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TWO-SIDED IDEALS IN GROUP NEAR-RINGS

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Abstract

The two-sided ideals of group near-rings are characterized and studied. Various examples are presented to illustrate the interplay between ideals in the base near-ring R and the corresponding group near-ring $R[G]$. Some results concerning the Jacobson radicals of $R[G]$ are also discussed.

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1. Introduction

In [2] group near-rings have been defined in their most general form. Since then, some work has been done on the ideal theory of group near-rings (see [1, 5]), but only for certain special cases, such as for near-rings which are distributively generated. This paper is meant to be the first step towards laying the groundwork for the ideal theory of general group near-rings. These ideals are characterized and some of their fundamental properties are revealed. Several results from matrix near-ring theory are utilized in order to do so.

Throughout this paper, R denotes a right near-ring with identity 1 and G denotes a (multiplicatively written) group with identity e and with $|G| \geq 2$. For general results on near-rings, the reader is referred to a standard textbook such as [9]. Recall that for any (additively written) group H, the set of all mappings $f : H \to H$ under the operations of pointwise addition and composition, forms a near-ring, denoted by $M(H)$. We need this in the following

DEFINITION 1.1 ([2]). Let R^G denote the direct sum of $|G|$ copies of the group $(R, +)$. The *group near-ring* constructed from R and G, denoted $R[G]$, is the

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subnear-ring of $M(R^G)$ generated by the set $\{[r, g] \in M(R^G) : r \in R, g \in G\}$, where $[r, g] : R^G \to R^G$ is defined by $([r, g](\mu))(h) = r\mu(hg)$, for all $\mu \in R^G$ and $h \in G$.

It follows that in case of a finite group G, with $|G| = n$, the near-ring $R[G]$ is closely related to the $n \times n$ matrix near-ring over R. Hence we pertinently give the following definition, due to Meldrum and van der Walt [6].

DEFINITION 1.2. Let R^n denote the direct sum of n copies of the group $(R, +)$. The *$n \times n$ matrix near-ring* over R, denoted $M_n(R)$, is the subnear-ring of $M(R^n)$ generated by the set $\{f_{ij}^r \in M(R^n) : r \in R, 1 \leq i, j \leq n\}$, where $f_{ij}^r : R^n \to R^n$ is defined by $f_{ij}^r(\alpha) = \iota_i(r\pi_j(\alpha))$, for all $\alpha \in R^n$. Here, $\iota_i : R \to R^n$ and $\pi_j : R^n \to R$ denote the i-th and j-th co-ordinate injection and projection functions respectively. For typographical reasons we also sometimes write $[r; i, j]$ for the matrix f_{ij}^r.

The interested reader should consult [1, 2, 5] for basic results on group near-rings and [6, 7] for general results on matrix near-rings. Note that when R happens to be a ring, then both $R[G]$ and $M_n(R)$ revert to the standard situation in ring theory.

Our first result relates $R[G]$ and $M_n(R)$ in case G is a finite group. As usual, S_n denotes the symmetric group on the set $\{1, 2, \ldots, n\}$.

THEOREM 1.3. *If G is a finite group with $|G| = n$, then $R[G]$ is a subnear-ring of $M_n(R)$, sharing the same identity element* $[1, e] = f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$.

PROOF. The elements of both $R[G]$ and $M_n(R)$ are mappings of the form $R^n \to R^n$. Hence it is sufficient to show that each mapping in $R[G]$ is also in $M_n(R)$. In fact, it is sufficient to show that each generator $[r, g]$ of $R[G]$ is an $n \times n$ matrix.

To this end, let $[r, g] \in R[G]$, where $G = \{g_1, g_2, \ldots, g_n\}$. We can use the elements of G to index the co-ordinates of any $\alpha \in R^n$, that is, the i-th co-ordinate of α is $\alpha(g_i) = \pi_i(\alpha)$. Now consider an arbitrary $\alpha = \langle s_{g_1}, s_{g_2}, \ldots, s_{g_n}\rangle \in R^n$, where $\alpha(g_i) = s_{g_i}$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
[r, g]\langle s_{g_1}, s_{g_2}, \ldots, s_{g_n}\rangle &= \langle rs_{g_1 g}, rs_{g_2 g}, \ldots, rs_{g_n g}\rangle \\
&= \langle rs_{g_{\rho(1)}}, rs_{g_{\rho(2)}}, \ldots, rs_{g_{\rho(n)}}\rangle \quad \text{for some } \rho \in S_n \\
&= \big([r; 1, \rho(1)] + [r; 2, \rho(2)] + \cdots + [r; n, \rho(n)]\big)\alpha.
\end{aligned}$$

Note that ρ depends on g only. It follows that $[r, g] = \sum_{i=1}^{n}[r; i, \rho(i)] \in M_n(R)$. □

2. Basic results on the ideal theory of $R[G]$

An important question now arises: Given an ideal A of R, how do we relate a corresponding ideal in $R[G]$? This problem has been studied for matrix near-rings,

