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NEW ALGORITHMS FOR DISCRETE-TIME OPTIMAL CONTROL PROBLEMS

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Abstract

The paper presents new demonstrably convergent first-order iterative algorithms for unconstrained discrete-time optimal control problems. The algorithms, which solve the linear-quadratic problem in one iterative step, are particularly suited for solving nonlinear problems with linear constraints *via* penalty function methods. The proofs of the reduction of cost at each iteration and convergence of the algorithms are provided.

1. Introduction

The purpose of this paper is to provide a new computational technique for solving discrete-time optimal control (or multi-stage decision) problems. Discrete-time systems are described by difference equations and involve choices or decisions at each of a finite set of times or stages. The optimal multi-stage decision problem is then to minimize the cost associated with each sequence of decisions.

Many economic and engineering problems in all sectors of business and industry can be viewed as multi-stage decision processes. Examples of this type of problem, where a finite change in control causes a finite change in the state of the system, include ecological systems, inventory control, resource allocation, production scheduling and the control of a system of water reservoirs.

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As differential equations are harder to solve than difference equations, discrete approximations are sometimes used for solving continuous-time problems. Since both continuous and discrete-time controls can be considered as points in appropriate spaces, in view of the fact that the theory of functional analysis reflects to a degree our abstractions of intuitive geometric properties, it is not surprising that there are great similarities in the analyses for continuous and discrete-time systems. In general, formulas for the discrete-time case turn out to be somewhat more complex than their continuous-time counterparts, but are easier to justify as problems regarding the existence of solutions of differential equations do not arise.

Although difference equations are easier to solve than the differential equations, optimization of discrete-time systems is still a formidable task when the number of state and control variables and the number of stages and constraints is large. In [8], for instance, the Tennessee Valley Authority, which manages a system of 40 water reservoirs, reported that the maximum size of the severely constrained problem of controlling a system of water reservoirs solved with the existing numerical methods involved a system of 6 reservoirs. A general analytic solution of the discrete-time problem does not exist and recourse must be made to numerical methods. Dynamic programming, although impressive in comparison with direct enumeration, is effective only when the number of variables is small. Consequently, many iterative methods have been developed. Dyer and McReynolds [1] devised a second order method known as the successive sweep (SV) algorithm. Jacobson and Mayne [4] invented the differential dynamic programming (DDP) and Gershwin and Jacobson [2] analysed DDP for constrained problems and applied it to optimal orbit transfer. Since the SV algorithm [1] and the second order DDP algorithms in [4] and [2] are generally not convergent as they require the inversion of matrices which may be singular, Ohno [7] modified the second order discrete-time algorithm for problems with and without constraints and proved its local convergence (when the starting point is sufficiently close to the optimum). Recently, Wong and Teo [10] devised a computational method for optimal control problems with discrete delays and bounded control region.

The general characteristic of the existing first order methods, in comparison with second order methods, is that they require a large number of iterations. However, the second order methods require the computation of second order derivatives which is time consuming and necessitates a large computer memory; in fact, the computational effort per iterative step and memory requirements increase as a cubic function of the number of state and control variables. It is clear that a fast first order method would substantially increase our ability to solve large problems. One such technique is the algorithm in [9] developed by Teo, Wong and Clements for solving time-lag optimal control problems with control and terminal constraints. Control parametrization technique is for continuous time optimal control problems. However, it can also be applied to discrete time optimal control problems.

The aim of this paper is to present a fast first-order method for discrete-time systems. A number of rapidly convergent algorithms, known as LQRE, for continuous-time optimal control problems, was presented in [6]. Here we show that it is possible to develop discrete-time versions of the continuous-time LQRE algorithms. As the continuous-time algorithm in [6] is a first order method with the speed of convergence comparable to that of second order methods, its discrete-time analog (which is a globally convergent first order algorithm and converges on the linear-quadratic problem in one step) should be seen as a step in that direction.

The organization of the paper is as follows. Section 2 contains the definition of the problem. A statement of the proposed algorithm is given in Section 3. Section 4 is devoted to the proof of reduction in the value of the cost functional, whilst global convergence of the algorithm to a point satisfying a first order necessary condition for optimality is proved in Sections 5 and 6. Finally in Section 7 we show how the discrete-time LQRE algorithm in conjunction with the penalty function method can be applied to constrained problems in general and to problems with linear constraints in particular.

2. Definitions and assumptions

Let $N \geqslant 1$ be a fixed integer, let U denote the space of admissible controls defined as the set of all ordered N-tuples $u \triangleq (u_0, u_1, \ldots, u_{N-1})$, $u_k \in R^m$, $k = 0, 1, \ldots, N-1$, and let $x \triangleq (x_0, x_1, \ldots, x_N)$, $x_k \in R^n$, $k = 0, 1, \ldots, N$, denote the state of a system governed by a first order ordinary difference equation

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, k), \qquad k = 0, 1, \ldots, N-1, \\ x_0 &= \chi_0, \qquad \chi_0 \text{ specified.} \end{aligned} \tag{2.1}$$

The problem to be solved is the following.

Minimize the cost functional $J\colon U \to R$ *defined by*

$$J(u) = \sum_{k=0}^{N-1} L(x_k, u_k, k) + F(x_N) \tag{2.2}$$

subject to the system equation (2.1).

Note that u_k and x_k are m and n dimensional vectors.

In the sequel $f(x_k, u_k, k)$ and $L(x_k, u_k, k)$ will also be denoted by $f_k(x_k, u_k)$ and $L_k(x_k, u_k)$, respectively. Since one can consider the above problem as a multi-stage process, it is customary to refer to u_k and x_k as the control and state of the system at stage k. We assume that F: $R^n \to R$ is continuously differentiable, that f: $R^n \times R^m \times \mathbf{N} \to R^n$ and L: $R^n \times R^m \times \mathbf{N} \to R$ are continuously differentiable in the first two arguments and that the solution to the discrete-time optimal control problem defined by (2.1) and (2.2) exists.

