CORRESPONDENCE

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A Budget of Paradoxes

Sirs,

It is quite common nowadays for actuarial and statistical authors to embellish their publications with the *cachet* of a historical reference. A harmless enough foible, to be sure, but it has its dangers. For all too often it turns out that a widely quoted statement about early work can be traced back to a single author's misinterpretation of the original source.

This contention may be illustrated by two frequently quoted 'inventions': (i) the measurement of mortality by contemporaries of Ulpianus, the Roman jurist who died A.D. 228, and (ii) the (continuous) Normal curve of errors by Abraham de Moivre. Neither of these claims stands up too well under close scrutiny.

Let us first consider the so-called *lex Falcidia* which was passed in Rome in 40 B.C. According to Braun (1921) it was intended to exercise some rein on the growing practice of disinheriting the heirs of estates. It stipulated that no testator could leave less than one-quarter of his estate to his legal heir.

In his A.D. 230 commentary on the law (see, for example, Charond, 1575), Aemilius Macer reported that it had been the custom to value a life tenant's annuity by multiplying it by a certain number of years m_x, where

$$m_x = \begin{cases} 30 & x < 30 \\ 60 - [x] & 30 \leqslant x \leqslant 60. \end{cases}$$

If the legatee were a municipal community the number of years to be used was 30. (Those who enjoy 'writing history backwards' might argue that this indicates that the underlying interest rate was 3·3%—cf. Hodge *J.I.A.* **6**, 315.)

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However, Macer reports, some years earlier the jurist Ulpianus had laid down a new rule. His multiplicative factor n_x was defined by

$$n_x = \begin{cases} 30 & x < 20 \\ 28 & 20 \leqslant x < 25 \\ 25 & 25 \leqslant x < 30 \\ 22 & 30 \leqslant x < 35 \\ 20 & 35 \leqslant x < 40 \\ 59-[x] & 40 \leqslant x < 50 \\ 9 & 50 \leqslant x < 55 \\ 7 & 55 \leqslant x < 60 \\ 5 & 60 \leqslant x. \end{cases}$$

This fragmentary commentary by Macer* is the only reference that has ever been found to mortality estimation in Roman times (van Haaften, 1943). And although Hodge (*loc. cit.*) states that the legal rate of interest in Rome was 4% in A.D. 230, there is no evidence that the Romans could calculate the present value of a number of future monetary payments. It is even more doubtful whether the crude arithmetic of Roman times would have extended to the calculation of an expectation of life from mortality data. Remember that the decimal system was only introduced into Europe in the twelfth century, that a proper theory of compound interest was not available until the fifteenth (Braun, 1925), and that the first published evidence of the 'concept of the mortality table'—l_x at integral ages—is to be found in Graunt (1662) (see, for example, Westergaard, 1901†).

Major Greenwood (*J.R. statist. Soc.* **103**, 1940, pp. 246–8) has referred to Ulpian's table as a 'statistical mare's nest'. A year

* Most of the Latin text is to be found in Trenerry (1926, p. 151). Although this doctoral thesis is very useful as a source-book, many of its conclusions are of the since-A-knew-the-meaning-of-x-he-must-have-understood-how-to-calculate-y variety.

† 'It should not be wondered at that this first attempt to find a mortality table gave only a most imperfect representation of mortality. But in spite of its imperfection it is right to designate this investigation as pioneering, not only in mortality but in the whole field of statistics. Graunt was the first who attempted to draw conclusions from statistical data....'

later he wrote (*Biometrika*, **32**, 101–27): 'There is not, I think, any reason to believe that the practical Romans had anticipated Graunt and Petty'.

We turn now to Simpson who is sometimes referred to as the author of a triangular curve of errors.

Simpson's work (1756,* 1757) should be particularly interesting to statisticians since it (1) is the first example of the application of probability theory to errors of measurement, and (2) contains the first reference to a continuous probability law. With the exception of some introductory sentences, the whole of the 1756 article is repeated in the 1757 publication. I have described most of the mathematical content of these papers elsewhere (1949) but the following additional remarks are relevant to the discussion, later in this letter, of the invention of the Normal curve of errors.