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and satisfactory results have been obtained (see [3, 4, 7, 8, 11]). Keeping in mind that when R is a ring (with identity), the complete set of (two-sided) ideals of $M_n(R)$ can be obtained by considering $M_n(A)$ for ideals A of R, the natural approach was to define ideals

$$A^+ = \mathrm{Id}\langle f_{11}^a : a \in A\rangle_{M_n(R)}$$

and

$$A^* = (A^n : R^n)_{M_n(R)} = \{U \in M_n(R) : U(R^n) \subseteq A^n\}$$

in $M_n(R)$, for an ideal A of R. We use the notation $\mathrm{Id}\langle X\rangle_R$ to denote the ideal of R generated by the subset $X \subseteq R$. Also note that, since $f_{ij}^a = f_{i1}^1 f_{11}^a f_{1j}^1$, our definition of A^+ agrees with the definition $A^+ = \mathrm{Id}\langle f_{ij}^a : a \in A,\ 1 \leq i, j \leq n\rangle_{M_n(R)}$ given in [11].

It is easily checked that $A^+ \subseteq A^*$. When R is a ring, $A^+ = A^* = M_n(A)$, but in the general near-ring situation, it can happen that $A^+ \subset A^*$, where '$\subset$' means 'proper inclusion'. Moreover, it turned out that, in certain cases, ideals strictly enveloped between A^+ and A^* for some A, and not equal to B^+ or B^* for any ideal B of R, exist. These ideals were termed *intermediate*, and it was shown in [3] that any ideal of a matrix near-ring must be of the form A^+ or A^* if it is not intermediate.

Following the same strategy for group near-rings leads to similar results, but we do (rather unexpectedly) also get something new. So let A be an ideal of R, and define

$$^{+}A = \mathrm{Id}\langle [a, e] : a \in A\rangle_{R[G]}$$

and

$$^{*}A = (A^G : R^G)_{R[G]} = \{U \in R[G] : U(R^G) \subseteq A^G\}.$$

Note that we use left superscripts to distinguish between the group near-ring and the matrix near-ring situation. The following result relates all these ideals in case G is finite.

THEOREM 2.1. *Let G be a finite group with $|G| = n$. For any ideal A of R, we have the following inclusions, denoted by arrows*:

$$\begin{array}{ccc} R[G] & \longrightarrow & M_n(R) \\ \uparrow & & \uparrow \\ {}^{*}A & \longrightarrow & A^* \\ \uparrow & & \uparrow \\ {}^{+}A & \longrightarrow & A^+ \end{array}$$

PROOF. The fact that each $[a, e]$, $a \in A$, belongs to $^{*}A$ and each f_{11}^a, $a \in A$, belongs to A^* forces $^{+}A \subseteq {}^{*}A$ and $A^+ \subseteq A^*$. The other inclusions follow from Theorem 1.3 and its proof. □

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It is also natural to ask how to construct an ideal in the base near-ring R from a given ideal $\mathcal{A}$ of $M_n(R)$ or $R[G]$. In the matrix near-ring case, the construction is as follows: For an ideal $\mathcal{A}$ of $M_n(R)$, define

$$\mathcal{A}_* = \{\pi_1 U \iota_1(1) : U \in \mathcal{A}\}.$$

Note that this definition is equivalent to the definition

$$\mathcal{A}_* = \{x \in R : x \in \mathrm{Im}(\pi_j U), \text{ for some } U \in \mathcal{A}, 1 \le j \le n\}$$

given in [6]: if $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in R^n$, and $U\alpha = \beta = \langle \beta_1, \beta_2, \ldots, \beta_n \rangle$ for some $U \in \mathcal{A}$, then $\pi_j(U\alpha) = \beta_j = \pi_1\left[f_{1j}^1 U(f_{11}^{\alpha_1} + f_{21}^{\alpha_2} + \cdots + f_{n1}^{\alpha_n})\right]\iota_1(1)$. It is clear that $f_{1j}^1 U(f_{11}^{\alpha_1} + f_{21}^{\alpha_2} + \cdots + f_{n1}^{\alpha_n}) \in \mathcal{A}$, since $\mathcal{A}$ is a two-sided ideal.

To make a similar construction in the group near-ring case, we need the analogous in R^G of the element $\iota_1(1)$ in R^n. This is given by $\varepsilon \in R^G$, where $\varepsilon(e) = 1$ and $\varepsilon(g) = 0$ if $g \neq e$. For the remainder of this paper, ε will always denote this particular element of R^G. Now let $\mathcal{A}$ be an ideal of $R[G]$. Then

$$_*\mathcal{A} = \{(U\varepsilon)(e) : U \in \mathcal{A}\}.$$

It follows that both $\mathcal{A}_*$ and $_*\mathcal{A}$ are ideals of R (see [6, Proposition 4.6] and [2, Lemma 4.8]). The following theorem summarizes the basic relationships amongst all these ideals.