For the purpose of the analysis in this paper, it is convenient to write equations (2.1) and (2.2) in a slightly different form. Let Q_N: $R^n \to R^{n\times n}$ be continuously differentiable and let A: $R^n \times R^m \times \mathbf{N} \to R^{n\times n}$, B: $R^n \times R^m \times \mathbf{N} \to R^{n\times m}$, Q: $R^n \times R^m \times \mathbf{N} \to R^{n\times n}$, P: $R^n \times R^m \times \mathbf{N} \to R^{m\times m}$ and R: $R^n \times R^m \times \mathbf{N} \to R^{m\times m}$ be continuously differentiable in the first two arguments. In addition, let $Q(\chi, \omega, k)$, $Q_N(\chi) \geqslant 0$ (positive semi-definite) and $R(\chi, \omega, k) > 0$ (positive definite) for all $\chi \in R^n$, $\omega \in R^m$ and $k \in I_{N-1} \triangleq \{0, 1, \dots, N-1\}$. Without any loss of generality $Q(\chi, \omega, k)$, $Q_N(\chi)$ and $R(\chi, \omega, k)$ can be considered as being symmetric. Then (2.1) and (2.2) can be rewritten as

$$\begin{aligned} x_{k+1} &= A(x_k, u_k, k)x_k + B(x_k, u_k, k)u_k + g(x_k, u_k, k), \qquad k \in I_{N-1}, \\ x_0 &= \chi_0, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} J(u) = \sum_{k=0}^{N-1} \big\{ &\tfrac{1}{2} x_k^T Q(x_k, u_k, k) x_k + u_k^T P(x_k, u_k, k) x_k \\ &+ \tfrac{1}{2} u_k^T R(x_k, u_k, k) u_k + q(x_k, u_k, k) \big\} + F(x_N) \end{aligned} \tag{2.4}$$

where the symbol T denotes transpose of a vector or matrix and the functions g and q are defined by

$$\begin{aligned} g(\chi, \omega, k) &\triangleq f(\chi, \omega, k) - A(\chi, \omega, k)\chi - B(\chi, \omega, k)\omega, \\ q(\chi, \omega, k) &\triangleq L(\chi, \omega, k) - \tfrac{1}{2}\chi^T Q(\chi, \omega, k)\chi \\ &\qquad - \omega^T P(\chi, \omega, k)\chi - \tfrac{1}{2}\omega^T R(\chi, \omega, k)\omega. \end{aligned}$$

If there is no confusion possible, to avoid cumbersome notation such as

$$A(x_k, u_k, k), \quad \frac{\partial f}{\partial x}(\chi, \omega, k)\Big|_{\omega = u_k}^{\chi = x_k}, \quad etc.,$$

we shall simply write

$$A_k, \frac{\partial f_k}{\partial x}, \textit{etc.}, \text{ respectively.}$$

3. Statement of the algorithm

Let $\bar{x} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N)$, the solution of (2.1) (or, equivalently of (2.3)) for some nominal control $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$, denote a nominal state of the system. Define $\bar{S}_k \in R^{n \times n}$, $k \in I_{N-1}$ as the solution of the matrix difference equation

$$\bar{S}_k = \bar{Q}_k + \bar{A}_k^T \bar{S}_{k+1} \bar{A}_k - \bar{W}_k^T \bar{V}_k^{-1} \bar{W}_k; \quad \bar{S}_N = \bar{Q}_N \tag{3.1}$$

and let

$$\bar{h}_k \in R^n, \qquad k \in I_{N-1}$$

be the solution of

$$\bar{h}_k^T = \bar{\phi}_k - \bar{\psi}_k^T \bar{V}_k^{-1} \bar{W}_k + \bar{h}_{k+1}\left(\frac{\partial \bar{f}_k}{\partial x} - \frac{\partial \bar{f}_k}{\partial u} \bar{V}_k^{-1} \bar{W}_k\right); \quad \bar{h}_N^T = \frac{de(\bar{x}_N)}{dx} \tag{3.2}$$

where

$$\bar{W}_k \triangleq \left(\bar{B}_k^T \bar{S}_{k+1} \bar{A}_k + \bar{P}_k\right), \tag{3.3}$$

$$\bar{V}_k \triangleq \bar{D}_k + \bar{B}_k^T \bar{S}_{k+1} \bar{B}_k, \tag{3.4}$$

$\bar{D}_k \in R^{m \times m}$ is a positive definite matrix, and

$$e(\chi) = F(\chi) - \tfrac{1}{2} T \bar{Q}_N \chi, \tag{3.5}$$

$$\bar{\phi}_k^T \triangleq \frac{\partial \bar{\rho}_k}{\partial x} - \bar{u}_k^T \bar{W}_k, \tag{3.6}$$

$$\bar{\psi}_k^T \triangleq \frac{\partial \bar{\rho}_k}{\partial u} - \bar{u}_k^T \bar{V}_k - \bar{x}_k^T \bar{W}_k, \tag{3.7}$$

$$\begin{aligned} \rho_k(\chi, \omega) \triangleq\ & \tfrac{1}{2} \bar{x}_k^T \left[Q_k(\chi, \omega) + A_k^T(\chi, \omega) \bar{S}_{k+1} A_k(\chi, \omega)\right] \bar{x}_k \\ & + \omega^T \left[B_k^T(\chi, \omega) \bar{S}_{k+1} A_k(\chi, \omega) + P_k(\chi, \omega)\right] \chi \\ & + \tfrac{1}{2} \omega^T \left[R_k(\chi, \omega) + B_k^T(\chi, \omega) \bar{S}_{k+1} B_k(\chi, \omega)\right] \omega \\ & + g_k^T(\chi, \omega) \bar{S}_{k+1} \left[A_k(\chi, \omega)\chi + B_k(\chi, \omega)\omega + g_k(\chi, \omega)\right] \\ & + q_k(\chi, \omega). \end{aligned} \tag{3.8}$$

The bars above the symbols indicate that the respective quantities are evaluated at $\chi = \bar{x}_k$ and $\omega = \bar{u}_k$. Also $A_k(\chi, \omega)$, *etc.*, denote $A(\chi, \omega, k)$, *etc.*, $\partial \bar{f}_k / \partial x$ and $\partial \bar{f}_k / \partial u$ denote the Jacobians, whilst $\partial \bar{\rho}_k / \partial x$ and $\partial \bar{\rho}_k / \partial u$ are considered row vectors. Observe that, for $\bar{D}_k > 0$, $\bar{V}_k > 0$.

