ON SOME DIOPHANTINE INEQUALITIES INVOLVING THE EXPONENTIAL FUNCTION

A. BAKER

1. Introduction. It is well known that for any real number θ there are infinitely many positive integers n such that

$$n||n\theta|| < 1.$$

Here $||\alpha||$ denotes the distance of α from the nearest integer, taken positively. Indeed, since $||\alpha|| < 1$, this implies more generally that if $\theta_1, \theta_2, \ldots, \theta_k$ are any real numbers, then there are infinitely many positive integers n such that

$$n||n\theta_1|| \, ||n\theta_2|| \ldots ||n\theta_k|| < 1.$$

It is also well known that if θ is a positive number other than 1 and $\log \theta$ is rational, then for every $\epsilon > 0$ there are only a finite number of positive integers n satisfying the inequality

$$n^{1+\epsilon}||n\theta|| < 1.$$

In fact sharper results of this type have been obtained, with ϵ replaced by a function of n which decreases to zero as n approaches infinity. (The work of Mahler **(9)** includes a result of this type in the case $\theta = e$, and the technique is easily modified to apply to the more general θ. Also, in certain cases, for example when $\theta = e^{1/q}$ where q is a positive integer, such results can be deduced from the known continued fraction for θ.) Nothing in this direction has hitherto been proved, however, about products containing more than two factors, and it is the object of the present paper to deduce such a generalization. Accordingly we shall prove the following:

THEOREM. *Suppose that k is a positive integer, that $\theta_1, \theta_2, \ldots, \theta_k$ are positive numbers other than* 1, *and that* $\log \theta_1, \log \theta_2, \ldots, \log \theta_k$ *are distinct rational numbers. Then there are only a finite number of positive integers n for which*

$$n^{1+\epsilon(n)}||n\theta_1|| \, ||n\theta_2|| \ldots ||n\theta_k|| < 1, \tag{1}$$

where $\epsilon(n) = c(\log \log n)^{-\frac{1}{2}}$ and $c > 0$ depends only on $k, \theta_1, \theta_2, \ldots, \theta_k$.

For the proof we use essentially the methods of Siegel **(13)**, as applied in his famous investigations on E-functions. (For further expositions, cf. **(8; 12; 14)**.) By this means we are able to prove that only a finite number of sets of non-zero integers $x_1, x_2, \ldots, x_k$ exist such that

$$|x_1 x_2 \ldots x_k(x_1 \theta_1 + x_2 \theta_2 + \ldots + x_k \theta_k)| < x^{1-\epsilon(x)}, \tag{2}$$

Received May 20, 1964.

https://doi.org/10.4153/CJM-1965-061-8 Published online by Cambridge University Press

where $x = \max|x_i|$ and $\epsilon(x)$ is of order $(\log\log x)^{-\frac{1}{2}}$. We then show, by modifying a well-known transference principle **(10)**, that this implies the assertion of the theorem. It may be observed that (2) includes, as a special case, a measure of transcendence for the number e. However, putting this in the more usual form, with the product $x_1 x_2 \ldots x_k$ replaced by x^k, we see that the result is then slightly weaker than the best so far established, the value of $\epsilon(x)$ in the latter being of order $(\log\log x)^{-1}$ **(9** and **11)**.

Finally we remark that the case $k = 2$ of the theorem provides explicit examples of transcendental numbers θ_1, θ_2 such that

$$n^{1+\epsilon(n)}||n\theta_1||\,||n\theta_2|| > 1$$

for all but a finite number of n. Spencer **(15**; see also **7)** has proved that almost all pairs of real numbers θ_1, θ_2, in the sense of Lebesgue measure, have this property, and indeed the same is true even with $\epsilon(n)$ of order $\log\log n/\log n$, but little appears to be known about the exact nature of the pairs θ_1, θ_2. (Spencer's result applies more generally with k real numbers θ_i, $k \geqslant 2$. We have in mind, however, a well-known problem of Littlewood on Diophantine approximation **(**see **4**, **5**, and for analogues **1**, **6)**, and thus we refer only to the special case.) For example it would be interesting to ascertain whether there are also algebraic numbers θ_1, θ_2 satisfying an inequality of the above type.

I should like to express my thanks to the referee for several helpful suggestions.