THEOREM 2.2. (a) *Let A be an ideal of R and let $\mathcal{A}$ be an ideal of $M_n(R)$. Then*

$$(A^+)_* = A = (A^*)_* \quad \text{and} \quad (\mathcal{A}_*)^+ \subseteq \mathcal{A} \subseteq (\mathcal{A}_*)^*.$$

(b) *Let A be an ideal of R and let $\mathcal{A}$ be an ideal of $R[G]$. Then*

$$_*(^+A) = A = {}_*(^*A) \quad \text{and} \quad \mathcal{A} \subseteq {}^*(_*\mathcal{A}).$$

(c) *The maps $A \mapsto A^+$, $A \mapsto A^*$, $A \mapsto {}^+A$ and $A \mapsto {}^*A$ (for A an ideal of R) are injective. The maps $\mathcal{A} \mapsto \mathcal{A}_*$ (for $\mathcal{A}$ an ideal of $M_n(R)$) and $\mathcal{A} \mapsto {}_*\mathcal{A}$ (for $\mathcal{A}$ an ideal of $R[G]$) are surjective.*

PROOF. (a) See, for example, [7].

(b) Let $a \in A$. Then $[a, e] \in {}^+A$, so that $a = ([a, e]\varepsilon)(e) \in {}_*(^+A)$. If $a \in {}_*(^+A)$, then $a = (U\varepsilon)(e)$ for some $U \in {}^+A \subseteq {}^*A$, which shows that $a \in A$. Hence, $A = {}_*(^+A)$. The same procedure is followed to show that $A = {}_*(^*A)$.

Furthermore, it follows from [2, Theorem 4.9] that $\mathcal{A} \subseteq {}^*(_*\mathcal{A})$.

(c) These properties follow in a straightforward manner from (a), (b), [7, Proposition 1.46] and [2, Theorems 4.4–4.5]. □

Unexpectedly (because of the corresponding result in (a)), the inclusion ${}^{+}({}_{*}\mathcal{A}) \subseteq \mathcal{A}$ is in general not valid for group near-rings; not even for a commutative group ring, as the next example shows.

EXAMPLE 2.3. Let R be a commutative ring and let G be an Abelian group which contains an element g of order 2. Consider the element $U = [1, e] + [1, g]$ of the commutative group ring $R[G]$, and let $\mathcal{A} = \mathrm{Id}\langle U\rangle_{R[G]}$. Then

$$(U\varepsilon)(e) = (([1, e] + [1, g])\varepsilon)(e) = \varepsilon(e) + \varepsilon(g) = 1.$$

So $1 \in {}_{*}\mathcal{A}$, forcing ${}_{*}\mathcal{A} = R$. But then ${}^{+}({}_{*}\mathcal{A}) = R[G]$.

We now show that $\mathcal{A}$ is a proper ideal of $R[G]$, from which the desired result ${}^{+}({}_{*}\mathcal{A}) \not\subseteq \mathcal{A}$ follows. Consider $\zeta \in R^G$, where $\zeta(e) = 1$; $\zeta(g) = -1$ and $\zeta(h) = 0$ if $h \in G \setminus \{e, g\}$. Then $(U\zeta)(h) = \zeta(h) + \zeta(hg) = 0$ for all $h \in G$. Hence $U \in \mathrm{Ann}_{R[G]}(\zeta)$ from which it follows that $\mathcal{A} \subseteq \mathrm{Ann}_{R[G]}(\zeta)$. But since there are elements in $R[G]$ which do not annihilate ζ (such as the identity $[1, e]$), our result follows.

3. Intermediate ideals

As in the case of matrix near-rings, the concept of an intermediate ideal also makes sense for group near-rings. As mentioned before, there is, in general, a gap between ${}^{+}A$ and ${}^{*}A$ for an ideal A of R. One way to measure the 'size' of this gap is to count the number of ideals which occur in this gap.

DEFINITION 3.1. An ideal $\mathcal{A}$ of $R[G]$ such that ${}^{+}A \subset \mathcal{A} \subset {}^{*}A$ for some ideal A of R, is called an *intermediate ideal* of $R[G]$. (Recall that '$\subset$' denotes proper inclusion.)

Our first task is to show that these ideals do indeed exist.

EXAMPLE 3.2. Consider the zero-symmetric near-ring $\mathbb{Z}_0[x]$ of polynomials over the integers with zero constant term. Addition is the usual addition of polynomials and multiplication is defined to be composition of polynomials. Fix $n \in \mathbb{Z}$, $n \geq 4$, and define R to be the subnear-ring of $\mathbb{Z}_0[x]$ of all polynomials of which the coefficients of $x^2, x^3, \ldots, x^{2n-1}$ are equal to 0, that is,

$$R = \{a_1 x + a_{2n} x^{2n} + a_{2n+1} x^{2n+1} + \cdots + a_k x^k : k \geq 2n, \\ a_i \in \mathbb{Z},\ i = 1, 2n, 2n+1, \ldots, k\}.$$

Also, if mR (for a positive integer m) denotes the set of all polynomials in R, the coefficients of which are divisible by m, then one easily checks that mR is an ideal of R.

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Let $G = \{e, g\}$. (We could use any finite group here, but the notation becomes more complicated and unnecessarily obscures the clarity of the arguments.) By Theorem 1.3, $R[G]$ is a subnear-ring of $M_2(R)$. Furthermore, by Theorem 2.1, if A is any ideal of R, then ${}^+A \subseteq A^+$. This implies that we can use the results of [8] to prove the following.