In addition, define

$$\bar{\tau}_k \triangleq \bar{u}_k + \bar{V}_k^{-1} \bar{W}_k \bar{x}_k, \tag{3.9}$$

$$\bar{\beta}_k^T \triangleq \bar{\psi}_k^T \bar{V}_k^{-1} + \bar{h}_{k+1}^T \frac{\partial \bar{f}_k}{\partial u} \bar{V}_k^{-1} + \bar{\tau}_k \tag{3.10}$$

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and

$$\Delta(\bar{u}) \triangleq \sum_{k=0}^{N-1} \{\bar{\beta}_k^T \bar{V}_k \bar{\beta}_k\}. \tag{3.11}$$

Then the proposed algorithm can be stated as follows.

Discrete time LQRE algorithm

STEP 1. Select some nominal control $\bar{u}$, compute the nominal state $\bar{x}$ according to (2.1) and the value of the cost functional $J(\bar{u})$ for that nominal control, given by (2.2).

STEP 2. Solve equations (3.1) and (3.2). Compute $\Delta(\bar{u})$ according to (3.11). Set $\varepsilon = 1$. If $\Delta(\bar{u}) = 0$ stop.

STEP 3. Apply control $\tilde{u}$: $\tilde{u}_k = \bar{u}_k + \delta u_k$, $k \in I_{N-1}$, and compute the corresponding state of the system $\tilde{x}$: $\tilde{x}_k = \bar{x}_k + \delta x_k$, $k \in I_N \triangleq \{0, 1, \ldots, N\}$, where $\delta u_k = -\varepsilon\bar{\beta}_k - \bar{V}_k^{-1}\bar{W}_k \delta x_k$. Compute the cost $J(\tilde{u})$.

STEP 4. If $J(\tilde{u}) - J(\bar{u}) + \frac{1}{2}\varepsilon\Delta(\bar{u}) < 0$, set $\bar{x} = \tilde{x}$, $\bar{u} = \tilde{u}$, $\bar{u} = \tilde{u}$, $J(\bar{u}) = J(\tilde{u})$ and go to Step 2. If the above criterion is not satisfied, set ε to $\varepsilon/2$ and go to Step 3.

REMARK 3.1. Although the proofs in Sections 4, 5 and 6 hold for an arbitrary $D_k > 0$, unless otherwise stated, it will be assumed that $\bar{D}_k = \bar{R}_k$.

REMARK 3.2. All continuous-time LQRE algorithms presented in [6] have their discrete-time analogs. Both first and second order algorithms can be obtained depending on the manner in which A_k, B_k, Q_k, P_k, B_k and Q_N are chosen. An algorithm worth explicit mention is the discrete-time analog of FORE-3 [6], which is a highly efficient first-order algorithm devised specifically for problems with nonlinear dynamics and quadratic cost; in the discrete version of FORE-3 we set $A_k = \partial f u_k/\partial x$, $B_k = \partial \bar{f}_k/\partial u$ ($Q_k > 0$, P_k, $R_k > 0$ and $Q_N > 0$ are given when the problem is specified).

4. Proof of reduction at each iteration

In this section we derive an expression (correct to the first order of δx_k and δu_k) for the increment of the cost functional at each iteration of the proposed algorithm and show that the increment represents a reduction in the value of the cost functional.

From the identity

$$\sum_{k=0}^{N-1} \left\{ \tfrac{1}{2}\left(x_{k+1}^T \bar{S}_{k+1} x_{k+1} - x_k^T \bar{S}_k x_k\right) \right\} - \tfrac{1}{2} x_N^T \bar{S}_N x_N + \tfrac{1}{2} x_0^T \bar{S}_0 x_0 = 0$$

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after x_{k+1} and $\bar{S}_k$ are substituted with the right-hand sides of (2.3) and (3.1), it follows that

$$\sum_{k=0}^{N-1} \tfrac{1}{2}\Big\{(A_k x_k + B_k u_k + g_k)^T \bar{S}_{k+1}(A_k x_k + B_k u_k + g_k) - x_k^T\big(\bar{Q}_k + \bar{A}_k \bar{S}_{k+1}\bar{A}_k - \bar{W}_k^T \bar{V}_k^{-1}\bar{W}_k\big)x_k\Big\} - \tfrac{1}{2}x_N^T \bar{Q}_N x_N + \tfrac{1}{2}\chi_0^T \bar{S}_0 \chi_0 = 0. \tag{4.1}$$

Adding this zero quantity to the expression (2.4) we have

$$J(u) = \sum_{k=0}^{N-1} \Big\{\tfrac{1}{2}x_k^T A_k^T \bar{S}_{k+1} A_k x_k + \tfrac{1}{2}x_k^T A_k^T \bar{S}_{k+1} B_k u_k + \tfrac{1}{2}u_k + g_k^T \bar{S}_{k+1} A_k x_k + g_k^T \bar{S}_{k+1} B_k u_k + \tfrac{1}{2} g_k^T \bar{S}_{k+1} g_k - \tfrac{1}{2}x_k^T\big((\bar{Q}_k + \bar{A}_k \bar{S}_{k+1}\bar{A}_k) - \bar{W}_k^T \bar{V}_k^{-1}\bar{W}_k\big)x_k + \tfrac{1}{2}x_k^T Q_k x_k + u_k^T P_k x_k + \tfrac{1}{2}u_k^T R_k u_k + q_k\Big\} + \tfrac{1}{2}\chi_0 \bar{S}_0 \chi_0 + e(x_N)$$

where e is defined by (3.5).