2. Lemmas. In the following lemmas we shall suppose that $k, \theta_1, \theta_2, \ldots, \theta_k$ satisfy the hypotheses of the theorem stated above, except that we allow the possibility that one of the θ_i is 1, and we shall denote by $c_1, c_2, \ldots$ positive numbers which depend only on $k, \theta_1, \theta_2, \ldots, \theta_k$. We use $f^{(i)}(x)$ to denote the ith derivative of $f(x)$ with respect to x, or $f'(x)$ in the case of the first derivative.

LEMMA 1. *Let m, n be positive integers with $n > m$. Suppose that a_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are integers with absolute values at most A. Then there are integers $x_1, x_2, \ldots, x_n$, not all zero, with absolute values at most $(nA)^{m/(n-m)} + 2$, such that*

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \qquad (i = 1, 2, \ldots, m).$$

Proof. See **(12**, p. 140, Hilfssatz 29**)**. The lemma is deduced easily by means of a box argument.

LEMMA 2. *Let $r_1, r_2, \ldots, r_k$ be positive integers and let $r > 1$ denote their maximum. Then there are polynomials $P_i(x)$ $(i = 1, 2, \ldots, k)$, not all identically zero, with the following properties:*

(i) *For each i, $P_i(x)$ has degree at most r, a zero at $x = 0$ of order at least $r - r_i$, and integer coefficients with absolute values at most*

$$r_i!\, c_1^{\,r(\log r)^{\frac{1}{2}}}.$$

(ii) *The following holds:*

$$\sum_{i=1}^{k} P_i(x)\theta_i^{x} = \sum_{h=m}^{\infty} \rho_h x^h, \tag{3}$$

where

$$m = r_1 + r_2 + \ldots + r_k + k - 1 - [r(\log r)^{-\frac{1}{2}}], \tag{4}$$

and, for each h,

$$|\rho_h| < r!\ (h!)^{-1} c_2^{h+r(\log r)^{\frac{1}{2}}}. \tag{5}$$

Proof. We denote by L the maximum of the absolute values of $\log \theta_1, \log \theta_2, \ldots, \log \theta_k$ and by l the least common multiple of their denominators. Let m be given by (4) and let $n = r_1 + r_2 + \ldots + r_k + k$. We take p_{ij} to be 0 for all integral values of i, j other than the n pairs given by $1 \leqslant i \leqslant k$, $r - r_i \leqslant j \leqslant r$, and then define the p_{ij} for these remaining values as integers, not all zero, satisfying the system of m equations

$$\sum_{i=1}^{k} \sum_{j=0}^{h} \binom{h}{j} (\log \theta_i)^{h-j} l^h p_{ij} = 0 \qquad (h = 0, 1, \ldots, m-1). \tag{6}$$

Such integers exist in virtue of Lemma 1, and indeed with absolute values at most

$$M = \{n(2lL)^m\}^{m/(n-m)} + 2.$$

We now prove that the polynomials given by

$$P_i(x) = r! \sum_{j=0}^{r} p_{ij}(j!)^{-1} x^j \qquad (i = 1, 2, \ldots, k)$$

have the required properties.

In part (i) we need only confirm the last estimate. Furthermore, it is easily verified, by expanding the θ_i^x as power series in x, that

$$\sum_{i=1}^{k} P_i(x)\theta_i^{x} = r! \sum_{h=0}^{\infty} \sigma_h (h!)^{-1} x^h,$$

where, for each h, $l^h\sigma_h$ is given by the left-hand side of (6). Hence (3) holds with $\rho_h = r!\ (h!)^{-1}\sigma_h$.

To estimate the coefficients of the polynomials $P_i(x)$ we note that $m < n < 2kr$ and that $n - m > r(\log r)^{-\frac{1}{2}}$. Then clearly

$$M < \{2kr(2lL)^{2kr}\}^{2k(\log r)^{\frac{1}{2}}} + 2 < c_3^{r(\log r)^{\frac{1}{2}}}. \tag{7}$$

Since $p_{ij} = 0$ for $j < r - r_i$, it follows that the coefficients of the $P_i(x)$ have absolute values at most

$$\frac{r!\, M}{(r - r_i)!} = \binom{r}{r_i} M(r_i!) \leqslant 2^r M(r_i!),$$

and this together with (7) gives the last part of (i).

https://doi.org/10.4153/CJM-1965-061-8 Published online by Cambridge University Press

Now, using the estimate (7) again, we see that the σ_h, with $h \geqslant m$, have absolute values at most

$$k(h+1)(2lL)^h M < c_2^{h+r(\log r)^{\frac{1}{2}}},$$

and hence (5) is satisfied. This completes the proof of the lemma.

