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# Classical solutions to the two-dimensional Euler equations and elliptic boundary value problems, an overview

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**Abstract**

Consider the classical initial, boundary-value problem for the 2D Euler equations, which describes the motion of an ideal, incompressible, fluid in a impermeable vessel. In the early eighties we introduced and studied a Banach space, denoted $C_*(\overline{\Omega})$, which enjoys the following property: if the curl of the initial velocity belongs to $C_*(\overline{\Omega})$, and the curl of the external forces is integrable in time with values in the above space $C_*(\overline{\Omega})$, then all derivatives appearing in the differential equations and in the boundary conditions are continuous in space-time, up to the boundary (we call these solutions *classical solutions*). At that time this conclusion was know if $C_*(\overline{\Omega})$ is replaced by a Hölder space $C^{0,\lambda}(\overline{\Omega})$. In the proof of the above result we appealed to a $C^2(\overline{\Omega})$ regularity result for solutions to the Poisson equation, vanishing on the boundary and with external forces in $C_*(\overline{\Omega})$. Actually, at that time, we have proved this regularity result for solutions to more general second-order linear elliptic boundary-value problems. However the proof remained unpublished. Recently, we have published an adaptation of the proof to solutions of the Stokes system. We recall these results in Section 1.1 below. On the other hand, attempts to prove the above regularity results for data in functional spaces properly containing $C_*(\overline{\Omega})$, have also been done. Bellow we prove some partial results in this direction. This possibly unfinished picture leads to interesting open problems.

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### 1.1 The Euler and Stokes equations with data in $C_*(\overline{\Omega})$.

In these notes we want to give an overview on some results, both old and new. Some are old, but remained unpublished for a long time. The starting point will be Beirão da Veiga (1981, 1982, 1984).

We start by introducing some notation. $\Omega$ is an open, bounded, connected set in $\mathbb{R}^n$, $n \geq 2$, locally situated on one side of its boundary $\Gamma$. We assume that $\Gamma$ is of class $C^{2,\lambda}(\overline{\Omega})$, for some positive $\lambda$. By $C(\overline{\Omega})$ we denote the Banach space of all real, continuous functions in $\overline{\Omega}$ with the norm

$$\|f\| \equiv \sup_{x \in \overline{\Omega}} |f(x)|.$$

In the sequel we use the notation

$$\|\nabla u\| = \sum_{i=1}^{n} \|\partial_i u\|, \quad \|\nabla^2 u\| = \sum_{i,j=1}^{n} \|\partial_{ij} u\|,$$

and appeal to the canonical spaces $C^1(\overline{\Omega})$ and $C^2(\overline{\Omega})$, with the norms

$$\|u\|_1 \equiv \|u\| + \|\nabla u\|, \quad \|u\|_2 \equiv \|u\| + \|\nabla^2 u\|$$

respectively. Further, for each $\lambda \in (0,1]$, we define the semi-norm

$$[f]_{0,\lambda} \equiv \sup_{x,y \in \Omega; x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\lambda}, \tag{1.1}$$

and the Hölder space $C^{0,\lambda}(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_{0,\lambda} < \infty\}$, with the norm

$$\|f\|_{0,\lambda} = \|f\| + [f]_{0,\lambda}.$$

In particular, $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions in $\overline{\Omega}$. By $C^\infty(\overline{\Omega})$ we denote the set of all restrictions to $\overline{\Omega}$ of infinitely differentiable functions in $\mathbb{R}^n$. We will use boldface notation to denote vectors, vector spaces, and so on. We denote the components of a generic vector $\boldsymbol{u}$ by $u_i$, and similarly for tensors. Norms in functional spaces whose elements are vector fields are defined in the usual way, by appealing to the corresponding norms of the components.

In considering the two-dimensional Euler equations we will introduce the following well-known simplification. For a scalar function $u(x)$ (identified here with the third component of a vector field, normal to the plane of motion) we define the vector field $\mathrm{Rot}u = (\partial_2 u, -\partial_1 u)$. For a vector field $\boldsymbol{v} = (v_1, v_2)$ we define the scalar field $\mathrm{rot}\boldsymbol{v} = \partial_1 \boldsymbol{v}_2 - \partial_2 \boldsymbol{v}_1$ (the normal component of the curl). One has $-\Delta = \mathrm{rotRot}$. Note that

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$\mathrm{Rot}u$ is the rotation of the gradient $\nabla u$ by $\pi/2$ in the counter-clockwise direction.

Next we describe the motivation and origin of this research. We follow Beirão da Veiga (1981, 1982, 1984) which were essentially written during a visiting professorship to the Mathematics Research Center and the Mathematics Department in Wisconsin-Madison, in the semester October 1981-March 1982. In the above references we consider the initial boundary value problem for the two dimensional Euler equations

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{v} - \nabla\pi \quad \text{in} \quad Q \equiv \mathbb{R} \times \Omega, \\ \mathrm{div}\boldsymbol{v} = 0 \quad \text{in} \quad Q, \\ \boldsymbol{v}_0 \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \mathbb{R} \times \Gamma, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

At that time our main interest was to determine minimal conditions on the data which imply that the global, unique, solutions to the above problem are *classical*. This means here that all derivatives appearing in the equations are continuous, up to the boundary, in the space-time cylinder. The main result on this problem was stated and proved in the preprint by Beirão da Veiga (1982), see the theorem 1.9 below. Exactly the same work was published in Beirão da Veiga (1984), to which we will refer in the sequel. To explain, in the simplest way, the main lines followed in our study, assume for now that no external forces are present, and that $\Omega$ is simply connected. In Beirão da Veiga (1984) we started by considering the Banach space

$$\boldsymbol{E}(\overline{\Omega}) \equiv \{\boldsymbol{v} \in \mathbf{C}(\overline{\Omega}) : \mathrm{div}\boldsymbol{v} = 0 \text{in } \Omega; \mathrm{rot}\boldsymbol{v} \in C(\overline{\Omega}); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}, \tag{1.3}$$