RESULT 3.2.1. For any $U \in R[G]$ and for any $\langle p, q\rangle \in R^2$ we have that $U\langle p, q\rangle = \langle \zeta_1(p, q), \zeta_2(p, q)\rangle$, where the ζ_i denote polynomials in two variables over the integers. Moreover, in both $\zeta_1(p, q)$ and $\zeta_2(p, q)$, the coefficients of $p^k q^{2n-k}$ are divisible by $\binom{2n}{k}$, $k = 0, 1, \ldots, 2n$.

RESULT 3.2.2. Let $A = mR$ for some positive integer m and let $U \in {}^+A$. Then, for any $\langle p, q\rangle \in R^2$ we have that $U\langle p, q\rangle = \langle \zeta_1(p, q), \zeta_2(p, q)\rangle$, where the ζ_i denote polynomials in two variables over the integers. Moreover, in both $\zeta_1(p, q)$ and $\zeta_2(p, q)$, the coefficients of $p^k q^{2n-k}$ are divisible by $m\binom{2n}{k}$, $k = 0, 1, \ldots, 2n$.

Now let $m = 2^n$ and consider the ideal $A = mR$. We show that ${}^+A \subset {}^*A$ and that there exists a chain of $n-2$ ideals $\mathcal{A}_i$, $i = 1, 2, \ldots, n-2$, such that

$$ {}^+A \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_{n-2} \subset {}^*A. $$

Consider the elements $W_1, W_2 \in R[G]$ where

$$ W_1 = [x^{2^{n-1}}, e]([x, e] + [x, g]), \qquad W_2 = [x^{2^{n-1}}, e]([-x, e] + [x, g]). $$

Define $W = W_1 - W_2$. Then $W\langle p, q\rangle = \langle \zeta(p, q), \zeta(p, q)\rangle$ where

$$ \zeta(p, q) = 2\sum_{i=1}^{2^{n-2}} \binom{2^{n-1}}{2i-1} p^{2^{n-1}-2i+1} q^{2i-1}, $$

for all $\langle p, q\rangle \in R^2$.

By using Results 3.2.1 and 3.2.2, the remainder of the proof follows exactly the same lines as the proof of [8, Proposition 3.2], except that we do not have 0 in the second co-ordinates of the elements of R^2, that is, we have here $\langle \zeta(p, q), \zeta(p, q)\rangle$ rather than $\langle \zeta(p, q), 0\rangle$, but this has no effect on what we want to show.

At this point one could raise the question: Although an intermediate ideal $\mathcal{A}$ has the property that ${}^+A \subset \mathcal{A} \subset {}^*A$ for some ideal A of R, isn't it possible that $\mathcal{A} = {}^+B$ or $\mathcal{A} = {}^*B$ for some other ideal B of R? As in the case of matrix near-rings, the answer is no:

THEOREM 3.3. *If $\mathcal{A}$ is an intermediate ideal of $R[G]$, then there is a unique ideal A of R such that ${}^+A \subset \mathcal{A} \subset {}^*A$. Moreover, $\mathcal{A}$ is not equal to ${}^+B$ or ${}^*B$ for any ideal B of R.*

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PROOF. By using Theorem 2.2 (b) together with the methods used in [3, Lemmas 2.2–2.3], the results follow. □

For a given intermediate ideal $\mathcal{A}$ of $M_n(R)$, it is known that $\mathcal{A}_*$ is the unique ideal A of R such that $A^+ \subset \mathcal{A} \subset A^*$ (see [3, Corollary 2.5]). It is, however, still an open question whether ${}_*\mathcal{A}$ is always the unique ideal of R enveloping the intermediate ideal $\mathcal{A}$ of $R[G]$.

4. Exceptional ideals

It was shown in [3, Lemma 2.3] that any ideal of $M_n(R)$ that is not intermediate, must be of the form A^+ or A^* for some ideal A of R. This gives a complete characterization of the two-sided ideals of $M_n(R)$.

Surprisingly, the situation is somewhat different for group near-rings. There are, in general, ideals of $R[G]$ that are not intermediate, but also not of the form ${}^+A$ or of the form ${}^*A$, for any ideal A of R.

DEFINITION 4.1. An ideal $\mathcal{A}$ of $R[G]$ that is not intermediate and also not of the form ${}^+A$ or of the form ${}^*A$, for any ideal A of R, is called an *exceptional* ideal of $R[G]$.

Lets continue to study Example 2.3.

EXAMPLE 4.2. In Example 2.3 it was found that ${}^+({}_*\mathcal{A}) \not\subseteq \mathcal{A}$ for the ideal $\mathcal{A} = \mathrm{Ann}_{R[G]}(\zeta)$. We proceed to show that $\mathcal{A}$ is an exceptional ideal of $R[G]$. Suppose that $\mathcal{A} \subseteq {}^*A$ for some ideal A of R. Then, since $(([1, e] + [1, g])\varepsilon)(e) = 1$, it follows that $1 \in A$, implying that $A = R$. This, in turn, implies that ${}^+A = {}^*A = R[G]$. For reference,

$$(1) \qquad \mathcal{A} \subseteq {}^*A \quad \text{implies} \quad {}^+A = {}^*A = R[G].$$

Now suppose that $\mathcal{A}$ is intermediate. Then ${}^+A \subset \mathcal{A} \subset {}^*A$ for an ideal A of R. By (1), ${}^+A = {}^*A$, a contradiction.