LEMMA 3. *Suppose the hypotheses of Lemma 2 hold, and let $P_i(x$ $(i = 1, 2, \ldots, k)$ be the polynomials given by the lemma. Let*

$$s = [r(\log r)^{-\frac{1}{2}}] + (k-1)^2, \tag{8}$$

and suppose that $r_i > 2s$ for all i. Let the polynomials $P_{ij}(x)$ $(i = 1, 2, \ldots, k;$ $j = 1, 2, \ldots)$ be defined inductively by the equations

$$\begin{aligned} P_{i1}(x) &= P_i(x), \\ P_{ij}(x) &= P_{i,\,j-1}'(x) + P_{i,j-1}(x)\log\theta_i. \end{aligned} \tag{9}$$

Then the determinant $\Delta(x)$ with $P_{ij}(x)$ in the ith row and jth column $(i, j = 1, 2, \ldots, k)$ cannot have a zero at $x = 1$ of order greater than s.

Proof. We shall first prove the lemma on the supposition that none of the $P_i(x)$ is identically zero and later show that this supposition is valid. Each of the $P_i(x)$ has then a non-zero leading coefficient and we shall denote this by p_i.

It is clear from the recurrence relations (9) that $P_{ij}(x)$ has degree at most r and leading coefficient $p_i(\log\theta_i)^{j-1}$. Hence $\Delta(x)$ represents a polynomial of degree at most kr and with leading coefficient $p_1 p_2 \ldots p_k \psi$, where ψ is a Vandermonde determinant of order k formed from the powers of the $\log\theta_i$. Since, by hypothesis, the θ_i are distinct, it follows that $\Delta(x)$ is not identically zero.

We suppose now, as we may without loss of generality, that $r = r_1$. Let $\Phi(x)$ denote the left-hand side of (3). The equations (9) clearly imply that

$$\Phi^{(j-1)}(x) = \sum_{i=1}^{k} P_{ij}(x)\theta_i^x, \tag{10}$$

and hence, by a linear combination of rows, we obtain

$$\theta_1^x \Delta(x) = \begin{vmatrix} \Phi(x) & \Phi^{(1)}(x) & \cdots & \Phi^{(k-1)}(x) \\ P_{21}(x) & P_{22}(x) & \cdots & P_{2k}(x) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P_{k1}(x) & P_{k2}(x) & \cdots & P_{kk}(x) \end{vmatrix}.$$

On differentiating (3), we see that $\Phi^{(j)}(x)$ has a zero at $x = 0$ of order at least $m - j$. Further, from (i) of Lemma 2 and (9), we deduce that $P_{ij}(x)$ has a zero at $x = 0$ of order at least $r - r_i - j + 1$. Hence $\Delta(x)$ has a zero at $x = 0$ of order at least

$$m - k + 1 + \sum_{i=2}^{k} (r - r_i - k + 1) = kr - s,$$

by virtue of (4), (8), and our supposition $r = r_1$. The lemma follows on noting that $\Delta(x)$ has degree at most kr.

It remains only to prove our original supposition. The argument used is similar to that just preceding. We suppose that exactly κ of the polynomials $P_i(x)$ do not vanish identically and, without loss of generality, we take these to be given by $i = 1, 2, \ldots, \kappa$. Also we may suppose that $r = r_i$ for at least one of $i = \kappa, \kappa + 1, \ldots, k$. We consider the minor $\Xi(x)$ in $\Delta(x)$ formed from the first κ rows and columns. It follows as above that $\Xi(x)$ is a polynomial, not identically zero, with degree at most κr. On the other hand, by a linear combination of rows, we see that $\Xi(x)$ has a zero at $x = 0$ of order at least

$$\begin{aligned} m - \kappa + 1 + \sum_{i=1}^{\kappa-1} (r - r_i - \kappa + 1) \\ &= (k - \kappa) + (\kappa - 1)r - (\kappa - 1)^2 - [r(\log r)^{-\frac{1}{2}}] + \sum_{i=\kappa}^{k} r_i \\ &\geqslant (\kappa - 1)r - s + \sum_{i=\kappa}^{k} r_i. \end{aligned}$$

Thus, in virtue of the hypothesis $r_i > 2s$ for all i, it follows that $\kappa = k$, and the lemma is proved.