Consequently,

$$J(\bar{u}) = \tfrac{1}{2}\chi_0^T \bar{S}_0 \chi_0 + e(\bar{x}_N) + \sum_{k=0}^{N-1} \Big\{\tfrac{1}{2}\bar{u}_k^T \bar{R}_k \bar{u}_k + \bar{u}_k^T \bar{B}_k^T \bar{S}_{k+1} \bar{A}_k \bar{x}_k + \bar{u}_k^T \bar{P}_k \bar{x}_k + \tfrac{1}{2}\bar{u}_k^T \bar{B}_k^T \bar{S}_{k+1} \bar{B}_k \bar{u}_k + \bar{g}_k^T \bar{S}_{k+1} \bar{A}_k \bar{x}_k + \bar{g}_k^T \bar{S}_{k+1} \bar{B}_k \bar{u}_k + \tfrac{1}{2}\bar{g}_k^T \bar{S}_{k+1} \bar{g}_k + \bar{q}_k + \tfrac{1}{2}\bar{x}_k^T \bar{W}_k^T \bar{V}_k^{-1} \bar{W}_k \bar{x}_k\Big\}$$

and

$$J(\tilde{u}) = \tfrac{1}{2}\chi_0^T \bar{S}_0 \chi_0 + e(\tilde{x}_N) + \sum_{k=0}^{N-1} \Big\{\tfrac{1}{2}\tilde{x}_k^T \tilde{Q}_k \tilde{x}_k - \tfrac{1}{2}\tilde{x}_k^T \bar{Q}_k \tilde{x}_k + \tfrac{1}{2}\tilde{x}_k^T \tilde{A}_k^T \bar{S}_{k+1} \tilde{A}_k \tilde{x}_k - \tfrac{1}{2}\tilde{x}_k^T \bar{A}_k^T \bar{S}_{k+1} \bar{A}_k x + \tfrac{1}{2}\tilde{u}_k^T \tilde{R}_k \tilde{u}_k + \tilde{u}_k^T \tilde{B}_k \bar{S}_{k+1} \tilde{A}_k \tilde{x}_k + \tilde{u}_k^T \tilde{P}_k \tilde{x}_k + \tfrac{1}{2}\tilde{u}_k^T \tilde{B}_k^T \bar{S}_{k+1} \tilde{B}_k \tilde{u}_k + \tilde{g}_k^T \bar{S}_{k+1} \tilde{A}_k \tilde{x}_k + \tilde{g}_k^T \bar{S}_{k+1} \tilde{B}_k \tilde{u}_k + \tfrac{1}{2}\tilde{g}_k^T \bar{S}_{k+1} \tilde{g}_k + \tilde{q}_k + \tfrac{1}{2}\tilde{x}_k^T \bar{W}_k^T \bar{V}_k^{-1} \bar{W}_k \tilde{x}_k\Big\}$$

where $\tilde{A}_k$, $\tilde{B}_k$, *etc.*, stand for $A_k(\tilde{x}_k, \tilde{u}_k)$, $B_k(\tilde{x}_k, \tilde{u}_k)$, *etc.*, respectively. It follows that

$$\begin{aligned}
\delta J &= J(\tilde{u}) - J(\bar{u}) = J(\bar{u} + \delta u) - J(\bar{u}) \\
&= e(\tilde{x}_N) - e(\bar{x}_N) + \sum_{k=0}^{N-1} \Big\{ \rho_k(\tilde{x}_k, \tilde{u}_k) - \rho_k(\bar{x}_k, \bar{u}_k) \\
&\qquad - \bar{u}_k^T \bar{V}_k \delta u_k - \bar{x}_k^T \overline{W}_k \delta u_k - \bar{u}_k^T \overline{W}_k \delta x_k \\
&\qquad + \bar{\tau}_k^T \bar{V}_k \left(\delta u_k + \bar{V}_k^{-1} \overline{W}_k \delta x_k \right) + \bar{z}_k^T \begin{bmatrix} Y_k & 0 \\ 0 & 0 \end{bmatrix} \delta z_k \\
&\qquad + \tfrac{1}{2} \delta z_k^T \begin{bmatrix} T_k & 0 \\ 0 & 0 \end{bmatrix} \delta z_k \Big\}
\end{aligned}$$

where

$$\begin{gathered}
\bar{\tau}_k \triangleq \bar{u}_k + \bar{V}_k^{-1} \overline{W}_k \bar{x}_k, \\
\bar{z}_k^T \triangleq \left[\bar{x}_k^T \bar{u}_k^T \right], \qquad \delta z_k^T \triangleq \left[\delta x_k^T \delta u_k^T \right], \\
Y_k \triangleq \left(\tilde{Q}_k + \tilde{A}_k \bar{S}_{k+1} \tilde{A}_k \right) - \left(\bar{Q}_k + \bar{A}_k \bar{S}_{k+1} \bar{A}_k \right), \\
T_k \triangleq Y_k + \overline{W}_k^T \bar{V}_k^{-1} \overline{W}_k
\end{gathered}$$

and ρ_k, $k \in I_{N-1}$ is a scalar function on $R^n \times R^m$ defined by (3.8).

As A_k and Q_k are continuous in both arguments, Y_k converges to 0 as $|\delta z_k| \to 0$, where $|\ |$ is the Euclidean norm in R^{n+m}. Expanding $\rho_k(\tilde{x}_k, \tilde{u}_k) = \rho_k(\bar{x}_k + \delta x_k, \bar{u}_k + \delta u_k)$ about $\bar{x}_k$ and $\bar{u}_k$ in δx_k and δu_k, we have

$$\begin{aligned}
\delta J = e(\tilde{x}_N) - e(\bar{x}_N) &+ \sum_{k=0}^{N-1} \left\{ \left(\frac{\partial \bar{\rho}_k}{\partial u} - \bar{u}_k^T \bar{V}_k - \bar{x}_k^T \overline{W}_k \right) \delta u_k \right. \\
&\qquad \left. + \left(\frac{\partial \bar{\rho}_k}{\partial x} - \bar{u}_k^T \overline{W}_k \right) \delta x_k + \bar{\tau}_k^T \bar{V}_k \delta u_k + \bar{\tau}_k^T \overline{W}_k \delta x_k \right\} \\
&+ \sum_{k=0}^{N-1} \sigma_k(\delta z_k) \qquad (4.2)
\end{aligned}$$

where

$$\sigma_k(\delta z_k) \triangleq \bar{z}_k^T \begin{bmatrix} Y_k & 0 \\ 0 & 0 \end{bmatrix} \delta z_k + \tfrac{1}{2} \delta z_k^T \begin{bmatrix} T_k & 0 \\ 0 & 0 \end{bmatrix} \delta z_k + \theta_k(\delta z_k).$$

and

$$\lim_{|\delta z_k| \to 0} \frac{|\theta_k(\delta z_k)|}{|\delta z_k|} = 0.$$

Clearly,

$$\lim_{|\delta z_k|\to 0} \frac{|\sigma_k(\delta z_k)|}{|\delta z_k|} = 0.$$

Further by expanding $e(\tilde{x}_N)$ about $\bar{x}_N$ in δx_N, (4.2) can be written in the form

$$\delta J = \frac{de(\bar{x}_N)}{dx}\delta x_N + \sigma_N(\delta x_N)$$
$$+ \sum_{k=0}^{N-1} \left\{ \bar{\psi}_k^T \delta u_k + \bar{\phi}_k^T \delta x_k + \bar{\tau}_k^T \bar{V}_k \delta u_k + \bar{\tau}_k^T \bar{W}_k \delta x_k \right\} + \sum_{k=0}^{N-1} \sigma_k(\delta z_k) \qquad (4.3)$$

where $\bar{\phi}_k$ and $\bar{\psi}_k$ are defined by (3.6) and (3.7), respectively and $\lim_{|\delta z_N|\to 0} |\sigma_N(\delta z_N)|/|\delta z_N| = 0$.