Suppose that $\mathcal{A} = {}^+A$ for some ideal A of R. Then, by (1) and Theorem 2.2 (b), $\mathcal{A} = {}^+A = R[G]$, a contradiction, because $\mathcal{A}$ is proper.

Finally, suppose that $\mathcal{A} = {}^*A$ for an ideal A of R. Again, by (1), it follows that $\mathcal{A} = {}^*A = R[G]$, a contradiction.

It is interesting to note that an exceptional ideal could be found in every group near-ring.

THEOREM 4.3. *The augmentation ideal Δ of $R[G]$ is always exceptional.*

PROOF. It was shown in [2, Theorem 4.13] that $\Delta = \mathrm{Id}\langle [1, g] - [1, e] : g \in G \rangle_{R[G]}$. For any $g \neq e$, $[1, g] - [1, e] \in \Delta$, so that $(([1, g] - [1, e])\varepsilon)(e) = -1 \in {}_*\Delta$, forcing ${}_*\Delta = R$. It follows that if $\Delta \subseteq {}^*A$ for an ideal A of R, then ${}^+A = {}^*A = R[G]$, because if $\Delta \subseteq {}^*A$, then ${}_*\Delta \subseteq {}_*({}^*A) = A$, according to Theorem 2.2 (b). Furthermore, because $R[G]/\Delta \cong R$, by [2, Corollary 4.12], and R is assumed to be a non-trivial near-ring, Δ is a proper ideal of $R[G]$. Now follow the same method as in Example 4.2. □

5. Modules over $R[G]$ and the Jacobson radicals

In this last section we would like to present some results regarding the $\mathcal{J}$-radicals of $R[G]$ which means that we need to study some module theory over $R[G]$. Since similar results have been obtained with respect to matrix near-rings, we certainly want to utilize these, henceforth we only focus on the case where G is finite. In particular, we let $G = \{g_1 = e, g_2, \dots, g_n\}$.

In what follows, the terminology 'ideal', 'R-subgroup', 'simple' and 'R-simple', has the same meaning as in [9, Definitions 1.27 (b), 1.21 (b) and 1.36]. Also note that, because of the way in which $R[G]$ (respectively, $M_n(R)$) is defined, R^n can be viewed in a natural way as a (left) $R[G]$-module (respectively, $M_n(R)$-module). This brings us to

THEOREM 5.1. *If L is an ideal of the module ${}_RR$, that is, L is a left ideal of the near-ring R, then L^n is an ideal of the module ${}_{R[G]}R^n$.*

PROOF. We know that L^n is an ideal of ${}_{M_n(R)}R^n$, by [6, Proposition 4.1]. But since $R[G]$ is a subnear-ring of $M_n(R)$ by Theorem 1.3, the result follows. □

The next step is to show how an arbitrary module over R can be extended to a module over $R[G]$. Since we are only interested in type 0 and type 2 modules, we will assume that all modules are monogenic, that is, if Γ is an R-module then there exists $\gamma \in \Gamma$ such that $R\gamma = \Gamma$. This implies that we can view Γ^n as an $R[G]$-module, as follows: Let $U \in R[G]$ and $\langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle \in \Gamma^n$. Then there are $r_1, r_2, \dots, r_n \in R$ such that $r_i\gamma = \gamma_i$, $i = 1, 2, \dots, n$. Define

$$U\langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle = (U\langle r_1, r_2, \dots, r_n \rangle)\gamma,$$

where $\langle s_1, s_2, \dots, s_n \rangle \gamma = \langle s_1\gamma, s_2\gamma, \dots, s_n\gamma \rangle$ for every $\langle s_1, s_2, \dots, s_n \rangle \in R^n$.

Note that this is exactly the way in which Γ^n has been defined as an $M_n(R)$-module (see [10]). Since $R[G]$ is a subnear-ring of $M_n(R)$, this definition makes sense, and ${}_{R[G]}\Gamma^n$ is well-defined.

THEOREM 5.2. *If Γ is a monogenic R-module, then Γ^n is a monogenic $R[G]$-module.*

PROOF. Suppose $R\gamma = \Gamma$ for some $\gamma \in \Gamma$. As before, we can index the coordinates of an $\alpha \in \Gamma^n$ with the elements of G, that is, $\alpha(g_i) = \pi_i(\alpha)$. Consider the element $\eta \in \Gamma^n$, where $\eta(g_1) = \gamma$ and $\eta(g_j) = 0$ for $j \neq 1$. We show that η is a generator for Γ^n over $R[G]$.

