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EXISTENCE OF TORUS BUNDLES ASSOCIATED TO COCYCLES

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A Kuga fibre variety is a fibre bundle over a locally symmetric space whose fibre is a polarized Abelian variety. We describe a complex torus bundle associated to a 2-cocycle of a discrete group, which may be regarded as a generalized Kuga fibre variety, and prove the existence of such a bundle.

1. INTRODUCTION

Families of Abelian varieties parametrised by an arithmetic quotient of a Hermitian symmetric domain play an important role in number theory. When such a family is viewed as a fibre bundle over an arithmetic variety its base space is essentially a Shimura variety and its total space is a Kuga fibre variety. This paper is concerned with a complex torus bundle associated to a 2-cocycle of a discrete group which may be regarded as a generalised Kuga fibre variety.

Let $\mathcal{H}_n$ be the Siegel upper half space of degree n on which the symplectic group $Sp(n,\mathbb{R})$ acts as usual. If Γ' is an arithmetic subgroup of $Sp(n,\mathbb{R})$, then the associated quotient space $\Gamma'\backslash\mathcal{H}_n$ can be regarded as a moduli space of a certain family of polarised Abelian varieties. Such a family of Abelian varieties can be considered as a fibre variety over the Siegel modular variety $X' = \Gamma'\backslash\mathcal{H}_n$.

Let $G = \mathbb{G}(\mathbb{R})$ be a semisimple Lie group of Hermitian type that can be realised as the set of real points of a linear algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$. Thus the quotient $\mathcal{D} = G/K$ by a maximal compact subgroup K of G has the structure of a Hermitian symmetric domain. We assume that there are a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}_n$ and a homomorphism $\rho : G \to Sp(n,\mathbb{R})$ of Lie groups defined over $\mathbb{Q}$ such that τ is equivariant with respect to ρ, meaning that $\tau(gz) = \rho(g)\tau(z)$ for all $z \in \mathcal{D}$ and $g \in G$. Let Γ be a torsion-free arithmetic subgroup of G such that $\rho(\Gamma) \subset \Gamma'$. Then the corresponding locally symmetric space $X = \Gamma\backslash\mathcal{D}$ has the structure of a complex variety and is called an arithmetic variety. The holomorphic map τ induces a morphism $\tau_X : X \to X'$ of varieties, and by pulling the fibre variety over X' back via τ_X we obtain a fibre variety over X whose fibres are again polarised Abelian varieties. Such fibre varieties over an arithmetic variety are called Kuga fibre varieties (see [**1, 5**]), and various geometric and

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arithmetic aspects of Kuga fibre varieties have been investigated in numerous papers over the years.

If L is a lattice in $\mathbb{R}^{2n}$, whose elements are regarded as row vectors, satisfying $\ell\rho(\gamma) \in L$ for all $\ell \in L$ and $\gamma \in \Gamma$, then the associated Kuga fibre variety can be considered as the quotient of $\mathcal{D} \times V$ by an action of the semidirect product $\Gamma \ltimes L$. Given a 2-cocycle $\psi : \Gamma \times \Gamma \to L$, we can consider a generalised semidirect product $\Gamma \ltimes_\psi L$ by modifying its multiplication operation by using ψ. Then we can define an action of this generalised semidirect product on $\mathcal{D} \times V$ by using a 1-cochain ξ for Γ with coefficients in the space $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued holomorphic functions which together with ψ satisfies a certain relation (see Section 2), and the associated quotient is also a complex torus bundle over $\Gamma\backslash\mathcal{D}$. In [**2**] some properties of such torus bundles were investigated (see also [**4**] and [**3**, Chapter 8]). Since then, there have been inquiries about the existence of a pair (ξ, ψ) satisfying the required condition. The purpose of this paper is to prove the existence of such a pair and provide an example.