LEMMA 4. *Suppose the hypotheses of Lemma 3 hold. Then there are k distinct suffixes $J(j)$ $(j = 1, 2, \ldots, k)$ between 1 and $k + s$ inclusive such that the determinant with $P_{i,J(j)}(x)$ in the ith row and jth column $(i, j = 1, 2, \ldots, k)$ does not vanish at $x = 1$.*

Proof. The proof proceeds on the lines indicated in the introduction to **(13)** (see also **12**, p. 118, Hilfssatz 22). We define linear forms in $w_1, w_2, \ldots, w_k$ by the equations

$$W_j = \sum_{i=1}^{k} P_{ij}(x) w_i \qquad (j = 1, 2, \ldots). \tag{11}$$

If $\Delta_{ij}(x)$ denotes the minor in $\Delta(x)$ formed by omitting the ith row and jth column, then for each $i = 1, 2, \ldots, k$ we have

$$w_i \Delta(x) = \sum_{j=1}^{k} (-1)^{i+j} W_j \Delta_{ij}(x). \tag{12}$$

By Lemma 3 we see that there is an integer $\tau \leqslant s$ such that $\Delta^{(\tau)}(1)$ is not zero, and we suppose that τ is the least such non-negative integer. We now regard the w_i as differentiable functions of x and differentiate (12) τ times, replacing the w_i' $(i = 1, 2, \ldots, k)$ occurring at each stage by $w_i \log \theta_i$, as is possible since the resulting equations hold identically in the w_i and w_i'. In this way we obtain equations of the form

$$w_i \left\{ \sum_{l=0}^{\tau} \binom{\tau}{l} (\log \theta_i)^{\tau - l} \Delta^{(l)}(x) \right\} = \sum_{j=1}^{k+\tau} W_j F_{ij}(x) \qquad (i = 1, 2, \ldots, k),$$

https://doi.org/10.4153/CJM-1965-061-8 Published online by Cambridge University Press

where the terms $F_{ij}(x)$ denote polynomials in x given by linear combinations of the $\Delta_{ij}(x)$ and their derivatives. Thus we see that the linear forms $W_j(j = 1, 2, \ldots, k + \tau)$ given by (11) with $x = 1$ include a set of k linearly independent forms. The lemma follows with $J(j)$ $(j = 1, 2, \ldots, k)$ given by the k associated suffixes.

LEMMA 5. *Suppose the hypotheses of Lemma 3 hold. Then there are k^2 integers q_{ij} $(i, j = 1, 2, \ldots, k)$ with the following properties:*

(i) *The determinant with q_{ij} in the ith row and jth column $(i, j = 1, 2, \ldots, k)$ is not zero.*

(ii) *For each pair i, j we have*

$$|q_{ij}| < r_i!\, c_4^{r(\log r)^{\frac{1}{2}}}. \tag{13}$$

(iii) *For each $j = 1, 2, \ldots, k$ we have*

$$\left| \sum_{i=1}^{k} q_{ij}\,\theta_i \right| < r!\, c_5^{r(\log r)^{\frac{1}{2}}} \left(\prod_{i=1}^{k} r_i! \right)^{-1}. \tag{14}$$

Proof. Let L, l be given as in the proof of Lemma 2. With the notation of Lemma 4 we define

$$q_{ij} = l^{k+s} P_{i,J(j)}(1) \qquad (i, j = 1, 2, \ldots, k),$$

and proceed to prove that the q_{ij} have the required properties.

First, in virtue of the recurrence relations (9), it is clear that the q_{ij} are integers. Also (i) is equivalent to the assertion of Lemma 4. To prove (ii) we note that, by (i) of Lemma 2, and by (8) and (9), the coefficients of $P_{ij}(x)$ with $j \leqslant k + s$ have absolute values at most

$$(r + L)^{k+s} (r_i!) c_1^{r(\log r)^{\frac{1}{2}}} < r_i!\, c_6^{r(\log r)^{\frac{1}{2}}},$$

and this clearly implies that (13) is satisfied.