Let $1 \leq i \leq n$ and let $r_i \in R$. Then

$$([r_i, g_i]\eta)(h) = \begin{cases} r_i\gamma & \text{if } h = g_i^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

But then $([r_1, g_1] + [r_2, g_2] + \cdots + [r_n, g_n])\eta = \langle r_1\gamma, r_2\gamma, \ldots, r_n\gamma\rangle$. By varying each r_i over the elements of R, we see that $R[G]\eta = \Gamma^n$. □

If Λ is an ideal of the monogenic module ${}_R\Gamma$, we can easily generalize Theorem 5.1 by showing that Λ^n is an ideal of ${}_{R[G]}\Gamma^n$. Also, by Theorem 5.2, since Γ/Λ is a monogenic R-module (via the natural action $r(\gamma + \Lambda) = r\gamma + \Lambda$), we have that $(\Gamma/\Lambda)^n$ is a monogenic $R[G]$-module. Moreover, $(\Gamma/\Lambda)^n \cong_{R[G]} \Gamma^n/\Lambda^n$, a fact which can be proved in a way similar to the proof of [7, Proposition 1.29], where the same result was proved for matrix near-rings.

The following result is needed in the example that follows:

THEOREM 5.3. *Let R be zero-symmetric and let ${}_R\Gamma$ be a monogenic module where $|\Gamma| = 2$. If $G = \{e, g\}$, then the diagonal of Γ^2, $d(\Gamma^2) = \{\langle\gamma, \gamma\rangle : \gamma \in \Gamma\}$, is a non-trivial, proper ideal of the module ${}_{R[G]}\Gamma^2$.*

PROOF. Since $|\Gamma^2| = 4$ and $|d(\Gamma^2)| = 2$, the diagonal is clearly non-trivial and proper. It is also trivially closed under addition. We use induction on the complexity of $U \in R[G]$ (see the discussion following Theorem 2.4 in [2]) to prove that

$$U(\langle\gamma, \gamma\rangle + \langle\alpha, \beta\rangle) - U\langle\alpha, \beta\rangle \in d(\Gamma^2), \tag{2}$$

for all $\langle\gamma, \gamma\rangle \in d(\Gamma^2)$, $\langle\alpha, \beta\rangle \in \Gamma^2$ and $U \in R[G]$. Note that if $R\gamma' = \Gamma$, then each of α and β in (2) vary over the set $\{0, \gamma'\}$.

Let $U \in R[G]$ have complexity 1, that is, $U = [r, e]$ or $U = [r, g]$ for some $r \in R$. Lets say $U = [r, e]$ (the case $U = [r, g]$ being treated similarly). Then

$$\begin{aligned} [r, e](\langle\gamma, \gamma\rangle + \langle\alpha, \beta\rangle) - [r, e]\langle\alpha, \beta\rangle &= \langle r(\gamma + \alpha) - r\alpha, r(\gamma + \beta) - r\beta\rangle \\ &= \langle r\gamma, r\gamma\rangle \in d(\Gamma^2). \end{aligned}$$

Now consider any $U \in R[G]$ with complexity greater than 1, and assume the result to be true for all elements of $R[G]$ which have complexity smaller than that of U. Then either $U = V + W$ or $U = VW$, where the complexity of both V and W are smaller than that of U. On the one hand,

$$\begin{aligned}
&U(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - U\langle\alpha,\beta\rangle\\
&\quad = (V+W)(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - (V+W)\langle\alpha,\beta\rangle\\
&\quad = V(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - V\langle\alpha,\beta\rangle + W(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - W\langle\alpha,\beta\rangle\\
&\quad \in d(\Gamma^2) + d(\Gamma^2) = d(\Gamma^2),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&U(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - U\langle\alpha,\beta\rangle\\
&\quad = (VW)(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - (VW)\langle\alpha,\beta\rangle\\
&\quad = V[W(\langle\gamma,\gamma\rangle + \langle\alpha,\beta\rangle) - W\langle\alpha,\beta\rangle + W\langle\alpha,\beta\rangle] - V(W\langle\alpha,\beta\rangle)\\
&\quad = V(\langle\delta,\delta\rangle + W\langle\alpha,\beta\rangle) - V(W\langle\alpha,\beta\rangle) \quad \text{for some } \langle\delta,\delta\rangle \in d(\Gamma^2)\\
&\quad \in d(\Gamma^2),
\end{aligned}$$

and the proof is complete. □

COROLLARY 5.4. *With the same assumptions as in Theorem* 5.3, *we have that both the* $R[G]$*-modules* $d(\Gamma^2)$ *and* $\Gamma^2/d(\Gamma^2)$ *are of type* 2, *hence also of type* 0.

PROOF. Both these modules have order 2 and are non-trivial. □

COROLLARY 5.5. *If* ${}_R\Gamma$ *is simple* (R*-simple*), *then* ${}_{R[G]}\Gamma^n$ *is not necessarily simple* ($R[G]$*-simple*).

There exists a very natural relationship between the $\mathcal{J}$-radicals of R and the corresponding matrix near-ring $M_n(R)$, namely $\mathcal{J}_\nu(M_n(R)) \subseteq \mathcal{J}_\nu(R)^*$, $\nu \in \{0, 2\}$ [7, Theorem 2.34]. When $\nu = 2$, we even have $\mathcal{J}_2(M_n(R)) = \mathcal{J}_2(R)^*$, which is, of course, a very useful tool.