2. Torus bundles

In this section we review the construction of a torus bundle associated to a 2-cocycle and a 1-cochain of a discrete subgroup of a semisimple Lie group considered in [**2**]. Let G be a semisimple Lie group of Hermitian type, so that its quotient $\mathcal{D} = G/K$ by a maximal compact subgroup K has the structure of a Hermitian symmetric domain. We assume that there are a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}_n$ and a homomorphism $\rho : G \to Sp(n, \mathbb{R})$ of Lie groups satisfying the equivariance condition

$$\tau(gz) = \rho(g)\tau(z)$$

for all $z \in \mathcal{D}$ and $g \in G$.

Let L be a lattice in $\mathbb{R}^{2n}$, and let Γ be a discrete subgroup of G such that $\ell\rho(\gamma) \in L$ for all $\ell \in L$ and $\gamma \in \Gamma$, where $\ell\rho(\gamma)$ is the matrix product of the row vector ℓ of $2n$ entries and the $2n \times 2n$ matrix $\rho(\gamma)$. Then L is a right Γ-module, and therefore we can consider the cohomology $H^*(\Gamma, L)$ of the group Γ with coefficients in L. We denote by $\mathfrak{C}^k(\Gamma, L)$ and $\mathfrak{Z}^k(\Gamma, L)$ the spaces of the associated k-cochains and k-cocycles, respectively, and choose an element ψ of $\mathfrak{Z}^2(\Gamma, L)$. Thus ψ is a map $\psi : \Gamma \times \Gamma \to L$ satisfying

$$\psi(\gamma_1, \gamma_2)\rho(\gamma_3) + \psi(\gamma_1\gamma_2, \gamma_3) = \psi(\gamma_2, \gamma_3) + \psi(\gamma_1, \gamma_2\gamma_3) \tag{2.1}$$

$$\psi(\gamma, 1) = 0 = \psi(1, \gamma)$$

for all $\gamma_1, \gamma_2, \gamma_3, \gamma \in \Gamma$, where 1 is the identity element of Γ. We note that an element $\alpha \in \mathfrak{Z}^2(\Gamma, L)$ is a coboundary if $\alpha = \partial\beta$ for some $\beta \in \mathfrak{C}^1(\Gamma, L)$, where

$$\partial\beta(\gamma_1, \gamma_2) = \beta(\gamma_2) - \beta(\gamma_1\gamma_2) + \beta(\gamma_1)\rho(\gamma_2) \tag{2.2}$$

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for all $\gamma_1, \gamma_2 \in \Gamma$. We now consider the generalised semidirect product $\Gamma \ltimes_\psi L$ associated to ψ, which consists of the elements (γ, ℓ) of $\Gamma \times L$ and is equipped with the multiplication operation defined by

$$(\gamma_1, \ell_1) \cdot (\gamma_2, \ell_2) = \big(\gamma_1\gamma_2, \ell_1\rho(\gamma_2) + \ell_2 + \psi(\gamma_1, \gamma_2)\big)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $\ell_1, \ell_2 \in L$. Then it can be shown that $\Gamma \ltimes_\psi L$ is a group with respect to this operation.

The symplectic group $Sp(n, \mathbb{R})$ acts on the Siegel upper half space $\mathcal{H}_n$ as usual by

$$g\zeta = (a\zeta + b)(c\zeta + d)^{-1}$$

for all $z \in \mathcal{H}_n$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{R})$. For such $g \in Sp(n, \mathbb{R})$ and $\zeta \in \mathcal{H}_n$, we set

$$j(g, \zeta) = c\zeta + d. \tag{2.3}$$

Then the resulting map $j : Sp(n, \mathbb{R}) \times \mathcal{H}_n \to GL(n, \mathbb{C})$ satisfies

$$j(g'g, \zeta) = j(g', g\zeta)j(g, \zeta) \tag{2.4}$$

for all $\zeta \in \mathcal{H}_n$ and $g, g' \in Sp(n, \mathbb{R})$. Given $z \in \mathcal{D}$ and $\gamma \in \Gamma \subset G$, we set

$$j_{\rho,\tau}(\gamma, z) = j\big(\rho(\gamma), \tau(z)\big). \tag{2.5}$$

Using (2.4) and the fact that τ is an equivariant with respect to ρ, we see that

$$j_{\rho,\tau}(\gamma'\gamma, z) = j_{\rho,\tau}(\gamma', \gamma z)j_{\rho,\tau}(\gamma, z) \tag{2.6}$$

for all $z \in \mathcal{D}$ and $\gamma, \gamma' \in \Gamma$.