Simpson starts his 1757 paper by making two 'suppositions' about the 'errors arising from the imperfection of instruments and of the organs of sense':

> '1. That there is nothing in the construction, or position of the instrument whereby the errors are constantly made to tend the same way, but that the respective chances for their happening in excess, and in defect, are either accurately, or nearly, the same.
>
> 2. That there are certain assignable limits between which all these errors may be supposed to fall; which limits depend on the goodness of the instrument and the skill of the observer.'†

The articles then consist of two propositions. In modern terminology the first of these derives the probability distribution of a sample mean of n observations from a *discrete* rectangular universe; the second provides the distribution of a sample mean from a discrete symmetric triangular universe—'much better adapted than if all the terms were to be equal'.

* This article was in the form of a letter to the Earl of Macclesfield and was dated 4 March 1755. There is no reference in it to a 'curve' of errors nor to a triangle.

† The copy of Simpson's *Tracts* in the library of Yale University is bound up with some of that author's other publications and, by a coincidence, bears on the flyleaf the ink signature: Francis Baily 1803. A pencil note indicates that it was purchased on 9 June of that year for £1. 11*s*. 6*d*.

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The 1756 article concludes with some numerical examples on the discrete triangular law. However, in his second paper Simpson goes on to 'show how the chances may be computed, when the error admits of any value whatever, whole or broken, within the proposed limits, or when the result of each observation is supposed to be *accurately* known'. To do this he allows the number of discrete ordinates of the distribution law to increase without limit, the range of the abscissae remaining unchanged. This limiting argument is illustrated by reference to a geometric figure—an isosceles triangle.

The 1757 article concludes with a problem which clearly shows that Simpson was aware that his limiting procedures had resulted in continuous error laws. He asks, in fact, what is the probability that $\bar{x}$, the mean of n observations from a continuous symmetric triangular law with (true) mean equal to zero (and a range of two units), numerically exceeds the value of a single observation, x.

Simpson writes (Seal, 1949), in effect,

$$\begin{aligned} Pr\,(|\bar{x}-x|>0) &= 2\int_0^1 Pr(x=\xi)\,Pr(|\bar{x}|>\xi)\,d\xi \\ &= 2\int_0^1 (1-\xi)\frac{2n^{2n}}{(2n)!}\sum_{j=0}^{[n-n\xi]}(-1)^j\binom{2n}{j}\left(1-\xi-\frac{j}{n}\right)^{2n}d\xi \\ &= \frac{4n^{2n}}{(2n)!}\sum_{j=0}^{n}(-1)^j\binom{2n}{j}\int_0^1 z\left(z-\frac{j}{n}\right)^{2n}dz. \end{aligned}$$

and proceeds to carry out the definite integration. Readers of this *Journal* will, no doubt, quickly perceive the grave errors involved in *two* of these three supposed identities.

Our statement that Simpson was the originator of the continuous probability distribution is in conflict with the oft-repeated statement that De Moivre invented the Normal curve of errors.

The theorem on which these allegations are based is to be found in the second (1738) and third (1756) editions of De Moivre's *The Doctrine of Chances*. As Pearson (1924) points out, it was first published in 1733 as a second Supplement to De Moivre's *Miscellanea Analytica* (1730).

This theorem provides an approximation to the sum of a

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number of terms of the discrete binomial distribution law. It is derived by De Moivre in the following steps:

$$\sum_{r=np-l}^{np+l} \binom{n}{r} p^r(1-p)^{n-r} \sim \sum_{k=-l}^{l} \{2\pi np(1-p)\}^{-\frac{1}{2}} \quad \exp[-k^2/2np(1-p)]$$

$$=\{2\pi np(1-p)\}^{-\frac{1}{2}} \sum_{k=-l}^{l} \sum_{j=0}^{\infty} (-1)^j\, k^{2j}/2^j n^j p^j\,(1-p)^j\, j!$$

$$\doteq 2\{2\pi np(1-p)\}^{-\frac{1}{2}} \sum_{j=0}^{\infty} (-1)^j\, l^{2j+1}/(2j+1)\, 2^j n^j p^j\,(1-p)^j\, j!,$$

where summation has been replaced by (approximate) integration but only *after* the exponential function has been expanded. De Moivre works out some numerical illustrations of this approximation but there is nowhere any indication that, at the first step above, the result is a continuous probability law with an overall integral of unity. (In fact, the integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ seems to have been evaluated by Laplace for the first time in 1778.)

In my opinion it is unjustified to see in the above series of approximations, the discovery of the Normal *curve of errors*. Neither 'error' nor 'curve' is involved in De Moivre's theorem.

Yours faithfully,

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