \cong (\mathcal{A} \times_\varphi \mathcal{B})^*$. Then

$$\begin{aligned}
\lambda \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}}((f, g)) &= ((\lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^* \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}})((f, g)), \\
&\qquad (\lambda_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^* \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}})((f, g)) \\
&\qquad\quad + (\varphi^* \circ \lambda_{\mathcal{A}} \circ (q_{\mathcal{A}} \otimes q_{\mathcal{A}})^* \circ \pi^*_{\mathcal{A} \times_\varphi \mathcal{B}})((f, g))) \\
&= ((\lambda_{\mathcal{A}} \circ \pi^*_{\mathcal{A}} \circ (q_{\mathcal{A}})^*)((f, g)), \\
&\qquad (\lambda_{\mathcal{B}} \circ \pi^*_{\mathcal{B}} \circ (s_{\mathcal{B}})^*)((f, g)) \\
&\qquad\quad + (\varphi^* \circ \lambda_{\mathcal{A}} \circ \pi^*_{\mathcal{A}} \circ (q_{\mathcal{A}})^*)((f, g))) \\
&= ((\mathrm{id}_{\mathcal{A}^*} \circ (q_{\mathcal{A}})^*)((f, g)), (\mathrm{id}_{\mathcal{B}^*} \circ (s_{\mathcal{B}})^*)((f, g)) \\
&\qquad + (\varphi^* \circ \mathrm{id}_{\mathcal{A}^*} \circ (q_{\mathcal{A}})^*)((f, g))) \\
&= (f, -f \circ \varphi + g + f \circ \varphi). \\
&= (f, g).
\end{aligned}$$

This completes the proof. □

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The converse of Theorem 5.1 is also valid. It is provided by the next theorem.

THEOREM 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\varphi \in \mathrm{hom}(\mathcal{B}, \mathcal{A})$. If $\mathcal{A} \times_\varphi \mathcal{B}$ is biflat, then $\mathcal{A}$ and $\mathcal{B}$ are also biflat.*

PROOF. By the hypothesis, there exists a bounded $(\mathcal{A} \times_\varphi \mathcal{B})$-bimodule map,

$$\lambda : ((\mathcal{A} \times_\varphi \mathcal{B}) \hat{\otimes} (\mathcal{A} \times_\varphi \mathcal{B}))^* \to (\mathcal{A} \times_\varphi \mathcal{B})^*,$$

such that $\lambda \circ \pi^*_{\mathcal{A}\times_\varphi\mathcal{B}} = \mathrm{id}_{(\mathcal{A}\times_\varphi\mathcal{B})^*}$. The following identities are easily shown:

$$(p_{\mathcal{B}} \otimes p_{\mathcal{B}})^* \circ \pi^*_{\mathcal{B}} = \pi^*_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (p_{\mathcal{B}})^*,$$
$$(r_{\mathcal{A}} \otimes r_{\mathcal{A}})^* \circ \pi^*_{\mathcal{A}} = \pi^*_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (r_{\mathcal{A}})^*.$$

Now define $\mu : (\mathcal{A} \hat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ and $\nu : (\mathcal{B} \hat{\otimes} \mathcal{B})^* \to \mathcal{B}^*$ by

$$\mu := (q_{\mathcal{A}})^* \circ \lambda \circ (r_{\mathcal{A}} \otimes r_{\mathcal{A}})^* \quad \text{and} \quad \nu := (q_{\mathcal{B}})^* \circ \lambda \circ (p_{\mathcal{B}} \otimes p_{\mathcal{B}})^*,$$

respectively. Direct verifications show that μ and ν are bounded $\mathcal{A}$-bimodule and bounded $\mathcal{B}$-bimodule maps, respectively. Moreover, we have

$$\begin{aligned}
\mu \circ \pi^*_{\mathcal{A}} &= ((q_{\mathcal{A}})^* \circ \lambda \circ (r_{\mathcal{A}} \otimes r_{\mathcal{A}})^*) \circ \pi^*_{\mathcal{A}} \\
&= (q_{\mathcal{A}})^* \circ \lambda \circ ((r_{\mathcal{A}} \otimes r_{\mathcal{A}})^* \circ \pi^*_{\mathcal{A}}) \\
&= (q_{\mathcal{A}})^* \circ \lambda \circ (\pi^*_{\mathcal{A}\times_\varphi\mathcal{B}} \circ (r_{\mathcal{A}})^*) \\
&= (q_{\mathcal{A}})^* \circ (r_{\mathcal{A}})^* \\
&= \mathrm{id}_{\mathcal{A}^*}.
\end{aligned}$$