Let $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ denote the space of $\mathbb{C}^n$-valued holomorphic functions on $\mathcal{D}$. We introduce on $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ a double Γ-module structure defined by

$$(\gamma \cdot f)(z) = f(z), \quad (f \cdot \gamma)(z) = f(\gamma z)j_{\rho,\tau}(\gamma, z) \tag{2.7}$$

for all $f \in \mathcal{A}(\mathcal{D}, \mathbb{C}^n)$, $\gamma \in \Gamma$ and $z \in \mathcal{D}$, where elements of $\mathbb{C}^n$ are considered as row vectors. Then we can consider the cohomology of the group Γ with coefficients in $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$, where its group $\mathfrak{C}^k\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ of k-cochains consists of all functions

$$\eta : \Gamma^k \to \mathcal{A}(\mathcal{D}, \mathbb{C}^n)$$

such that $\eta(\gamma_1, \dots, \gamma_k) = 0$ whenever at least one of the γ_i is 1. If δ denotes the coboundary operator and if η is a 1-cochain belonging to $\mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$, then we have

$$\delta\eta(\gamma_1, \gamma_2) = \gamma_1 \cdot \eta(\gamma_2) - \eta(\gamma_1\gamma_2) + \eta(\gamma_1) \cdot \gamma_2 \tag{2.8}$$

for all $\gamma_1, \gamma_2 \in \Gamma$, where the right and left actions of Γ are given by (2.7).

We now assume that there exist a 1-cochain $\xi \in \mathfrak{C}^1(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n))$ and a 2-cocycle $\psi \in \mathfrak{Z}^2(\Gamma, L)$ satisfying

$$\delta\xi(\gamma_1, \gamma_2)(z) = \psi(\gamma_1, \gamma_2)\begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} \tag{2.9}$$

for all $z \in \mathcal{D}$, where the right hand side is the matrix product of the row vector $\psi(\gamma_1, \gamma_2)$ $\in L \subset \mathbb{R}^n \times \mathbb{R}^n$ and the complex $2n \times n$ matrix $\begin{pmatrix} \tau(z) \\ 1 \end{pmatrix}$. The existence of such a pair (ξ, ψ) is the main theorem in this paper and will be proved in the next section. Given elements $(\gamma, (\mu, \nu)) \in \Gamma \ltimes_\psi L$ and $(z, w) \in \mathcal{D} \times \mathbb{C}^n$, we set

$$(\gamma, (\mu, \nu)) \cdot (z, w) = \Big(\gamma z, \big(w + \mu\tau(z) + \nu + \xi(\gamma)(z)\big) j_{\rho,\tau}(\gamma, z)^{-1}\Big), \tag{2.10}$$

where $j_{\rho,\tau} : \Gamma \times \mathcal{D} \to GL(n, \mathbb{C})$ is given by (2.5). Then it can be shown that the operation given by (2.10) determines an action of the group $\Gamma \ltimes_\psi L$ on the space $\mathcal{D} \times \mathbb{C}^n$.