For the proof of inequality (14) we use (ii) of Lemma 2. As in the proof of Lemma 3, we denote the left-hand side of (3) by $\Phi(x)$ so that equations (10) hold. Then by the definition of the q_{ij}, the sum on the left of (14) is equal to $l^{k+s}\Phi^{J(j)-1}(1)$. Now on differentiating (3) j times, where $j \leqslant k + s - 1$, and using (5) we obtain

$$\left| \Phi^{(j)}(1) \right| = \left| \sum_{h=m}^{\infty} \rho_h (h!)((h - j)!)^{-1} \right| < r!\, c_2^{r(\log r)^{\frac{1}{2}}} \sum_{h=m}^{\infty} c_2^{h}((h - j)!)^{-1}.$$

By removing a factor $(m - j)!$ from the denominator of each term, we see that the sum on the right is not greater than $e^{c_2} c_2^{m}((m - j)!)^{-1}$, and, since $j \leqslant k + s - 1$, it follows from (4) that this is less than

$$e^{c_2} c_2^{2kr}((r_1 + r_2 + \cdots + r_k - 2s)!)^{-1} < c_8^{r}(kr)^{2s} \left(\prod_{i=1}^{k} r_i! \right)^{-1}.$$

Thus, using (8), we deduce that (14) holds, and this completes the proof of the lemma.

3. Proof of the principal inequality. Let k, $\theta_1, \theta_2, \ldots, \theta_k$ satisfy the hypotheses of the preceding section, and let c_4, c_5 denote the positive numbers which appear in Lemma 5. We suppose, as clearly we may, that both c_4 and c_5 exceed 1. Let

$$c_9 = \max(1, |\theta_1|^{-1}, |\theta_2|^{-1}, \ldots, |\theta_k|^{-1}),$$

put $\mu = (4kc_4 c_5 c_9)^{16k}$, and then define ν, λ by the equations

$$\nu = 32k \log \mu, \qquad \log\log\lambda = 4(\log\mu)^4. \tag{15}$$

It is the object of the present section to prove that if $x_1, x_2, \ldots, x_k$ are non-zero integers satisfying (2), where

$$x = \max(|x_1|, |x_2|, \ldots, |x_k|) \quad \text{and} \quad \epsilon(x) = \nu(\log\log x)^{-\frac{1}{2}},$$

then $x < \lambda$. Accordingly we suppose that $x \geqslant \lambda$ and we shall deduce a contradiction.

Let integers $u_1, u_2, \ldots, u_k$ be defined by the equations $u_i = x_i[X]$, where

$$X = x^{64(\log\log x)^{-\frac{1}{2}}}.$$

Clearly $u = x[X]$ is the maximum of the absolute values of these integers. Further, noting that $x < u$, that $\epsilon(x) < 1$, and that X increases with x if $x \geqslant \lambda$, we obtain

$$[X]^{k+1} x^{1-\epsilon(x)} < u^{1+\{64(k+1)/\nu - 1\}\epsilon(u)} < u^{1-\frac{1}{2}\epsilon(u)},$$

and from (2) it follows that

$$|u_1 u_2 \ldots u_k(u_1\theta_1 + u_2\theta_2 + \ldots + u_k\theta_k)| < u^{1-\frac{1}{2}\epsilon(u)}. \tag{16}$$

Also, from the inequalities $u^{\frac{1}{2}} \leqslant (xX)^{\frac{1}{2}} < x$, we have

$$|u_i| > X^{\frac{1}{2}} > u^{16(\log\log u)^{-\frac{1}{2}}} \qquad \text{for all } i. \tag{17}$$

We now take r to be the smallest positive integer for which

$$u < r!\,\mu^{-r(\log r)^{\frac{1}{2}}}. \tag{18}$$

That the integer r exists and is greater than 1 is clear, for as r tends to infinity, so does the number on the right, which is 1 for $r = 1$. From the definition of r it follows that

$$(r-1)!\,\mu^{-r(\log r)^{\frac{1}{2}}} \leqslant (r-1)!\,\mu^{-(r-1)\{\log(r-1)\}^{\frac{1}{2}}} \leqslant u.$$