endowed with the norm (in the simply connected case)

$$|||\boldsymbol{v}||| = \|\mathrm{rot}\boldsymbol{v}\|, \tag{1.4}$$

and show the global boundedness, strong-continuous dependence on the data, and other basic properties with respect to data in the above space $\boldsymbol{E}(\overline{\Omega})$ (see the theorems 1.1, 1.2, and 1.3, in the above reference). These preliminary results were obtained by improving techniques already used by other authors; see for instance Kato (1967), and Schaeffer (1937). However these results do not imply that solutions are classical under the given assumption on the initial data, since

$$\mathrm{rot}\boldsymbol{v}_0 \in C(\overline{\Omega})$$

leads to $\mathrm{rot}\boldsymbol{v}(t,\cdot) \in C(\overline{\Omega})$, but this last property does not imply $\nabla\boldsymbol{v}(t,\cdot) \in$

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$\mathbf{C}(\overline{\Omega})$. This gap is strictly related to a corresponding gap for solutions to elliptic equations, namely, the solution $\boldsymbol{v}$ to the system (see equation (1.3) in Beirão da Veiga, 1982)

$$\begin{cases} \mathrm{rot}\boldsymbol{v} = f & \text{in} \quad \Omega, \\ \mathrm{div}\boldsymbol{v} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.5}$$

does not necessarily belong to $\mathbf{C}^1(\overline{\Omega})$, whenever $f \in C(\overline{\Omega})$. On the other hand, at that time, it was already well known that if $f$ belongs to a Hölder space $C^{0,\lambda}(\overline{\Omega})$, then $\boldsymbol{v} \in \mathbf{C}^{1,\lambda}(\overline{\Omega})$. This result, together with a clever use of Lagrangian coordinates, makes it possible to prove that solutions to the system (1.2) are classical under the hypothesis

$$\mathrm{rot}\boldsymbol{v}_0 \in C^{0,\lambda}(\overline{\Omega}).$$

This was a well known result at that time, see Bardos (1972), Judovich (1963), Kato (1967), and Schaeffer (1937).

Having the above picture in mind, it seemed natural to start our approach to the Euler equations by studying the system (1.5). We wanted to single out a Banach spaces $C_*(\overline{\Omega})$, strictly contained in the Hölder spaces $C^{0,\lambda}(\overline{\Omega})$, such that solutions $\boldsymbol{v}$ to the first order system (1.5) are classical under the assumption $f \in C_*(\overline{\Omega})$. On the other hand, a classical argument shows that the solution $\boldsymbol{v}$ to the system (1.5) can be obtained by setting $\boldsymbol{v} = -\mathrm{Rot}u$, where $u$ solves the problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.6}$$

It follows that solutions $\boldsymbol{v}$ to system (1.5) belong to $\mathbf{C}^1(\overline{\Omega})$ if the solutions $\boldsymbol{u}$ to the system (1.6) belong to $\mathbf{C}^2(\overline{\Omega})$. This situation led us to look for a Banach space $C_*(\overline{\Omega})$, for which the following result holds.

**Theorem 1.1.1.** *Let $f \in C_*(\overline{\Omega})$ and let $u$ be the solution to problem* (1.6). *Then $u \in C^2(\overline{\Omega})$, moreover, $\|u\|_2 \leq c_0 \|f\|_*$.*

The above theorem was stated in Beirão da Veiga (1984) as Theorem 4.5. For convenience, the space $C_*(\overline{\Omega})$ will be defined at the end of this section.

Having obtained the above result, we succeeded in proving that the solutions to the Euler equations (1.2) are classical under the assumption

$$\mathrm{rot}\boldsymbol{v}_0 \in C_*(\overline{\Omega}).$$

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This is the main result in Beirão da Veiga (1984). More precisely, we proved the following statement.

**Theorem 1.1.2.** *Let* $\mathrm{rot}\boldsymbol{u}_0 \in C_*(\overline{\Omega})$ *and* $\mathrm{rot}\boldsymbol{v} \in L^1(\mathbb{R}^+; C_*(\overline{\Omega}))$. *Then, the global solution* $\boldsymbol{v}$ *to problem* (1.2) *is continuous in time with values in* $\mathbf{C}^1(\overline{\Omega})$, *that is*

$$\boldsymbol{v} \in C(\mathbb{R}^+; \mathbf{C}^1(\overline{\Omega})). \tag{1.7}$$

*Furthermore, the estimate*

$$\|\boldsymbol{v}(t)\|_{\mathbf{C}^1(\overline{\Omega})} \leq c\mathrm{e}^{c_1 B_t t}\{\|\mathrm{rot}\boldsymbol{v}_0\|_{C_*(\overline{\Omega})} + \|\mathrm{rot}\boldsymbol{v}\|_{L^1(0,t;C_*(\overline{\Omega}))}\} \tag{1.8}$$

*holds for all* $t \in \mathbb{R}^+$, *where*

$$B_t = \|\mathrm{rot}\boldsymbol{v}_0\| + \|\mathrm{rot}\boldsymbol{v}\|_{L^1(0,t;C(\overline{\Omega}))}. \tag{1.9}$$

*Moreover,* $\partial_t \boldsymbol{v}$ *and* $\nabla\pi$ *are continuous in* $\overline{Q}$ *if both terms* $\boldsymbol{v}_0$ *and* $\nabla F$, *in the canonical Helmholtz decomposition* $\boldsymbol{v} = \boldsymbol{v}_0 + \nabla F$ *separately satisfy this same continuity property. Then all derivatives that appear in equations* (1.2) *are continuous in* $\overline{Q}$, *that is, we have a classical solution.*

The conclusion of the theorem is false in general for data $\boldsymbol{v}_0 \in \mathbf{C}^1(\overline{\Omega})$, or $\boldsymbol{v} \in L^1(\mathbb{R}^+; \mathbf{C}^1(\overline{\Omega}))$.