We assume that the discrete subgroup $\Gamma \subset G$ does not contain elements of finite order, so that the quotient space $X = \Gamma\backslash\mathcal{D}$ has the structure of a complex manifold, and set

$$Y_{\xi,\psi} = \Gamma \ltimes_\psi L\backslash\mathcal{D} \times \mathbb{C}^n,$$

where the quotient is taken with respect to the action in (2.10). Given $z \in \mathcal{D}$, let L_z denote the lattice in $\mathbb{C}^n$ defined by

$$L_z = \{\mu\tau(z) + \nu \mid (\mu, \nu) \in L\}.$$

Then the map $\pi : Y_{\xi,\psi} \to X$ induced by the natural projections $\mathcal{D} \times \mathbb{C}^n \to \mathcal{D}$ and $\Gamma \ltimes_\psi L \to \Gamma$ has the structure of a fibre bundle over X whose fibre over the point in X corresponding to $z \in \mathcal{D}$ is isomorphic to the complex torus $\mathbb{C}^n/L_z$. If $\psi = 0$ and $\xi = 0$, then the the corresponding torus bundle $Y_{0,0}$ is a family of Abelian varieties known as a Kuga fibre variety (see [**1**, **5**]).

3. Existence

Let the Hermitian symmetric domain $\mathcal{D}$ associated to a semisimple Lie group G, the discrete subgroup Γ of G, and the lattice $L \subset \mathbb{C}^n$ be as in Section 2. In this section we prove the existence of a 1-cochain $\xi \in \mathfrak{C}^1(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n))$ and a 2-cocycle $\psi \in \mathfrak{Z}^2(\Gamma, L)$ considered in Section 2 satisfying (2.9), where $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ is the space of $\mathbb{C}^n$-valued holomorphic functions on $\mathcal{D}$. We also discuss an example.

We consider the space $\mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued holomorphic functions on the Siegel upper half space $\mathcal{H}_n$ as a double Γ-module by defining the left and right Γ-actions by

$$(g \cdot F)(\zeta) = F(\zeta), \quad (F \cdot g)(\zeta) = F(g\zeta)j(g, \zeta) \tag{3.1}$$

for all $F \in \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)$, $g \in Sp(n,\mathbb{R})$ and $\zeta \in \mathcal{H}_n$, where $j(g,\zeta)$ is as in (2.3). Let $\widetilde{\Gamma}$ be a discrete subgroup of $Sp(n,\mathbb{R})$ such that $\rho(\Gamma) \subset \widetilde{\Gamma}$. Then we can consider the cohomology of the group $\widetilde{\Gamma}$ with coefficients in $\mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)$, and the coboundary operator

$$\widetilde{\delta} : \mathfrak{C}^1\big(\widetilde{\Gamma}, \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)\big) \to \mathfrak{C}^2\big(\widetilde{\Gamma}, \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)\big)$$

is given by

$$\widetilde{\delta}\widetilde{\eta}(g_1, g_2) = g_1 \cdot \widetilde{\eta}(g_2) - \widetilde{\eta}(g_1 g_2) + \widetilde{\eta}(g_1) \cdot g_2 \tag{3.2}$$

for $\widetilde{\eta} \in \mathfrak{C}^1\big(\widetilde{\Gamma}, \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)\big)$ and $\gamma_1, \gamma_2 \in \Gamma$, where the left and right actions of Γ are given by (3.1).

Theorem 3.1. *There exist a 1-cochain* $\xi \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ *and a 2-cocycle* $\psi \in \mathfrak{Z}^2(\Gamma, L)$ *satisfying* (2.9).

Proof: We shall first prove that there exist a 2-cocycle $\widetilde{\psi} \in \mathfrak{Z}^2(\widetilde{\Gamma}, L)$ and a 1-cochain $\widetilde{\xi}$ belonging to $\mathfrak{C}^1\big(\widetilde{\Gamma}, \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)\big)$ satisfying

$$\widetilde{\delta}\widetilde{\xi}(g_1, g_2)(\zeta) = \widetilde{\psi}(g_1, g_2)\begin{pmatrix}\zeta \\ 1\end{pmatrix} \tag{3.3}$$

for all $\zeta \in \mathcal{H}_n$ and $g_1, g_2 \in \widetilde{\Gamma}$. Let $\widetilde{\xi} : \widetilde{\Gamma} \to \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)$ be an element of $\mathfrak{C}^1\big(\widetilde{\Gamma}, \mathcal{A}(\mathcal{H}_n, \mathbb{C}^n)\big)$ defined by