The key result which enables us to prove these relationships, is the fact that ${}_R\Gamma$ is simple (R-simple) if and only if ${}_{M_n(R)}\Gamma^n$ is simple ($M_n(R)$-simple) [10, Corollary 3.8]. We have just seen in Corollary 5.5 that this flow of simplicity does not necessarily occur between R-modules and $R[G]$-modules. The consequences of this are reflected in the following example, where we construct a finite, Abelian, zero-symmetric near-ring R such that (for $\nu \in \{0, 2\}$) both $\mathcal{J}_\nu(R[G]) \not\subseteq {}^*\mathcal{J}_\nu(R)$ and ${}^*\mathcal{J}_\nu(R) \not\subseteq \mathcal{J}_\nu(R[G])$, where $|G| = 2$. It turns out, though, that ${}^+\mathcal{J}_\nu(R) \subset \mathcal{J}_\nu(R[G])$ for this example. It is still an open question whether ${}^+\mathcal{J}_\nu(R) \subseteq \mathcal{J}_\nu(R[G])$ holds in general.

EXAMPLE 5.6. Consider the (additive) groups

$$M = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad N = M \oplus \mathbb{Z}_2, \quad H = N \oplus \mathbb{Z}_2.$$

Let M_i, $1 \le i \le 3$, be the two-element subgroups of M and let N_j, $1 \le j \le 4$, be the two-element subgroups of N which are not contained in M. Also, let $m_i \in M_i$, $1 \le i \le 3$, and $n_j \in N_j$, $1 \le j \le 4$, denote the non-zero elements in these groups. Finally, let $h_1, h_2, \dots, h_8$ denote the elements of $H \setminus N$. Define the near-ring R as follows:

$$\begin{aligned} R = \{f \in M_0(H) : f(M_i) \subseteq M_i,\ 1 \le i \le 3;\ f(N_j) \subseteq N_j,\ 1 \le j \le 4;\\ h, h' \in H \text{ and } h - h' \in M \text{ implies } f(h) - f(h') \in M;\\ h, h' \in H \text{ and } h - h' \in N \text{ implies } f(h) - f(h') \in N\}, \end{aligned}$$

where $M_0(H)$ is the subnear-ring of $M(H)$ containing the zero-preserving mappings. It turns out that R is a zero-symmetric, Abelian near-ring with identity and R is finite with $|R| = 2^{23}$. We also note that each M_i $(1 \le i \le 3)$, each N_j $(1 \le j \le 4)$, as well as the group H/N can be viewed as an R-module because of the way that R has been defined. We study the group near-ring $R[G]$ where G is the group $\{e, g\}$.

First, define the following ideals of ${}_R R$:

$$\begin{aligned} K &= \{f \in R : f(h_i) \in M,\ 1 \le i \le 8;\ 0 \text{ otherwise}\},\\ L &= \{f \in R : f(h_i) \in N,\ 1 \le i \le 8;\ 0 \text{ otherwise}\}. \end{aligned}$$

Our first observation is that

$$\mathcal{J}_0(R) = \mathcal{J}_2(R) = \operatorname{Ann}_R N \cap \operatorname{Ann}_R(H/N) = L. \tag{3}$$

This follows from the fact that all M_i's, all N_j's, as well as H/N, are R-modules of type 0, since they are all of order 2 and non-trivial (hence also of type 2), the fact that

$$\operatorname{Ann}_R N = \left[\bigcap_{i=1}^{3} \operatorname{Ann}_R M_i\right] \cap \left[\bigcap_{j=1}^{4} \operatorname{Ann}_R N_j\right],$$

and also from the fact that L is nilpotent (see [9, Theorem 5.37 (d)]). From now on, we simply write $\mathcal{J}(R)$ for $\mathcal{J}_0(R) = \mathcal{J}_2(R)$.

An easy application of Corollary 5.4 and by arguments similar to the above leads us to

$$\begin{aligned} \mathcal{J}_0(R[G]) &= \mathcal{J}_2(R[G])\\ &= \left[\bigcap_{i=1}^{3} \operatorname{Ann}_{R[G]}(d((M_i)^2))\right] \cap \left[\bigcap_{i=1}^{3} \operatorname{Ann}_{R[G]}(M_i^2/d(M_i^2))\right]\\ &\cap \left[\bigcap_{j=1}^{4} \operatorname{Ann}_{R[G]}(d((N_j)^2))\right] \cap \left[\bigcap_{j=1}^{4} \operatorname{Ann}_{R[G]}(N_j^2/d((N_j^2))\right]\\ &\cap \operatorname{Ann}_{R[G]}(d((H/N)^2)) \cap \operatorname{Ann}_{R[G]}((H/N)^2/d((H/N)^2)), \end{aligned}$$

which, from now on, will simply be denoted by $\mathcal{J}(R[G])$.