Moreover, in virtue of the inequalities

$$\lambda \leqslant x < u < r! < r^r < e^{r^2}, \tag{19}$$

and the definition of λ by (15), we deduce that $\log r > 2(\log\mu)^4$. Then from Stirling's formula it follows that u is certainly greater than

$$r^{r-\frac{1}{2}} e^{-r} \mu^{-r(\log r)^{\frac{1}{2}}} > r^{\frac{1}{2}r}.$$

On the other hand, (19) gives $\log\log u < 2\log r$, and hence from (17) we obtain

$$|u_i| > e^{4r(\log r)^{\frac{1}{2}}} \qquad \text{for all } i. \tag{20}$$

Also it is easily seen that (16) and (18) imply that

$$|u_1 u_2 \ldots u_k(u_1\theta_1 + u_2\theta_2 + \ldots + u_k\theta_k)| < r!\,\mu^{-4kr(\log r)^{\frac{1}{2}}}. \tag{21}$$

Let now $r_1, r_2, \ldots, r_k$ be the integers defined by

$$(r_i - 1)! \leqslant |u_i|\mu^{r(\log r)^{\frac{1}{2}}} < r_i! \qquad (i = 1, 2, \ldots, k). \tag{22}$$

Then clearly r is the maximum of $r_1, r_2, \ldots, r_k$. From (20) and the right-hand inequality of (22) we deduce that

$$r_i \log r_i > \log|u_i| > 4r(\log r)^{\frac{1}{2}},$$

and, since $r^2 > \log\lambda$ and $r \geqslant r_i$, it follows that

$$r_i > 4r(\log r)^{-\frac{1}{2}} > 2s,$$

where s is given by (8). Further, from (21) and the left-hand inequality of (22), we obtain

$$|\Theta| < r!\,\mu^{-r(\log r)^{\frac{1}{2}}}\left(\prod_{i=1}^{k} r_i!\right)^{-1}, \tag{23}$$

where, for brevity, we put

$$\Theta = u_1\theta_1 + u_2\theta_2 + \ldots + u_k\theta_k.$$

We have verified that $r_1, r_2, \ldots, r_k$, and r satisfy all the hypotheses of Lemma 5, and we denote by $q_{ij}(i, j = 1, 2, \ldots, k)$ the corresponding set of k^2 integers given by the lemma. We consider the linear forms in $\theta_1, \theta_2, \ldots, \theta_k$ given by

$$\Psi_j = \sum_{i=1}^{k} q_{ij}\theta_i \qquad (j = 1, 2, \ldots, k).$$

Since the matrix (q_{ij}) is non-singular, there are $k - 1$ of these forms which together with the linear form Θ make up a linearly independent set. Without loss of generality we can take them to be the last $k - 1$ forms. Also without loss of generality we shall suppose that $r = r_1$. By a linear combination of rows we have

$$\begin{vmatrix} u_1 & q_{12} & \cdots & q_{1k} \\ u_2 & q_{22} & \cdots & q_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ u_k & q_{k2} & \cdots & q_{kk} \end{vmatrix} = \theta_1^{-1} \begin{vmatrix} \Theta & \Psi_2 & \cdots & \Psi_k \\ u_2 & q_{22} & \cdots & q_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ u_k & q_{k2} & \cdots & q_{kk} \end{vmatrix}.$$

The determinant on the left is a non-zero integer. Hence, expanding the determinant on the right by the first row and estimating the corresponding co-factors by means of (13), we obtain

$$1 \leqslant |\theta_1|^{-1}\left(\prod_{i=2}^{k} r_i!\, c_4^{r(\log r)^{\frac{1}{2}}}\right)\left\{(k-1)!\,|\Theta| + (k-2)! \sum_{i=2}^{k} \sum_{j=2}^{k} |u_i \Psi_j| (r_i!\, c_4^{r(\log r)^{\frac{1}{2}}})^{-1}\right\}.$$

Then from (14) and (23) it follows that

$$1 \leqslant (k-1)!\, c_9 \left\{ (c_4^{k} \mu^{-1})^{r(\log r)^{\frac{1}{2}}} + (c_4^{k} c_5)^{r(\log r)^{\frac{1}{2}}} \sum_{i=2}^{k} |u_i| (r_i!)^{-1} \right\},$$

and thus, using the right-hand inequality of (22), we deduce that

$$1 \leqslant k!\, c_9 \{ (c_4^{k} \mu^{-1})^{r(\log r)^{\frac{1}{2}}} + (c_4^{k} c_5 \mu^{-1})^{r(\log r)^{\frac{1}{2}}} \}.$$

However, this is impossible, in virtue of the definition of μ, and the contradiction gives the required result.