On the other hand if $\bar{h}_k \in R^n$ is defined as the solution of (3.2) and δx_k is considered as the solution of

$$\delta x_{k+1} = \frac{\partial \bar{f}_k}{\partial x}\delta x_k + \frac{\partial \bar{f}_k}{\partial u}\delta u_k + \eta_k(\delta z_k), \qquad k \in I_{N-1}, \quad \delta x_0 = 0, \qquad (4.4)$$

where $\lim_{|\delta z_k|\to 0} |\eta_k(\delta z_k)|/|\delta z_k| = 0$, then, from the identity

$$\bar{h}_0^T \delta x_0 - \bar{h}_N^T \delta x_N + \sum_{k=0}^{N-1} \left(\bar{h}_k^T \delta x_k - \bar{h}_{k+1}^T \delta x_{k+1} \right) = 0,$$

it follows that

$$\frac{-de(\bar{x}^N)}{dx}\delta x_N + \sum_{k=0}^{N-1} \left\{ -\bar{\phi}_k^T + \bar{\psi}_k^T \bar{V}_k^{-1} \bar{W}_k + \bar{h}_{k+1}^T \left(-\frac{\partial \bar{f}_k}{\partial x} + \frac{\partial \bar{f}_k}{\partial u} \bar{V}_k^{-1} \bar{W}_k \right) \delta x_k \right.$$
$$\left. + \bar{h}_{k+1}^T \left(\frac{\partial \bar{f}_k}{\partial x}\delta x_k + \frac{\partial \bar{f}_k}{\partial u}\delta u_k + \eta_k(\delta z_k) \right) \right\} = 0. \qquad (4.5)$$

Adding (4.5) and (4.3), the expression for δJ becomes

$$\delta J = \sum_{k=0}^{N-1} \left\{ \bar{\psi}_k^T \delta u_k + \bar{\psi}_k^T \bar{V}_k^{-1} \bar{W}_k \delta x_k + \bar{h}_{k+1}^T \frac{\partial \bar{f}_k}{\partial u} \delta u_k \right.$$
$$\left. + \bar{h}_{k+1}^T \frac{\partial \bar{f}_k}{\partial u} \bar{V}_k^{-1} \bar{W}_k \delta x_k + \bar{\tau}_k^T \bar{V}_k \delta u_k + \bar{\tau}_k^T \bar{W}_k \delta x_k \right\} + \left(\sum_{k=0}^{N-1} r_k \right) + \sigma_N(\delta z_N)$$

where $r_k \triangleq \sigma_k(\delta z_k) + \bar{h}_{k+1}^T \eta_k(\delta z_k)$. Clearly, $\lim_{|\delta z_k|\to 0} |r_k|/|\delta z_k| = 0$. Let

$$\bar{\beta}_k^T \triangleq \bar{\psi}_k^T \bar{V}_k^{-1} + \bar{h}_{k+1}^T \frac{\partial \bar{f}_k}{\partial u} \bar{V}_k^{-1} + \bar{\tau}_k^T$$

and

$$r \triangleq \sum_{k=0}^{N-1} r_k + \sigma_N(\delta z_N).$$

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Then

$$\delta J = \sum_{k=0}^{N-1} \left\{ \bar{\beta}_k^T \bar{V}_k \left(\delta u_k + \bar{V}_k^{-1} \bar{W}_k \delta x_k \right) \right\} + r. \tag{4.6}$$

Finally, by setting

$$\delta u_k = -\varepsilon \bar{\beta}_k - \bar{V}_k^{-1} \bar{W}_k \delta x_k, \qquad \varepsilon \in [0,1], \tag{4.7}$$

we have

$$\delta J = -\varepsilon \sum_{k=0}^{N-1} \left\{ \bar{\beta}_k^T \bar{V}_k \bar{\beta}_k \right\} + r, \tag{4.8}$$

$$\delta J = -\varepsilon \Delta(\bar{u}) + r \tag{4.9}$$

where

$$\Delta(\bar{u}) \triangleq \sum_{k=0}^{N-1} \bar{\beta}_k^T \bar{V}_k \bar{\beta}_k. \tag{4.10}$$

If δu_k is chosen according to (4.7), δx_k, the solution of

$$\bar{x}_{k+1} + \delta x_{k+1} = f_k(\bar{x}_k + \delta x_k, \bar{u}_k + \delta u_k), \qquad k \in I_{N-1}, \delta x_0 = 0,$$

with $\bar{x}_k$, $k \in I_N = \{0, 1, \dots, N\}$ and $\bar{u}_k$, $k \in I_{N-1}$, specified, is a continuous function of ε. Therefore, arguing along the lines similar to those in [6] for the continuous case, it is not difficult to show that $\lim_{\varepsilon \to 0} |r|/\varepsilon = 0$.

Since $\bar{V}_k$ is positive definite it is immediate from (4.8) that for $\bar{\beta}_k \neq 0$ and ε sufficiently small,

$$\delta J = -\varepsilon \sum_{k=0}^{N-1} \bar{\beta}_k^T \bar{V}_k \bar{\beta}_k + r < 0,$$

i.e. under the above assumptions the variation δJ represents a reduction in the value of the cost functional. Clearly,

$$\sum_{k=0}^{N-1} \bar{\beta}_k^T \bar{V}_k \bar{\beta}_k = 0 \tag{4.11}$$

or, equivalently,

$$\bar{\beta}_k = 0, \qquad k \in I_{N-1} \tag{4.12}$$

are necessary conditions for optimality. In the sequel any control which satisfies (4.12) will be referred to as a desirable control or desirable point. (Note that if $\bar{\beta}_k = 0$ for $k \in I_{N-1}$ is not satisfied, $J(u)$ can be further reduced by the LQRE algorithm.) In Section 5, we shall show that the above condition is equivalent to the first order necessary condition $\partial H_k/\partial u = 0$, $k \in I_{N-1}$.