If $\Omega$ is not simply connected the results still apply, as remarked in Beirão da Veiga (1984), by appealing to well known devices. See, for instance, the appendix 1 in the above reference.

Concerning the 2D Euler equations, we also refer the reader to Koch (2002). In this interesting work the author considers not only the 2D Euler equations but also many other central problems. However, the claims and proofs that followed to treat the particular two-dimensional problem considered in reference Beirão da Veiga (1984) are not very dissimilar to those previously showed by us in this last reference. Related results can also be found in reference Vishik (1998).

In Beirão da Veiga (1984) it was remarked that Theorem 1.1.1 could also be extended to solutions to more general linear elliptic boundary value problems. In fact, in Beirão da Veiga (1981) we proved the following regularity result.

**Theorem 1.1.3.** *For every* $f \in C_*(\overline{\Omega})$ *the solution* $u$ *to the problem*

$$\begin{cases} \mathcal{L}u = f & in \quad \Omega, \\ \mathcal{B}u = 0 & on \quad \Gamma, \end{cases} \tag{1.10}$$

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*belongs to* $C^2(\overline{\Omega})$. *Moreover, there is a constant* $c_0$ *such that the estimate*

$$\|u\|_2 \leq c_0 \|f\|_*, \quad \forall f \in C_*(\overline{\Omega}). \tag{1.11}$$

*holds.*

In the above theorem $\mathcal{L}$ is a second order partial differential elliptic operator with smooth coefficients, and $\mathcal{B}$ is a linear differential operator, of order less or equal to one, acting on the boundary $\Gamma$. In Beirão da Veiga (1981) we assumed that $\mathcal{L}$, $\mathcal{B}$, and $\Omega$ are such that, for each $f \in C(\overline{\Omega})$, problem (1.10) has a unique solution $u \in C^1(\overline{\Omega})$, given by

$$u(x) = \int_\Omega g(x,y) f(y)\, \mathrm{d}y, \tag{1.12}$$

where $g$ is the Green function associated with he above boundary value problem. Our hypotheses on $\mathcal{L}$, $\mathcal{B}$, and $\Omega$ are given by the following two requirements:

– For each $f \in C(\overline{\Omega})$ the solution $u$ of problem (1.10) is unique, belongs to $C^1(\overline{\Omega})$, and is is given by (1.12). Furthermore, if $f \in C^\infty(\overline{\Omega})$ then $u \in C^2(\overline{\Omega})$.

– The above Green's function $g(x,y)$ satisfies the estimates

$$\Big|\frac{\partial g}{\partial x_i}\Big| \leq \frac{k}{|x-y|^{n-1}}, \quad \Big|\frac{\partial^2 g}{\partial x_i \partial x_j}\Big| \leq \frac{k}{|x-y|^n}, \tag{1.13}$$

where $i, j = 1, ..., n$.

The above estimates for Green's functions have been well known for a large class of problems for a long time. Classical works are due to, for instance, Levi (1908, 1909), Hadamard (1914), Lichtenstein (1918), Eidus (1958), Levy (1920), and many other authors. We refer in particular to (Miranda, 1955, Chap. III, Sections 21, 22, and 23), and references therein (in particular, to Giraud's references). For much more general results on Green functions see Solonnikov (1970, 1971).

It is worth noting that the proof of Theorem 1.1.3 may be extended to a larger class of problems, like non-homogeneous boundary-value problems, elliptic systems, and in particular the Stokes system, higher order problems, etc. The main point is that solutions $u$ are given by expressions like (1.12), where the Green's functions $g$ satisfy suitable estimates, which extend that shown in equation (1.13). Recently, we have adapted the unpublished proof of theorem 1.1.3 to show a similar regularity result for solutions to the Stokes system (1.10). Actually, in Beirão da Veiga (2014) we prove the following result.

**Theorem 1.1.4.** *For every $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$ the solution $(\boldsymbol{u}, p)$ of the Stokes system*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & in\,\Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & in\,\Omega, \\ \boldsymbol{u} = 0 & on\,\Gamma, \end{cases} \tag{1.14}$$

*belongs to $\boldsymbol{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$. Moreover, there exists a constant $c_0$, depending only on $\Omega$, such that the estimate*

$$\|\boldsymbol{u}\|_2 + \|\nabla p\| \leq c_0 \|\boldsymbol{f}\|_*, \quad \forall \boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega}), \tag{1.15}$$

*holds.*

In the final part of the section we define the Banach space $C_*(\overline{\Omega})$.
If $f \in C(\overline{\Omega})$ set, for each $r > 0$,

$$\omega_f(r) \equiv \sup_{x,y\in\Omega;0<|x-y|\leq r} |f(x) - f(y)|, \tag{1.16}$$

and define the semi-norm

$$[f]_* = [f]_{*,\delta} \equiv \int_0^\delta \omega_f(r) \frac{\mathrm{d}r}{r}. \tag{1.17}$$

If $0 < \delta < R$, one has

$$[f]_{*,\delta} \leq [f]_{*,R} \leq [f]_{*,\delta} + 2\Big( \log \frac{R}{\delta} \Big) \|f\|. \tag{1.18}$$

It follows that norms (obtained by the addition of $\|f\|$, see (1.20) below), are equivalent.

In the literature, the condition

$$\int_0^\delta \omega_f(r) \frac{\mathrm{d}r}{r} < +\infty$$

is called *Dini's continuity condition*, see Gilbarg & Trudinger (1977), equation (4.47). In Gilbarg & Trudinger (1977), problem 4.2, it is remarked that if $f$ satisfies Dini's condition in $\mathbb{R}^n$, then its Newtonian potential is a $C^2$ solution of Poisson's equation $\Delta u = f$ in $\mathbb{R}^n$.