Similar arguments show that $\nu \circ \pi^*_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}^*}$, and consequently $\mathcal{A}$ and $\mathcal{B}$ are biflat. □

We conclude this work with the following remark, which provides some useful applications of our results related to amenability and contractibility of $\mathcal{A} \times_\varphi \mathcal{B}$. Recall from [7] that a Banach algebra $\mathcal{A}$ is called amenable if it has a bounded approximate diagonal, that is, if there is a bounded net (m_λ) in the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $\|am_\lambda - m_\lambda a\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \to_\lambda 0$ and $\|a\pi_{\mathcal{A}}(m_\lambda) - a\|_{\mathcal{A}} \to_\lambda 0$, for each $a \in \mathcal{A}$. Similarly, $\mathcal{A}$ is contractible if and only if it has a diagonal, that is, there is an element $m \in \mathcal{A} \widehat{\otimes} \mathcal{A}$ for which $am = ma$ and $\pi_{\mathcal{A}}(m)a = a$, for all $a \in \mathcal{A}$ (see [6]).

REMARK 5.3. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi \in \mathrm{hom}(\mathcal{B}, \mathcal{A})$.

(1) We show that our results imply part (1) of [2, Theorem 4.1] in the general case. Let $\mathcal{A} \times_\varphi \mathcal{B}$ be amenable. Then $\mathcal{A} \times_\varphi \mathcal{B}$ is biflat and has a bounded approximate identity; see, for example, [11, Section 4, Exercise 4.3.15]. By Proposition 3.2 and Theorem 5.2, we obtain that $\mathcal{A}$ and $\mathcal{B}$ are biflat and have bounded approximate identities. Again [11, Section 4, Exercise 4.3.15] implies that $\mathcal{A}$ and $\mathcal{B}$ are amenable. Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are amenable. Then by [11, Section 4, Exercise 4.3.15] they are biflat and admit bounded approximate identities. Again Proposition 3.2 and Theorem 5.1 yield that $\mathcal{A} \times_\varphi \mathcal{B}$ is biflat and has a bounded approximate identity, which is equivalent to the amenability of $\mathcal{A} \times_\varphi \mathcal{B}$.

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(2) We show that $\mathcal{A} \times_\varphi \mathcal{B}$ is contractible if and only if $\mathcal{A}$ and $\mathcal{B}$ are contractible. Let $\mathcal{A} \times_\varphi \mathcal{B}$ be contractible. Then $\mathcal{A} \times_\varphi \mathcal{B}$ is biprojective and unital; see, for example, [11, Section 4, Exercises 4.1.1, 4.3.1]. By Proposition 3.2 and Theorem 4.1, $\mathcal{A}$ and $\mathcal{B}$ are biprojective and have identities. Thus $\mathcal{A}$ and $\mathcal{B}$ are contractible by [11, Section 4, Exercises 4.1.1, 4.3.1]. For the converse, suppose that $\mathcal{A}$ and $\mathcal{B}$ are contractible. Then, again by [11, Section 4, Exercises 4.1.1, 4.3.1], $\mathcal{A}$ and $\mathcal{B}$ are biprojective and have identities. So by Proposition 3.2 and Theorem 4.2, and the fact that $\mathcal{A}$ is unital, $\mathcal{A} \times_\varphi \mathcal{B}$ is biprojective and has an identity. Consequently, $\mathcal{A} \times_\varphi \mathcal{B}$ is contractible, again by [11, Section 4, Exercises 4.1.1, 4.3.1].

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