REMARK 4.1. Using similar arguments to those in [6], page 877, it can easily be shown that the discrete-time LQRE algorithm converges in one step on the linear-quadratic problem defined in [1], page 42.

REMARK 4.2. Similarly, it can be shown that the discrete-time LQRE algorithm exhibits one step convergence on the problem with the system dynamics and cost functional given by

$$x_{k+1} = A_k x_k + B_k u_k + a_k, \qquad k \in I_{N-1};\ x_0 = x_0,$$

$$J(u) = \sum_{k=0}^{N-1} \left\{ \tfrac{1}{2} x_k^T Q_k x_k + u_k^T P_k x_k + \tfrac{1}{2} u_k^T R_k u_k + b_k^T x_k + c_k^T u_k + d_k \right\} + \tfrac{1}{2} x_N^T Q_N x_N + p_N^T x_N + s_N,$$

where $A_k, B_k, Q_k \geqslant 0$, $P_k, R_k > 0$, $Q_N \geqslant 0$, a_k, b_k, c_k, d_k, p_N and s_N are constants (independent of x_k and u_k) of appropriate dimensions.

5. Criterion for optimality

For convenience "bars" above symbols will be dropped in this section. The control u,

$$u^T = \left[u_0^T u_1^T \cdots u_{N-1}^T\right] = \left[u_0^1 \cdots u_0^m u_1^1 \cdots u_1^m \cdots u_{N-1}^1 \cdots u_{N-1}^m\right]$$

and the state of the system x,

$$x^T = \left[x_0^T x_1^T \cdots x_N^T\right] = \left[x_0^1 \cdots x_0^n x_1^1 \cdots x_1^n \cdots x_N^1 \cdots x_N^n\right]$$

can be considered as points in $R^{(N-1)m}$ and R^{Nn}. From the equation

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}) = f_{k-1}\big(f_{k-2}(x_{k-2}, u_{k-2}), u_{k-1}\big) \\ &= f_{k-1} f_{k-2}(\ldots, u_{k-2}, u_{k-1}) \end{aligned}$$

and from the assumptions concerning the differentiability of f_k, $k \in I_{N-1}$, it follows then that x_k is differentiable with respect to $u_0, u_1, \ldots, u_{k-1}$ which, in turn, implies that x_k and x are differentiable in u. (Here $\partial x_{k+1}/\partial u_i = 0$, $k < i \leqslant N-1$.)

Thus, J given by (2.2) is differentiable in $u \in R^{Nm}$ and, consequently, in the discrete-time case the Gateaux and Fréchet differentials coincide with ordinary

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differentials. Therefore, the first order necessary condition for optimality of discrete-time systems can be stated as $dJ(u)/du = 0$ or in an equivalent form $(\partial H_k/\partial u)(x_k, u_k, \lambda_{k+1}) = 0$, where $H_k(\chi, \omega, \lambda_{k+1}) \triangleq L_k(\chi, \omega) + \lambda_{k+1}^T f_k(\chi, \omega)$ and λ_k is the solution of the difference equation $\lambda_k^T = (\partial H_k/\partial x)(x_k, u_k, \lambda_{k+1})$, $k \in I_{N-1}$, $\lambda_N^T = \partial F(x_N)/\partial x$ (*cf.* [1], page 67).

Let ν_k be the solution of the difference equation

$$\nu_k^T = \phi_k^T + \tau_k^T W_k + \nu_{k+1}^T \frac{\partial f_k}{\partial x}, \qquad k \in I_{N-1}, \nu_N^T = \frac{de(X_N)}{dx}. \tag{5.1}$$

As δx_k is the solution of (4.4), from the identity

$$\nu_0^T \delta x_0 - \nu_N^T \delta x_N + \sum_{k=0}^{N-1} \left(\nu_{k+1}^T \delta x_{k+1} - \nu_k^T \delta x_k \right) = 0,$$

it follows that

$$-\nu_N^T \delta x_N + \sum_{k=0}^{N-1} \left[\nu_{k+1}^T \left(\frac{\partial f_k}{\partial x} \delta x_k + \frac{\partial f_k}{\partial u} \delta u_k + \eta_k \right) - \left(\phi_k^T + \tau_k^T W_k + \nu_{k+1}^T \frac{\partial f_k}{\partial x} \right) \delta x_k \right] = 0.$$

Adding this to the expression (4.3) for δJ, gives

$$\delta J = \sigma_N(\delta x_N) + \sum_{k=0}^{N-1} \left[\nu_{k+1}^T \frac{\partial f_k}{\partial u} + \psi_k^T + \tau_k^T V_k \right] \delta u_k + \sum_{k=0}^{N-1} \left[\nu_{k+1}^T \eta_k + \sigma_k(\delta z_k) \right]$$

or

$$\delta J = \sum_{k=0}^{N-1} \left[\xi_k^T \delta u_k + \alpha_k(\delta z_k) \right] + \sigma_N(\delta x_N)$$

where

$$\xi_k^T \triangleq \nu_{k+1}^T \frac{\partial f_k}{\partial u} + \psi_k^T + \tau_k^T V_k$$

and

$$\alpha_k(\delta z_k) \triangleq \nu_{k+1}^T \eta_k(\delta z_k) + \sigma_k(\delta z_k).$$

Since $\lim_{|\delta u| \to 0} |\alpha_k(z_k)|/|\delta u| = 0$ (*cf.* (4.2), (4.3) and (4.4)) and the expansion

$$\delta J = \sum_{k=0}^{N-1} \left(\frac{\partial H_k(x_k, u_k, \lambda_{(k+1)})}{\partial u} \right) \delta u_k$$