**Definition 1.1.5.**

$$C_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_* < \infty\}. \tag{1.19}$$

As claimed in Beirão da Veiga (1984), $C_*(\overline{\Omega})$ endowed with the norm

$$\|f\|_* \equiv [f]_* + \|f\|, \tag{1.20}$$

is a Banach space, compactly embedded in $C(\overline{\Omega})$. Furthermore, $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$. We have appealed to these properties in reference Beirão da Veiga (1984), however the complete proofs were written only in an unpublished manuscript Beirão da Veiga (1981). For the complete proofs see the recent publication Beirão da Veiga (2014).

In Beirão da Veiga (1981) we introduced a functional space $B_*(\overline{\Omega})$, which strictly contains $C_*(\overline{\Omega})$, for which we have proven that the second order derivatives of the solutions to the system (1.10) are bounded in $\Omega$ for all $f \in B_*(\overline{\Omega})$. However, we did not succeed in proving, or disproving, the full result, namely, the continuity up to the boundary of the second order derivatives. This led us to leave unpublished the statements concerning the space $B_*(\overline{\Omega})$. In the next sections we show some of these results and proofs, and related open problems. Some results are proved below for data in a larger space $D_*(\overline{\Omega}) \supset B_*(\overline{\Omega})$.

As remarked in Beirão da Veiga (2014), another significant candidate could be obtained by replacing in the definition of $C_*(\overline{\Omega})$ given in (1.17) by the quantity $\omega_f(x;r)$ by

$$\widetilde{\omega}_f(x;r) = \sup_{x\in\Omega} \Big| f(x) - |\Omega(x;r)|^{-1} \int_{\Omega(x;r)} f(y)\,\mathrm{d}y \Big|. \tag{1.21}$$

## 1.2 The functional spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$.

In this section we define the spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$. We start with $B_*(\overline{\Omega})$. Set

$$\omega_f(x;r) = \sup_{y\in\Omega(x;r)} |f(x) - f(y)|, \tag{1.22}$$

and define, for each $x \in \overline{\Omega}$, the "point-wise" semi-norms

$$p_x(f) \equiv \int_0^\delta \omega_f(x;r)\frac{\mathrm{d}r}{r}, \tag{1.23}$$

and also the "global" semi-norm

$$\langle f\rangle_* = \sup_{x\in\overline{\Omega}} \int_0^\delta \omega_f(x;r)\frac{\mathrm{d}r}{r} = \sup_{x\in\overline{\Omega}} p_x(f). \tag{1.24}$$

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Note that

$$[f]_* = \int_0^\delta \sup_{x\in\overline{\Omega}} \omega_f(x;r)\frac{\mathrm{d}r}{r}. \tag{1.25}$$

**Definition 1.2.1.**

$$B_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : \langle f\rangle_* < +\infty\}. \tag{1.26}$$

The space $B_*(\overline{\Omega})$ endowed with

$$\|f\|^* \equiv \|f\| + \langle f\rangle_*, \tag{1.27}$$

is a normed linear space. Clearly $\langle f\rangle_* \leq [f]_*$. Further, in Beirão da Veiga (1981), we proved that the embedding $B_*(\overline{\Omega}) \subset C_*(\overline{\Omega})$ is strict, by constructing an oscillating function which belongs to $B_*(\overline{\Omega})$ but not to $C_*(\overline{\Omega})$; for the counterexample we take $\overline{\Omega} = [0,1]$. We show this construction in Section 1.7 below.

Next we define $D_*(\overline{\Omega})$. Set

$$S(x;r) = \{y \in \Omega : |x-y| = r\},$$

and define

$$\mu_f(x;r) = \sup_{y\in S(x;r)} |f(x) - f(y)|, \tag{1.28}$$

for each fixed $x \in \overline{\Omega}$ and $r > 0$. Further, fix a real positive $\delta$, and define the semi-norms

$$q_x(f) \equiv \int_0^\delta \mu_f(x;r)\frac{\mathrm{d}r}{r}, \tag{1.29}$$

for each $x \in \overline{\Omega}$. As in (1.18), the particular positive value $\delta$ is not significant here. Note that the continuity of $f$ at single point $x$ follows necessarily from the finiteness of the integral in equation (1.29). To avoid unnecessary complications, we assume in the sequel that $f \in C(\overline{\Omega})$. Next define the semi-norm

$$(f)_* = \sup_{x\in\overline{\Omega}} \int_0^\delta \omega_f(x;r)\frac{\mathrm{d}r}{r} = \sup_{x\in\overline{\Omega}} q_x(f). \tag{1.30}$$

It is worth noting that all the semi-norms introduced above enjoy property (1.18).

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**Definition 1.2.2.**

$$D_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : (f)_* < +\infty\}. \tag{1.31}$$

The linear space $D_*(\overline{\Omega})$ endowed with

$$|||f|||^* \equiv \|f\| + (f)_*, \tag{1.32}$$

is a normed linear space. Obviously, $B_*(\overline{\Omega}) \subset D_*(\overline{\Omega})$. Finally, note that (1.18) holds for the above two functional spaces, with the obvious modifications.

## 1.3 Results and open problems.

**Theorem 1.3.1.** *Let $f \in D_*(\overline{\Omega})$, and let $u$ be the solution to problem* (1.10)*. Then the first order derivatives of the solution $u$ are Lipschitz continuous in $\overline{\Omega}$. Furthermore, the estimate*

$$\|\nabla^2 u\|_{L^\infty(\Omega)} \leq c_0 \; |||f|||^* \tag{1.33}$$

*holds.*

The proof of this result is an extension of the unpublished proof given in Beirão da Veiga (1981) for data $f \in B_*(\overline{\Omega})$. The proof will be shown in Section 1.4.

It remains an open problem whether the Theorems 1.1.3 and 1.1.4 hold with $C_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$ or by $D_*(\overline{\Omega})$. Let us discuss this point. Below we prove the following *conditional* result.