$$\widetilde{\xi}(g)(\zeta) = \kappa j(g, \zeta)$$

for all $\zeta \in \mathcal{H}_n$ and $g \in \widetilde{\Gamma}$, where $\kappa \in \mathbb{C}$ is a constant and $j(g,\zeta)$ is as in (2.3). Then for this $\widetilde{\xi}$ the condition (3.3) can be written as

$$\begin{aligned} \widetilde{\psi}(g_1, g_2)\begin{pmatrix}\zeta \\ 1\end{pmatrix} &= \widetilde{\xi}(g_2)(\zeta) - \widetilde{\xi}(g_1 g_2)(z) + \widetilde{\xi}(g_2)(g_2\zeta) j(g_2, \zeta) \\ &= j(g_2, \zeta) - j(g_1 g_2, \zeta) + j(g_1, g_2\zeta) j(g_2, \zeta) = j(g_2, \zeta) \end{aligned} \tag{3.4}$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $z \in \mathcal{D}$, where we used (2.4) and (3.2). Here we note that it is possible choose a map $\psi : \Gamma \times \Gamma \to V$ satisfying (3.4). For example, if $\widetilde{\psi}$ is given by

$$\widetilde{\psi}(g_1, g_2) = \left((0,1)g_2\begin{pmatrix}1 \\ 0\end{pmatrix}, (0,1)g_2\begin{pmatrix}1 \\ 0\end{pmatrix}\right),$$

then for $g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\widetilde{\psi}(g_1, g_2)\begin{pmatrix}\zeta \\ 1\end{pmatrix} = (c, d)\begin{pmatrix}\zeta \\ 1\end{pmatrix} = c\zeta + d = j(g_2, \zeta).$$

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Let $\widetilde{\psi} : \widetilde{\Gamma} \times \widetilde{\Gamma} \to L$ be a map satisfying (3.4). Using (2.1) and (2.4), we have

$$\begin{aligned}\left(\widetilde{\psi}(g,g')g'' + \widetilde{\psi}(gg',g'') - \widetilde{\psi}(g',g'') - \widetilde{\psi}(g,g'g'')\right)\begin{pmatrix}\zeta\\1\end{pmatrix}\\ = \widetilde{\psi}(gg')\begin{pmatrix}g''\zeta\\1\end{pmatrix} j(g'',\zeta) + j(g'',\zeta) - j(g'',\zeta) - j(g'g'',\zeta)\\ = j(g',g''\zeta)j(g'',\zeta) - j(g'g'',\zeta) = 0\end{aligned}$$

for $\zeta \in \mathcal{H}_n$ and $g, g', g'' \in \Gamma$. This implies that $\widetilde{\psi}$ satisfies (2.1); hence $\widetilde{\psi} \in \mathfrak{Z}^2(\widetilde{\Gamma}, L)$. We now set

$$\psi(\gamma_1,\gamma_2) = \widetilde{\psi}(\rho(\gamma_1),\rho(\gamma_2)), \quad \xi(\gamma)(z) = \widetilde{\xi}(\rho(\gamma))(\tau(z))$$

for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $z \in \mathcal{D}$. Then $\psi \in \mathfrak{Z}^2(\Gamma, L)$ and $\xi \in \mathfrak{C}^1(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n))$, and by using (2.2) and (3.2) as well as (3.3) we obtain

$$\begin{aligned}\delta\xi(\gamma_1,\gamma_2)(z) &= \xi(\gamma_2)(z) - \xi(\gamma_1\gamma_2)(z) + \xi(\gamma_2)(\gamma_2 z)j_{\rho,\tau}(\gamma_2,z)\\ &= \widetilde{\xi}(\rho(\gamma_2))(\tau(z)) - \widetilde{\xi}(\rho(\gamma_1)\rho(\gamma_2))(\tau(z))\\ &\qquad + \widetilde{\xi}(\rho(\gamma_2))(\tau(\gamma_2 z))j(\rho(\gamma_2),\tau(z))\\ &= \widetilde{\delta\xi}(\rho(\gamma_1),\rho(\gamma_2))(\tau(z))\\ &= \widetilde{\psi}(\rho(\gamma_1),\rho(\gamma_2))\begin{pmatrix}\tau(z)\\1\end{pmatrix} = \psi(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}.\end{aligned}$$