4. Proof of the theorem. Suppose that $k, \theta_1, \theta_2, \ldots, \theta_k$ satisfy the hypotheses of the theorem stated in §1, and let M denote the maximum of the numbers μ defined in §3, corresponding to all the different subsets, not necessarily proper, of $\theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1} = 1$. Let ν, λ be defined by (15) with $\mu = M$, let $c = 2k\nu$, and let ξ be given by the equation $\log\log \xi = (\nu\omega)^2$, where $\omega = 4k\lambda\eta$ and

$$\eta = \max(1, |\theta_1|, |\theta_2|, \ldots, |\theta_k|).$$

We shall prove that if $\epsilon(n) = c(\log\log n)^{-\frac{1}{2}}$, then there is no integer $n > \xi$ satisfying (1). We suppose the opposite, namely that integers $n > \xi, n_1, n_2, \ldots, n_k$ exist for which

$$n^{1+\epsilon(n)} |n\theta_1 - n_1|\, |n\theta_2 - n_2| \ldots |n\theta_k - n_k| < 1,$$

and we shall deduce a contradiction.

First we put

$$\chi = n^{1+\frac{1}{2}\epsilon(n)} |n\theta_1 - n_1|\, |n\theta_2 - n_2| \ldots |n\theta_k - n_k|,$$

and note that

$$\chi < n^{-\frac{1}{2}\epsilon(n)} < e^{-\frac{1}{2}c(\log n)^{\frac{1}{2}}} < \omega^{-4k^2}. \tag{24}$$

Next let $\phi_1, \phi_2, \ldots, \phi_k$ be defined by

$$\phi_i = (\chi n^{-\frac{1}{2}\epsilon(n)})^{1/k} |n\theta_i - n_i|^{-1} \qquad (i = 1, 2, \ldots, k). \tag{25}$$

Then clearly $\phi_1\phi_2 \ldots \phi_k = n$. We suppose, as we may without loss of generality, that $\phi_1 \geqslant \phi_2 \geqslant \ldots \geqslant \phi_k$, and we take $\kappa \leqslant k$ to be the smallest integer for which the product $\phi_{\kappa+1}\phi_{\kappa+2} \ldots \phi_k$ is not greater than 1. By Minkowski's theorem on linear forms (**2**, p. 151 or **3**, p. 73) there are integers $x_1, x_2, \ldots, x_{\kappa+1}$, not all 0, satisfying the inequalities

$$|x_i| \leqslant \phi_i \quad (i = 1, 2, \ldots, \kappa - 1), \qquad |x_\kappa| \leqslant \phi_\kappa \phi_{\kappa+1} \ldots \phi_k, \quad |X| < 1, \tag{26}$$

https://doi.org/10.4153/CJM-1965-061-8 Published online by Cambridge University Press

where

$$X = n_1 x_1 + n_2 x_2 + \ldots + n_\kappa x_\kappa + n x_{\kappa+1}.$$

Clearly the last inequality implies that $X = 0$, and the preceding inequality shows that $|x_\kappa| \leqslant \phi_\kappa$. Hence from (25), (26), and the identity

$$X = nY - \sum_{i=1}^{\kappa} (n\theta_i - n_i)x_i,$$

where

$$Y = x_1\theta_1 + x_2\theta_2 + \ldots + x_\kappa\theta_\kappa + x_{\kappa+1},$$

we obtain

$$|nY| \leqslant k(\chi n^{-\frac{1}{2}\epsilon(n)})^{1/k}. \tag{27}$$

From $|X| < 1$ it is clear that at least one of the $x_i (i = 1, 2, \ldots, \kappa)$ differs from zero. We now suppose, again without loss of generality, that the first K, and only these, of the integers $x_1, x_2, \ldots, x_\kappa$, are non-zero. (Thus $x_1, \ldots, x_K$ are non-zero, $x_{K+1}, \ldots, x_\kappa$ are zero, and $x_{\kappa+1}$ may be zero or non-zero.) Then, since $\phi_i > 1$ for $i = K+1, K+2, \ldots, \kappa$, and since $\phi_\kappa \phi_{\kappa+1} \ldots \phi_k > 1$, it follows from (26) that