**Theorem 1.3.2.** *Let $u$ be the solution of problem* (1.10) *with a given data $f \in D_*(\overline{\Omega})$. Assume that there is a sequence of data $f_m \in D_*(\overline{\Omega})$, convergent to $f$ in $D_*(\overline{\Omega})$, such that the solutions $u^m$ of problem* (1.10) *with data $f_m$ belong to $C^2(\overline{\Omega})$. Then $u \in C^2(\overline{\Omega})$, and moreover*

$$\|\nabla^2 u\| \leq c_0 |||f|||^*. \tag{1.34}$$

Theorem 1.3.2 will be proven in Section 1.5. It is worth noting that, since $B_*(\overline{\Omega}) \subset D_*(\overline{\Omega})$, the above two theorems hold with $D_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$, and $|||f|||^*$ replaced by $\|f\|^*$.

**Corollary 1.3.3.** *If $C^\infty(\overline{\Omega})$ is dense in $B_*(\overline{\Omega})$, then solutions $u$ with data $f \in B_*(\overline{\Omega})$, belong to $C^2(\overline{\Omega})$, moreover*

$$\|u\|_2 \leq c_0 \|f\|^*, \quad \forall f \in B_*(\overline{\Omega}). \tag{1.35}$$

*The result holds with $B_*(\overline{\Omega})$ replaced by $D_*(\overline{\Omega})$, and $\|f\|^*$ replaced by $|||f|||^*$.*

In the above corollary we may replace $C^\infty(\overline{\Omega})$ by $C^{0,\lambda}(\overline{\Omega})$ (or even by $C_*(\overline{\Omega})$, as a consequence of Theorem 1.1.3). However we put here $C^\infty(\overline{\Omega})$ functions since there is a well known "two steps" argument used to prove density of $C^\infty(\overline{\Omega})$ in larger functional spaces. The first step consists of constructing a linear continuous map $f \to \widetilde{f}$, from $B_*(\overline{\Omega})$ to $B_*(\Omega_\delta)$, where

$$\Omega_\delta \equiv \{x : \text{dist}(x, \Omega) < \delta\}, \tag{1.36}$$

for some $\delta > 0$, such that the restriction of $\widetilde{f}$ to $\overline{\Omega}$ coincides with $f$. In the second step, we appeal to the usual mollification technique to prove the desired density result in compact subsets of $\Omega_\delta$, so in $\overline{\Omega}$. The extension step is necessary, since approximation by mollification may hold only in compact subsets.

Concerning the first step, we prove the following "extension" result. For the proof, see the Section 1.6 below.

**Proposition 1.3.4.** *There exists $\delta > 0$, depending only on $\Omega$, such that the following statement holds. There is a linear continuous map $f \to \widetilde{f}$, from $B_*(\overline{\Omega})$ to $B_*(\Omega_\delta)$, such that the restriction of $\widetilde{f}$ to $\overline{\Omega}$ coincides with $f$.*

Theorem 1.3.1, together with the ideas introduced by Beirão da Veiga (1984), provides new regularity results for solutions to the 2D Euler equations. This point will be considered in a forthcoming paper. For the time being, we merely state the following result.

**Theorem 1.3.5.** *Let $\text{rot}\,\boldsymbol{v}_0 \in B_*(\overline{\Omega})$, and assume that the external forces $\boldsymbol{v}$ vanish. Then, the global solution $\boldsymbol{v}$ to problem (1.2) satisfies*

$$\nabla \boldsymbol{u} \in L^\infty(\overline{Q}_T), \tag{1.37}$$

*for all $T > 0$.*

The conclusion of the theorem is false in general for data $\boldsymbol{v}_0 \in \mathbf{C}^1(\overline{\Omega})$.

It is worth noting that we have no reason to conjecture density of $C^\infty(\overline{\Omega})$ in $B_*(\overline{\Omega})$. In fact, our advise to readers interested in the subject is to try to prove the full Theorem 1.3.1 in the framework of the norm obtained by means of (1.21). In this context, it would be of interest to show that $C_*(\overline{\Omega})$ in strictly contained in the above new space.

## 1.4 Proof of Theorem 1.3.1.

We start by estimating the differential quotients of the first order derivatives of the solutions $u$.

**Proposition 1.4.1.** *Let $f \in C(\overline{\Omega})$, and let $u$ be the solution to problem (1.10). Assume that for every given $x_0 \in \overline{\Omega}$, there exists a real $\delta_0 > 0$ such that*

$$q_{x_0}(f) \equiv \int_0^{\delta_0} \mu_f(x_0; r) \frac{\mathrm{d}r}{r} < \infty. \tag{1.38}$$

*Then*

$$\Big| \frac{\partial_i u(x) - \partial_i u(x_0)}{x - x_0} \Big| \le c(q_{x_0}(f) + \|f\|), \tag{1.39}$$

*for all $x \in \frac{\delta_0}{2}$, and $i = 1, ..., n$.*

*Proof.* Let us introduce the auxiliary function $v(x)$ defined by

$$\begin{cases} \mathcal{L}v = 1 & \text{in} \quad \Omega, \\ \\ \mathcal{B}v = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.40}$$

In particular $v \in C^{1,1}(\overline{\Omega})$, by assumption (i). Define

$$k_1 \equiv \|v\|_{1,1}. \tag{1.41}$$

Clearly $k_1$ depends only on $\mathcal{L}$, $\mathcal{B}$, and $\Omega$, since $v$ is completely determined by these data. Actually, $k_1$ depends only on some parameters related to the above elements (like the ellipticity constant of $\mathcal{L}$, for instance).

Define, in $\Omega$, the functions $v^0(x) \equiv f(x_0)v(x)$, and $w(x) \equiv u(x) - v^0(x)$. Clearly, for each index $i = 1, ..., n$, we have

$$\partial_i w(x) = \int_\Omega \partial_i g(x, y)[f(y) - f(x_0)] \, \mathrm{d}y, \quad \forall x \in \Omega.$$

Consequently,

$$|\partial_i w(x) - \partial_i w(x_0)| \le \int_\Omega |\partial_i g(x, y) - \partial_i g(x_0, y)||f(y) - f(x_0)| \, \mathrm{d}y,$$

for all $x \in \Omega$.