(correct to the first order of δu) is unique. It follows that

$$\xi_k^T = \frac{\partial H_k(x_k, u_k, \lambda_{k+1})}{\partial u}.$$

THEOREM 1. *For the discrete-time control problem defined in Section 2, the optimality criterion* $\beta_k = 0$ *is equivalent to the criterion*

$$\frac{\partial H_k}{\partial u} \triangleq \nu_{k+1}^T \frac{\partial f_k}{\partial u} + \psi_k^T + \tau_k^T V_k = 0, \qquad k \in I_{N-1}.$$

PROOF. As $V_k > 0$,

$$\beta_k^T = \psi_k^T V_k^{-1} + h_{k+1}^T \frac{\partial f_k}{\partial u} V_k^{-1} + \tau_k^T = 0$$

yields

$$\psi_k^T + h_{k+1}^T \frac{\partial f_k}{\partial u} + \tau_k^T V_k = 0 \tag{5.2}$$

with h_k defined by (3.2). From (5.2),

$$-\tau_k^T W_k = \psi_k^T V_k^{-1} W_k + h_{k+1}^T \frac{\partial f_k}{\partial u} V_k^{-1} W_k$$

which implies that (3.2) can be written in the form

$$h_k^T = \phi_k^T + \tau_k^T W_k + h_{k+1}^T \frac{\partial f_k}{\partial x}, \qquad k \in I_{N-1}, h_N^T = \frac{de(x_N)}{dx}.$$

Comparing this with (5.1), we see that in this case $h_k = \nu_k$ and, consequently, $\partial H_k/\partial u = 0$ follows from (5.2). On the other hand if $\partial H_k/\partial u = \nu_{k+1}^T \partial f_k/\partial u + \psi_k^T + \tau_k^T V_k = 0$ is satisfied, $\tau_k W_k = -\nu_{k+1}^T(\partial f_k/\partial u)V_k^{-1}W_k - \psi_k^T V_k^{-1} W_k$ and (5.1) can be written in the form

$$\nu_k^T = \phi_k^T - \psi_k^T V_k^{-1} W_k + \nu_{k+1}^T \left(\frac{\partial f_k}{\partial x} - \frac{\partial f_k}{\partial u} V_k^{-1} W_k \right).$$

Comparing this with (3.2), it follows (by inspection) that in this case $\nu_k = h_k$. Thus (5.2) follows from $\partial H_k/\partial u = 0$. Since we have proved that both $\beta_k = 0$, $k \in I_{N-1}$ implies $\partial H_k/\partial u = 0$, $k \in I_{N-1}$ and $\partial H_k/\partial u = 0$, $k \in I_{N-1}$ implies $\beta_k = 0$, $k \in I_{N-1}$, the two criteria are equivalent.

6. Proof of convergence

The LQRE algorithm for discrete-time systems utilizes the search function α: $U \to U$ defined by

$$u^T(i+1) = \alpha^T(u(i)) = \left[\alpha_0^T(u_0(i)) \cdots \alpha_{N-1}^T(u_{N-1}(i)) \right]$$

where i denotes the iteration number and

$$\alpha_k(u_k(i)) = u_k(i) - \varepsilon(i)\beta_k(i) - V_k^{-1}(i) W_k(i) \delta x_k(i), \qquad k \in I_{N-1}.$$

Define x_k^0 as the solution of

$$x_{k+1}^0 = x_k^0 + L_k(x_k, u_k) + F(f_k(x_k, u_k)) - F(x_k), \qquad k \in I_{N-1}, x_0^0 = F(\chi_0),$$

and let

$$y^T \triangleq [y_0 \cdots y_N], \quad y_k \triangleq \beta_k^T V_k \beta_k, \qquad k \in I_{N-1},$$
$$\check{x}_k^T = [x_k^0 x_k^T], \qquad k \in I_N,$$
$$\check{x}^T = [\check{x}_0^T \check{x}_1^T \cdots \check{x}_N^T], \qquad \check{\chi}_0 = [F(\chi_0)\,\chi_0]$$
$$f_k^0(\check{x}_k, u_k) \triangleq x_k^0 + L_k(x_k, u_k) + F(f_k(x_k, u_k)) - F(x_k), \qquad k \in I_N,$$
$$\check{f}_k^T(\check{x}_k, u_k) = [f_k^0(\check{x}_k, u_k) f_k^T(x_k, u_k)], \qquad k \in I_N.$$

Then on the assumption that the increment of the control is in the form (4.7), it follows that

$$\check{x}_{k+1} = \check{f}_k(\check{x}_k, \bar{u}_k - \varepsilon\bar{\beta}_k - \bar{V}_k^{-1}\bar{W}_k \delta x_k), \qquad k \in I_{N-1}; \check{x}_0 = \check{\chi}_0.$$

With the above definitions, the convergence proof for the discrete case is a verbatim repetition of the convergence analysis for continuous time systems in Appendix III of [6], provided that

(i) $U = \mathscr{L}_\infty[0,1]$ and W in [6] are formally replaced with R^{Nm},
(ii) $\mathscr{L}_\infty^n[0,1]$ is replaced with $R^{(N+1)n}$,
(iii) $\mathscr{L}_\infty^{n\times n}[0,1]$ is replaced with $R^{(N+1)(n\times n)}$,
(iv) $C[0,1]$ with R^{N+1},
(v) $\|\ \|_\infty$ with $|\ |$,
(vii) $\bar{R}$, $\bar{B}^T\bar{S}$, $\bar{\beta}_1$ are replaced with $\bar{V}_k, \bar{W}_k, \bar{\beta}_k$,

and, in general, the value $v(t)$ of any function v is replaced with its discrete-time counterpart v_k.

Note also that all the Fréchet derivatives referred to in Appendix III of [6], reduce to ordinary derivatives in the discrete case.

Thus, Theorem 1 in [6] holds for the discrete-time problem and the sequence $\{u(i)\}$ constructed by the discrete-time LQRE algorithm is either finite and its last element is desirable or else it is infinite and any accumulation pont of $\{u(i)\}$ is desirable. Under the assumption

(H1) If $\{u(i)\}$ is any sequence of admissible controls and $|u(i)| \to \infty$, then $J(u(i)) \to \infty$

the sequence $\{u(i)\}$ generated by the algorithm is bounded. Suppose that the opposite is true, *i.e.* that the generated sequence $\{u(i)\}$ is not bounded. Then there exists a subsequence $\{u(i_k)\}$ of $\{u(i)\}$ such that $|u(i_k)| \to \infty$ and, by virtue of (H1), $J(u(i_k)) \to \infty$ which contradicts the fact that $\{J(u(i))\}$ is a monotone decreasing sequence as shown in Section 4. Thus, the generated sequence $\{u(i)\}$ must be bounded and there exists a closed bounded subset Ω of $U = R^{Nm}$ such that $u(i) \in \Omega$ for all i. Since the set Ω is sequentially compact, the existence of accumulation points of the infinite sequence $\{u(i)\}$ is guaranteed and, consequently, the convergence of the algorithm is proved.