Thus we see that the pair (ξ, ψ) satisfies (2.9), and therefore the proof of the theorem is complete. ∎

EXAMPLE 3.2. Let $\xi_0 \in \mathfrak{C}^1(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n))$ be the 1-cochain given by

$$\xi_0(\gamma)(z) = \kappa j_{\rho,\tau}(\gamma, z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{D}$, where $\kappa \in \mathbb{C}$ is a constant and $j_{\rho,\tau}(\gamma, z)$ is as in (2.5). If $\psi_0 \in \mathfrak{Z}^2(\Gamma, L)$ is the 2-cocycle given by

$$\psi_0(\gamma_1,\gamma_2) = \left((0,1)\rho(\gamma_2)\begin{pmatrix}1\\0\end{pmatrix}, (0,1)\rho(\gamma_2)\begin{pmatrix}1\\0\end{pmatrix}\right)$$

for all $\gamma_1, \gamma_2 \in \Gamma$, then from the proof of Theorem 3.1 we see that the pair (ξ_0, ψ_0) satisfies (2.9). By using this pair we can construct other pairs as follows. Let ε be a 1-cochain belonging to $\mathfrak{C}^1(\Gamma, L)$, and consider the elements $\xi_\varepsilon \in \mathfrak{C}^1(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n))$ and $\psi_\varepsilon \in \mathfrak{Z}^2(\Gamma, L)$ given by

$$\xi_\varepsilon(\gamma)(z) = \xi_0(\gamma)(z) + \varepsilon(\gamma)\begin{pmatrix}\tau(z)\\1\end{pmatrix}, \quad \psi_\varepsilon = \psi + \partial\varepsilon \tag{3.5}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{D}$, where $\partial : \mathfrak{C}^1(\Gamma, L) \to \mathfrak{C}^2(\Gamma, L)$ is the coboundary operator. Then for $\gamma_1, \gamma_2 \in \Gamma$ and $z \in \mathcal{D}$ we have

$$\begin{aligned}\partial\xi_\varepsilon(\gamma_1,\gamma_2)(z) &= \xi_\varepsilon(\gamma_2)(z) - \xi_\varepsilon(\gamma_1\gamma_2)(z) + \xi_\varepsilon(\gamma_1)(\gamma_2 z)j_{\rho,\tau}(\gamma_2, z)\\ &= \partial\xi(\gamma_1,\gamma_2)(z) + \varepsilon(\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}\\ &\qquad - \varepsilon(\gamma_1\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix} + \varepsilon(\gamma_1)\begin{pmatrix}\tau(\gamma_2 z)\\1\end{pmatrix}j_{\rho,\tau}(\gamma_2, z).\end{aligned}$$

However, we have

$$\begin{aligned}\begin{pmatrix}\tau(\gamma_2 z)\\1\end{pmatrix}j_{\rho,\tau}(\gamma_2, z) &= \begin{pmatrix}(a\tau(z)+b)(c\tau(z)+d)^{-1}\\1\end{pmatrix}(c\tau(z)+d)\\ &= \begin{pmatrix}a\tau(z)+b\\c\tau(z)+d\end{pmatrix} = \rho(\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}\end{aligned}$$

if $\rho(\gamma_2) = \begin{pmatrix}a & b\\c & d\end{pmatrix}$. Using this, (2.2), and (3.5), we see that

$$\partial\xi_\varepsilon(\gamma_1,\gamma_2)(z) = \psi(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix} + (\partial\varepsilon)(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix} = \psi_\varepsilon(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}.$$

Thus the pair $(\xi_\varepsilon, \psi_\varepsilon)$ satisfies (2.9).

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