Define

$$\Omega_c(x_0; r) = \Omega - \Omega(x_0; r),$$

and set $\rho = |x - x_0|$. We have

$$\begin{aligned}
&|\partial_i w(x) - \partial_i w(x_0)| \\
&\le \int_{\Omega(x_0;2\rho)} |\partial_i g(x, y) - \partial_i g(x_0, y)||f(y) - f(x_0)| \,\mathrm{d}y \\
&+ \int_{\Omega(x_0;\delta_0)-\Omega(x_0;2\rho)} |\partial_i g(x, y) - \partial_i g(x_0, y)||f(y) - f(x_0)| \,\mathrm{d}y \\
&+ \int_{\Omega_c(x_0;\delta_0)} |\partial_i g(x, y) - \partial_i g(x_0, y)||f(y) - f(x_0)| \,\mathrm{d}y \\
&\equiv I_1 + I_2 + I_3, \qquad (1.42)
\end{aligned}$$

where $\delta_0 > 2\rho$.

Further, by appealing to the first estimate (1.13), we show that

$$\begin{aligned}
I_1 \le& 2\|f\| \left\{ \int_{\Omega(x_0;2\rho)} |\partial_i g(x_0, y)| \,\mathrm{d}y + \int_{\Omega(x_0;2\rho)} |\partial_i g(x, y)| \,\mathrm{d}y \right\} \\
\le& c\|f\| \left\{ \int_{\Omega(x_0;2\rho)} \frac{\mathrm{d}y}{|x_0 - y|^{n-1}} + \int_{I(x;3\rho)} \frac{\mathrm{d}y}{|x - y|^{n-1}} \right\}. \qquad (1.43)
\end{aligned}$$

Hence

$$I_1 \le c\rho\|f\|.$$

On the other hand, by appealing to the mean-value theorem and the second estimate in (1.13), we find that

$$|\partial_i g(x, y) - \partial_i g(x_0, y)| \le c\rho|x' - y|^{-n} \le c\rho 2^n |x_0 - y|^{-n},$$

for each $y \in \Omega_c(x_0; 2\rho)$, where the point $x'$ belongs to the straight segment joining $x_0$ to $x$. Consequently,

$$I_2 \leq c\rho \int_{\Omega(x_0;\delta_0)-\Omega(x_0;2\rho)} |f(y) - f(x_0)| \frac{\mathrm{d}y}{|x_0 - y|^n} \tag{1.44}$$

$$\leq c\rho \int_{2\rho}^{\delta_0} \frac{\mathrm{d}r}{r^n} \int_{S(x_0;r)} \mu_f(x_0;r)\,\mathrm{d}S \tag{1.45}$$

$$\leq c\rho S_n \int_{2\delta}^{\delta_0} \mu_f(x_0;r) \frac{\mathrm{d}r}{r}, \tag{1.46}$$

where $S_n$ is the area of the surface of the $n$-dimensional unit sphere. It follows that

$$I_2 \leq \; c\rho q_{x_0}(f).$$

Finally, a crude estimate for $I_3$ shows that

$$I_3 \leq c\rho \int_{\Omega_c(x_0;\delta_0)} \frac{1}{|x_0 - y|^n} |f(y) - f(x_0)|\,\mathrm{d}y \leq 2c\delta_0^{-n} \rho \|f\|.$$

By appealing to equation (1.42) and to the estimates proved above for $I_1$, $I_2$, and $I_3$, we obtain that

$$|\partial_i w(x) - \partial_i w(x_0)| \leq c\rho(q_{x_0}(f) + \|f\|).$$

Since

$$|\partial_i v^0(x) - \partial_i v^0(x_0)| \leq k_1 \rho |f(x_0)|,$$

it follows that

$$\begin{aligned} c|\partial_i u(x) - \partial_i u(x_0)| \leq & |\partial_i w(x) - \partial_i w(x_0)| + |\partial_i v^0(x) - \partial_i v^0(x_0)| \\ \leq & c\rho\big(q_{x_0}(f) + \|f\| + k_1 |f(x_0)|\big). \end{aligned}$$

So,

$$\frac{|\partial_i u(x) - \partial_i u(x_0)|}{|x - x_0|} \leq c(q_{x_0}(f) + \|f\|), \quad \forall x \in \Omega, x \neq x_0. \tag{1.47}$$

This shows (1.39), completing the proof.

□

The proof of Theorem 1.3.1 follows immediately from proposition 1.4.1. Note that, by appealing to the first equation (1.13), we obtain

$$|\partial_i u(x)| \leq \|f\| \int_\Omega |\partial_i g(x,y)|\,\mathrm{d}y \leq c\|f\|, \quad \forall x \in \Omega.$$

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Hence,

$$\|\nabla u\| \le c\|f\|, \tag{1.48}$$

where $u$ is the solution of problem (1.10).

## 1.5 Proof of Theorem 1.3.2.

Due to Theorem 1.3.1, it is sufficient to show that the second order classical derivatives of the solution $u(x)$ exist and are continuous, everywhere in $\overline{\Omega}$.

Consider the solutions $u^m$ of problems

$$\begin{cases} \mathcal{L}u^m = f_m \quad \text{in} \quad \Omega, \\ \mathcal{B}u^m = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{1.49}$$

Clearly $u_m \to u$ in $C(\overline{\Omega})$. Further, by assumption, $u^m \in C^2(\overline{\Omega})$. By applying the result stated in Theorem 1.3.1 to the system

$$\begin{cases} \mathcal{L}(u^m - u^n) = f_m - f_n \quad \text{in} \quad \Omega, \\ \mathcal{B}(u^m - u^n) = 0 \quad \text{on} \quad \Gamma, \end{cases} \tag{1.50}$$

we obtain $\|\partial_{ij} u^m - \partial_{ij} u^n\| \le c_0 \|f_m - f_n\|^*$. This proves that $\partial_{ij} u^m$ is a Cauchy sequence in $C(\overline{\Omega})$. Hence, by the completeness of $C(\overline{\Omega})$, there exists an element $v_{ij} \in C(\overline{\Omega})$ such that the sequence $\partial_{ij} u^m$ is uniformly convergent in $\overline{\Omega}$ to $v_{ij}$. Furthermore, by applying estimate (1.48) to $u^m - u$, it follows that

$$\|\partial_i u^m - \partial_i u\| \le c\|f_m - f\|.$$

Hence $\partial_i u^m$ converges uniformly in $\overline{\Omega}$ to $\partial_i u$.