Next, observe that, since $(\mathcal{J}(R))^2 = 0$ (by (3)), we have that $({}^{+}\mathcal{J}(R))^2 = 0$ in $R[G]$. This follows from [1, Lemma 3.1] and the fact that ${}^{+}\mathcal{J}(R) \subseteq {}^{*}\mathcal{J}(R)$ (Theorem 2.1). Consequently, ${}^{+}\mathcal{J}(R) \subseteq \mathcal{J}(R[G])$.

We now show that there are also elements of nilpotency degree 3 in $\mathcal{J}(R[G])$, implying that

$$(4) \qquad {}^{+}\mathcal{J}(R) \subset \mathcal{J}(R[G]).$$

To this end, consider the ideal

$$\mathcal{A} = \mathrm{Ann}_{R[G]} K^2 \cap \mathrm{Ann}_{R[G]}(L^2/K^2) \cap \mathrm{Ann}_{R[G]}(R^2/L^2).$$

Since $\mathcal{A}^3 = 0$, we have that $\mathcal{A} \subseteq \mathcal{J}(R[G])$.

Also consider the elements $a, b, c, d \in R$, defined as follows:

$$\begin{aligned} a(h_i) &= n_1, \ 1 \le i \le 8; \ 0 \text{ otherwise}, \\ b(h_i) &= n_2, \ 1 \le i \le 8; \ 0 \text{ otherwise}, \\ c(m_3) &= m_3; \ 0 \text{ otherwise}, \\ d(n_j) &= n_j, \ 1 \le j \le 4; \ 0 \text{ otherwise}, \end{aligned}$$

where $n_1 = (0, 1, 1, 0)$, $n_2 = (1, 0, 1, 0)$ and $m_3 = (1, 1, 0, 0)$.

Direct computation shows that $V = [a, e] + [b, g] + [c, e]([d, e] + [d, g]) \in R[G]$ is an element of $\mathcal{A}$, hence an element of $\mathcal{J}(R[G])$. It is, however, not an element of ${}^{+}\mathcal{J}(R)$, because $V^2 \neq 0$. (Note that $V^2\langle 1, 0\rangle = \langle c(da + db), c(da + db)\rangle \neq \langle 0, 0\rangle$, since $c(da + db)(h_1) \neq 0$.) So (4) is proved.

Our next task is to show that

$$(5) \qquad \mathcal{J}(R[G]) \not\subseteq {}^{*}\mathcal{J}(R).$$

Consider $U = [1, e] + [1, g]$. Since R (hence $R[G]$) has characteristic 2, the diagonal of any (Abelian) $R[G]$-module Γ^2 is mapped to 0, and all other (non-diagonal) elements are mapped into the diagonal ($U\langle \gamma_1, \gamma_2\rangle = \langle \gamma_1 + \gamma_2, \gamma_1 + \gamma_2\rangle$). It follows that $U \in \mathcal{J}(R[G])$. But since $U\langle 1, 0\rangle = \langle 1, 1\rangle \notin (\mathcal{J}(R))^2$, it is immediate that $U \notin {}^{*}\mathcal{J}(R)$, thus (5) follows.

We finally show that there are elements in ${}^{*}\mathcal{J}(R)$ which are not in $\mathcal{J}(R[G])$. One such element is $W = [s, e]([t, e] + [t, g])$, where

$$\begin{aligned} s(m_3) &= m_3; \ 0 \text{ otherwise}, \\ t(m_i) &= m_i, \ i = 1, 2; \ 0 \text{ otherwise}. \end{aligned}$$

To see this, let K_0 be the R-subgroup of K generated by $k_1, k_2 \in K$, where

$$k_1(h_1) = m_1;\ 0 \text{ otherwise},$$
$$k_2(h_1) = m_2;\ 0 \text{ otherwise}.$$

In other words, $K_0 = \{f \in R : f(h_1) \in M;\ 0 \text{ otherwise}\}$.

It is easy to see that the ideal generated by any non-zero element of the $R[G]$-module K_0^2, is all of K_0^2, which means that the module is simple. It is also monogenic with generator $\langle k_1, k_2\rangle$, hence a type 0 module. But this implies that

$$\mathcal{J}(R[G]) \subseteq \operatorname{Ann}_{R[G]} K_0^2. \tag{6}$$

We find that $W\langle r, r'\rangle = \langle s(tr + tr'), s(tr + tr')\rangle$ for any $\langle r, r'\rangle \in R^2$. Furthermore, direct computation shows that $s(tr + tr')(N) = 0$ and $s(tr + tr')(H) \subseteq N$, and it follows that $W \in {}^*\mathcal{J}(R)$, by (3).

However, $W\langle k_1, k_2\rangle = \langle s(tk_1 + tk_2), s(tk_1 + tk_2)\rangle$ where

$$s(tk_1 + tk_2)(h_1) = s(t(m_1) + t(m_2)) = s(m_1 + m_2) = s(m_3) = m_3 \neq 0.$$

Consequently, $W \notin \operatorname{Ann}_{R[G]} K_0^2$, and, by (6), ${}^*\mathcal{J}(R) \not\subseteq \mathcal{J}(R[G])$ is proved.

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