$$|x_1 x_2 \ldots x_K| \leqslant n. \tag{28}$$

On the other hand, if x denotes the maximum of all the $|x_i|$ for $i = 1, 2, \ldots, \kappa + 1$, then $|x_1 x_2 \ldots x_K|$ is greater than $(2k\eta)^{-1}x$, for (24) and (27) imply that $|Y| < 1$, and thus

$$|x_{\kappa+1}| < 1 + |x_1\theta_1 + \ldots + x_\kappa\theta_\kappa| < 2k\eta \max(|x_1|, |x_2|, \ldots, |x_\kappa|).$$

Hence we see that $v = 2[\lambda]x$ is less than ωn, and from this and our supposition $n > \xi$ we deduce that

$$n^{-\frac{1}{2}\epsilon(n)} < \omega(\omega n)^{-\frac{1}{2}\epsilon(n)} < \omega(\omega n)^{-\frac{1}{2}\epsilon(\omega n)} < \omega v^{-\frac{1}{2}\epsilon(v)}.$$

Then from (24), (27), and (28) we obtain

$$|x_1 x_2 \ldots x_K\, Y| \leqslant k\omega^{-4k+1} v^{-\epsilon(v)/(2k)}. \tag{29}$$

If now we take $v_i = 2[\lambda]x_i$ for $i = 1, 2, \ldots, \kappa + 1$, then, in virtue of the definitions of ω and c, we see that (29) gives

$$\begin{aligned} &|v_1 v_2 \cdots v_K\, v(v_1\theta_1 + v_2\theta_2 + \cdots + v_K\theta_K + v_{\kappa+1})| \\ &\qquad < k(2\lambda)^{k+1}\omega^{-4k+1} v^{1-\epsilon(v)/(2k)} < v^{1-\nu(\log\log v)^{-\frac{1}{2}}}. \end{aligned}$$

However, since $v \geqslant \lambda$ is the maximum of $|v_i|$, this contradicts the result of the previous section, and the contradiction proves the theorem.

References

1. A. Baker, *On an analogue of Littlewood's Diophantine approximation problem*, Michigan Math. J., *11* (1964), 247–250.

2. J. W. S. Cassels, *An introduction to Diophantine approximation* (Cambridge, 1957).

3. ——— *An introduction to the geometry of numbers* (Berlin, Göttingen, Heidelberg, 1959).
4. J. W. S. Cassels and H. P. F. Swinnerton-Dyer, *On the product of three homogeneous linear forms and indefinite ternary quadratic forms*, Philos. Trans. Roy. Soc. London Ser. A, *248* (1955), 73–96.
5. H. Davenport, *Note on irregularities of distribution*, Mathematika, *3* (1956), 131–135.
6. H. Davenport and D. J. Lewis, *An analogue of a problem of Littlewood*, Michigan Math. J., *10* (1963), 157–160.
7. P. Gallagher, *Metric simultaneous Diophantine approximation*, J. London Math. Soc., *37* (1962), 387–390.
8. A. O. Gelfond, *Transcendental and algebraic numbers* (New York, 1960).
9. K. Mahler, *Zur Approximation der Exponentialfunktion und des Logarithmus*, I and II, J. Reine Angew. Math., *166* (1932), 118–136, 137–150.
10. ——— *Ein Übertragungsprinzip für lineare Ungleichungen*, Časopis Pěst. Mat., *68* (1939), 85–92.
11. J. Popken, *Zur Transzendenz von e*, Math. Z. *29* (1929), 525–541.
12. Th. Schneider, *Einführung in die transzendenten Zahlen* (Berlin, Göttingen, Heidelberg, 1957).
13. C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss. (1929), Nr 1.
14. ——— *Transcendental numbers* (Princeton, 1949).
15. D. C. Spencer, *The lattice points of tetrahedra*, J. Math. Phys., *21* (1942), 189–197.

Trinity College, Cambridge, England