7. Problems with constraints

Consider the problem defined in Section 2, *i.e.*

$$\begin{aligned} &\textit{minimise}\, J(u) = \sum_{k=0}^{N-1} \{L_k(x_k, u_k) + F(x_N)\}, \\ &x_{k+1} = f_k(x_k, u_k, u_k), \qquad k \in I_{N-1}, x_0 = \chi_0, \end{aligned} \tag{7.1}$$

subject to the constraint

$$G_k(x_k, u_k) = 0, \qquad k \in I_{N-1}, \tag{7.2}$$

where G_k: $R^n \times R^m \to R^p$ is continuously differentiable in both arguments. It is assumed that an optimal solution exists. The problem can be solved by adding the penalty function $\frac{1}{2}\mathscr{K}_i\Sigma_{k=0}^{N-1} G_k(x_k, u_k)G_k(x_k, u_k)$ to the cost functional and solving progressively the unconstrained approximating problem

$$\textit{minimise}\, J_a(u, K_i) = \sum_{k=0}^{N-1} \{L_k(x_k, u_k) + \tfrac{1}{2}\mathscr{K}_i G_k^T(x_k, u_k)G_k(x_k, u_k)\} + F(x_N),$$

$$x_{k+1} = f_k(x_k, u_k), \qquad k \in I_{N-1}, \qquad x_0 = \chi_0, \tag{7.3}$$

for a sequence of positive scalars $\{\mathscr{K}_i\}$ which tends to infinity. By (5.1) $J_a(u, \mathscr{K}_i)$ can be considered as a function of $u \in R^{mN}$ and $\mathscr{K}_i$ only and, consequently, the results on convergence of penality function methods in [5] are directly applicable in this case.

In theory at least, then, any globally convergent method for unconstrained optimization of discrete-time systems could be used for solving the approximating problem (7.3). However, for many algorithms that might be applied, the structure of (7.3) becomes increasingly unfavourable as $\mathscr{K}_i$ is increased which is reflected in a poor convergence rate. In implementation of the penality function method it is, therefore, very important to select an efficient algorithm for unconstrained problems when the cost functional contains a penality term.

We shall now examine how the LQRE algorithm is affected when constraint (7.2) is linear by the addition of the penalty term to the cost functional. Linear constraints are extremely important from a practical viewpoint and they are also simplest to analyse. Assume first that the unconstrained problem (7.1) has been solved by means of the LQRE algorithm. Then, in order to solve the constrained problem by the LQRE and penalty function method, we write the cost functional in (7.3) in the form

$$J_a(u, \mathscr{K}_i) = \sum_{k=0}^{N-1} \left\{ \tfrac{1}{2} x_k^T Q_k x_k + u_k^T P_k x_k + \tfrac{1}{2} u_k^T R_k u_k + q_k(x_k, u_k) \right\}$$
$$+ \tfrac{1}{2}\mathscr{K}_i \left\{ G_k^T(x_k, u_k) G_k(x_k, u_k) \right\} + F(x_N) \tag{7.4}$$

(*cf.* 2.4) where $Q_k \geqslant 0$, $R_k > 0$ denote the matrices used for solving the unconstrained problem. (It is assumed that $D_k = R_k$; *cf.* (3.4) and Remark 3.1.) As the constraints are linear, we have

$$G_k(x_k, u_k) \triangleq [\Gamma_k \Pi_k] \begin{bmatrix} x_k \\ u_k \end{bmatrix} - \gamma_k = 0,$$

thus

$$J_a(u, \mathscr{K}_i) = \sum_{k=0}^{N-1} \left\{ \tfrac{1}{2} x_k^T (Q_k + \mathscr{K}_i \Gamma_k^T \Gamma_k) x_k + u_k^T (P_k + \mathscr{K}_i \Pi_k^T \Gamma_k) x_k \right.$$
$$\left. + \tfrac{1}{2} u_k^T (R_k + \mathscr{K}_i \Pi_k^T \Pi_i) u_k + q_k'(x_k, u_k) \right\} + F(x_N) \tag{7.5}$$

where $\Gamma_k \in R^{n \times p}$, $\Pi_k \in R^{m \times p}$ and $\gamma_k \in R^p$ are constant matrices and vectors, respectively and

$$q'(x_k, u_k) \triangleq q_k(x_k, u_k) + \gamma_k^T \gamma_k - \tfrac{1}{2}\mathscr{K}_i \gamma_k^T \Gamma_k x_k - \tfrac{1}{2}\mathscr{K}_i \gamma_k^T \Pi_k u_k.$$

Since $\Gamma_k^T \Gamma_k \geqslant 0$ and $\Pi_k^T \Pi_k \geqslant 0$ for arbitrary Γ_k and Π_k, it follows that $Q_k + \mathscr{K}_i \Gamma_k^T \Gamma_k \geqslant 0$ and $R_k + \mathscr{K}_i \Pi_k^T \Pi_k > 0$. Thus instead of Q_k and R_k which were used for solving the unconstrained problem (7.1), to solve the approximating problem (7.3) one may use $Q_k' \triangleq Q_k + \mathscr{K}_i \Gamma_k^T \Gamma_k$ and $R_k' \triangleq R_k + \mathscr{K}_i \Pi_k^T \Pi_k$. Consequently, since the difference between $q_k'(x_k, u_k)$ and $q_k(x_k, u_k)$ is only in the linear terms (with regard to the LQRE algorithm), the structure of the cost functional in (7.3) is as suitable as that of the cost functional in (7.1). Furthermore, if the problem (7.1) is linear-quadratic and constraint (7.2) linear, LQRE solves the approximating problem in one iteration (*cf.* Remark 4.2).

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