The above results guarantee that the second order derivatives $\partial_{ij} u$ exist and are given by $v_{ij}$, for $i, j = 1, ..., n$.

## 1.6 Proof of Proposition 1.3.4.

In this section we prove the Proposition 1.3.4. The density of smooth functions is a crucial ingredient in proving full regularity. As already remarked, Proposition 1.3.4 is a typical first step to try to prove that smooth functions are dense in $B_*(\overline{\Omega})$. We start with some preliminary results. Recall that

$$\Omega_\delta \equiv \{x : \mathrm{dist}(x, \Omega) < \delta\}.$$

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It is well known that, for sufficiently small, positive $\delta$, we can construct a suitable system of parallel surfaces $\Gamma_r$, where $-2\delta < r < 2\delta$, and $\Gamma_0 = \Gamma$. The surface $\Gamma_r$ is at distance $|r|$ from $\Gamma$. It lies inside or outside $\Omega$, depending on the negative or positive sign of the parameter $r$. Furthermore, a one to one correspondence between pairs of points in the opposite surfaces $\Gamma_r$ and $\Gamma_{-r}$ is defined by imposing that they belong to the same straight segment, orthogonal to $\Gamma$. We say that these points, denoted here by $x$ and $\overline{x}$, are obtained from each other, by reflection (with respect to $\Gamma$).

Note that a positive lower bound for $\delta$ depends on the upper bound of the absolute values of the principal curvatures of the boundary $\Gamma$. We denote such a positive lower bound by $\delta$, and use this same value in definitions (1.23) and (1.24).

**Lemma 1.6.1.** *There exist $\delta > 0$ and $k \geq 1$ (which depend only on the given set $\Omega$), such that the following holds. Given $f \in C(\overline{\Omega})$, there is an extension $\widetilde{f} : \Omega_{2\delta} \to \mathbb{R}$,*

$$\widetilde{f}(x) = f(x) \quad \textit{for} \quad x \in \overline{\Omega},$$

*such that, for each $r \in (0, \delta)$,*

$$\omega_{\widetilde{f}}(x; r) \leq \begin{cases} \omega_f(x; kr), & \textit{if} \quad x \in \overline{\Omega}, \\ \omega_f(\overline{x}; kr), & \textit{if} \quad x \in \overline{\Omega}_\delta - \overline{\Omega}. \end{cases} \tag{1.51}$$

*Proof.* We define

$$\omega_f(x; r) = \sup_{y \in \Omega(x; r)} |f(x) - f(y)|, \tag{1.52}$$

and similarly,

$$\omega_{\widetilde{f}}(x; r) = \sup_{y \in \Omega_{2\delta}(x; r)} |\widetilde{f}(x) - \widetilde{f}(y)|. \tag{1.53}$$

Obviously, if $x \in \Omega$, and $\mathrm{dist}(x, \Gamma) \geq \delta$, it follows that

$$\omega_{\widetilde{f}}(x; r) = \omega_f(x; r).$$

Hence we assume below that $\mathrm{dist}(x, \Gamma) \leq \delta$.

In the sequel we show the basic ideas that lead to a more formal proof, whish is left to the interested reader.

We start by considering the particular case $x_0 \in \Gamma$, and by assuming that inside the $\delta$-neighborhood of $x_0$ the boundary $\Gamma$ is flat. Under these assumptions, compare the quantity

$$\omega_{\widetilde{f}}(x_0; r) = \sup_{y \in I(x_0; r)} |\widetilde{f}(x_0) - \widetilde{f}(y)|,$$

with that defined by (1.52) for $x = x_0$. In the case of $\widetilde{f}$, the points $y$ describe a full ball, while, in the case of $f$, the points $y$ describe a half ball. However, the set of numerical values $f(y)$ into play are, in both cases, exactly the same. The reader may verify that a similar situation occurs whenever the sphere $I(x; r)$ intersects $\Gamma$, in the flat case. The numerical values $f(y)$ into play are still the same, for $y \in I(x; r)$ and $y \in \overline{\Omega}(\overline{x}; r)$. The above remarks show that, in the locally-flat boundary case, (1.51) holds with $k = 1$.

In the general, non-flat, case, the geometrical situation is simply a deformation of the above one. Let us start by assuming that $x \notin \overline{\Omega}$, and $I(x, r)$ does not intersect $\Gamma$. Here, contrary to the flat boundary case, the reflection of $I(x, r)$ is not $I(\overline{x}, r)$. However, the reflection is contained in a (possibly large) sphere $I(\overline{x}, kr)$ (roughly speaking, $k \leq 1$, if $\Gamma$ is locally convex, and $k \geq 1$, if $\Gamma$ is locally concave). Since we assume that $\Omega$ is regular (in particular, locally situated in one side of the boundary) the local values obtained for $k$ are uniformly bounded from above.

Finally, if $I(x, r)$ intersects the boundary $\Gamma$, the more general picture is simply an overlap of the two single situations, already described. Details are left to the reader.

□

*Proof of proposition 1.3.4.* By Lemma 1.6.1, we may construct an extension $\widetilde{f} : \Omega_{2\delta} \to \mathbb{R}$ of the given function $f$, such that (1.51) holds. It follows that, for each $x \in \overline{\Omega}$,

$$\int_0^\delta \omega_{\widetilde{f}}(x; r) \frac{\mathrm{d}r}{r} \leq \int_0^\delta \omega_f(x; kr) \frac{\mathrm{d}r}{r},$$

and, for each $x \in \overline{\Omega}_\delta - \overline{\Omega}$,

$$\int_0^\delta \omega_{\widetilde{f}}(x; r) \frac{\mathrm{d}r}{r} \leq \int_0^\delta \omega_f(\overline{x}; kr) \frac{\mathrm{d}r}{r} = \int_0^{k\delta} \omega_f(\overline{x}; r) \frac{\mathrm{d}r}{r}.$$

In particular, it follows (note that extension of formula (1.18) holds) that

$$\langle \widetilde{f} \rangle_{*,\delta} \leq \langle f \rangle_{*,k\delta} \leq \langle f \rangle_{*,\delta} + 2\Big(\log k\Big) \|f\|.$$

$\square$

## 1.7 The embedding $B_*(\overline{\Omega}) \subset C_*(\overline{\Omega})$ is strict.

Below we construct an oscillating function in the interval $[0,1]$ which belongs to $B_*(\overline{\Omega})$ but not to $C_*(\overline{\Omega})$.

**Proposition 1.7.1.** *The inclusion $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega})$ is proper.*

*Proof.* For each non-negative integer $n$ set

$$r_n = e^{-2^n},$$

and define, in the interval $I = [0,1]$, the real continuous function

$$f_n(x) = \begin{cases} \frac{2^{-n}}{r_n}(x - (2^{-n} - r_n)), & \text{if} \quad 2^{-n} - r_n \le x \le 2^{-n}; \\[2ex] \frac{2^{-n}}{r_n}((2^{-n} + r_n) - x), & \text{if} \quad 2^{-n} \le x \le 2^{-n} + r_n; \\[2ex] 0, \quad \text{if} \quad |x - 2^{-n}| \ge r_n. \end{cases} \tag{1.54}$$

Note that $f_n$ is linear in the intervals $[2^{-n} - r_n, 2^{-n}]$ and $[2^{-n}, 2^{-n} + r_n]$. Below we work with $\delta = r_0 = \frac{1}{\mathrm{e}}$ in Definition 1.17.

In $I$, define the function

$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

Note that $0 \le f(x) \le x$. We start by showing that $f \notin C_*(\overline{\Omega})$. For convenience, we set

$$\omega(r) = \omega_f(r).$$

Clearly, if $r_{n+1} \le r \le r_n$ then

$$2^{-(n+1)} = \omega(r_{n+1}) \le \omega(r) \le \omega(r_n) = 2^{-n}.$$

Hence

$$\int_0^{\delta} \omega(r) \frac{\mathrm{d}r}{r} \ge \sum_{n=0}^{\infty} \int_{r_{n+1}}^{r_n} \omega(r_{n+1}) \frac{\mathrm{d}r}{r} \ge \sum_{n=0}^{\infty} \frac{1}{2} = \infty.$$

This shows that $f \notin C_*(\overline{\Omega})$.

Next we prove that $f \in B_*(\overline{\Omega})$. We want to show that there exists a constant $c_0$ such that, for all $x \in I$,

$$\int_0^{\delta} \omega_f(x; r) \frac{\mathrm{d}r}{r} \le c_0. \tag{1.55}$$

We start by considering the points $x_n = 2^{-n}$ where the function $f(x)$ attain local maximum values. Actually, $f(2^{-n}) = 2^{-n}$. Other points $x \in I$ can be treated by following a similar argument.

Let $x_n = 2^{-n}$ be fixed. One has

$$\omega_f(x_n; r) \le \begin{cases} 2^{-n}\frac{r}{r_n}, & \text{if} \quad 0 < r \le r_n; \\ \\ 2^{-n+1}, & \text{if} \quad r_n \le r \le 2^{-n}; \\ \\ 2^{-n} + r, & \text{if} \quad 2^{-n} \le r \le r_0. \end{cases} \tag{1.56}$$

Note that equality holds in the first row of equation (1.56). Further, in the second row, the inequality $r \le 2^{-n}$ should be interpreted as $x_n + r \le 2^{-n+1}$. The second and third rows follow from the inequality

$$\omega(x, r) \le x + r,$$

which holds for all $x \in I$ and $r > 0$.

Next, in accordance with (1.56), we decompose the integral on the left hand side of equation (1.55) as the sum of three integrals, and we estimate each of the integrals by appealing to the related region in (1.56). After some elementary calculations (left to the reader) it is east to see that equation (1.55) holds with $c_0 = 3$, for each point $x_n$. Finally note that, for $x = 0$, equation (1.55) holds with $c_0 = 1$.

Assume now that

$$x_{n-1} < x < x_n, \tag{1.57}$$

for some index $n$. We will follow the argument used above. As before,

$$\omega_f(x; r) \le \begin{cases} 2^{-n+1}, & \text{if} \quad r_n \le r \le 2^{-n}; \\ \\ 2^{-n} + r, & \text{if} \quad 2^{-n} \le r \le r_0. \end{cases} \tag{1.58}$$

It remains to consider the case in which $0 < r \le r_n$. Under this assumption, and by taking into account (1.57), it readily follows that points $y$ satisfying $|y - x| < r_n$ do not reach the supports of $f_{n+2}$ and $f_{n-1}$. Hence we may replace here $f$ by $f_n + f_{n+1}$. It follows that

$$\omega_f(x; r) \le \omega_{(f_n + f_{n+1})}(x; r) \le \omega_{f_n}(x; r) + \omega_{f_{n+1}}(x; r),$$

for $0 < r \le r_n$. The above considerations lead to (1.55).

□

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We remark that the above function $f$ is not the limit in the $B_*(\overline{\Omega})$ norm of the sequence of Lipschitz continuous functions

$$F_N(x) = \sum_{n=1}^{N} f_n(x),$$

since $\langle f_n \rangle_* \geq \frac{1}{2}$, for every $n$. This fact does not exclude the possibility of approximating $f$ by a sequence of elements belonging to $C_*(\overline{\Omega})$.

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