COMPLETENESS IN SEMI-LATTICES

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1. Introduction. Let $(X, \leqslant)$ be a partially ordered set, that is, X is a set and $\leqslant$ is a reflexive, anti-symmetric, transitive, binary relation on X. We write

$$M(x) = \{a{:}x \leqslant a\}, \quad L(x) = \{a{:}a \leqslant x\},$$

for each $x \in X$. If, moreover,

$$x \wedge y = \sup L(x) \cap L(y)$$

exists for each x and y in X, then $(X, \leqslant)$ is said to be a *semi-lattice*. If $(X, \leqslant)$ and $(X, \geqslant)$ are semi-lattices, then $(X, \leqslant)$ is a *lattice*.

The lattice $(X, \leqslant)$ is *complete* if, for each non-empty subset A of X, elements

$$\wedge A = \sup \bigcap \{L(a){:}a \in A\}, \tag{1}$$

$$\vee A = \inf \bigcap \{M(a){:}a \in A\} \tag{2}$$

exist. Lattice-completeness has been characterized in various ways; in particular Frink **(4)** showed it equivalent to compactness relative to a natural sort of topology, and Anne C. Davis **(3)** proved it equivalent to an agreeable fixed point condition.

Let us say that a semi-lattice $(X, \leqslant)$ is *complete* provided (1) exists for each non-empty subset A of X. To avoid ambiguity, we shall refer to a structure $(X, \leqslant)$ as being *lattice-complete* or *semi-lattice-complete* whenever it is not clear from context whether $(X, \leqslant)$ is to be regarded as a lattice or a semi-lattice. In what follows, semi-lattice analogues of theorems on lattices due to Frink **(4)**, Tarski **(5)**, and Davis **(3)**, are proved.

2. Topology in partially ordered sets; Frink's theorem. Let $(X, \leqslant)$ be a partially ordered set. The *interval topology* **(2**, p. 60**)** is that topology generated by taking all of the sets $L(x)$ and $M(x)$, $x \in X$, as a subbasis for the closed sets. An element of X is *maximal* (*minimal*) if it has no proper successor (predecessor). A *zero* (unit) of X is an element which precedes (succeeds) all other elements of X.

LEMMA 1. *Let A be a non-empty subset of X, where $(X, \leqslant)$ is a semi-lattice. If $L(a)$ is compact in the interval topology, for some $a \in A$, then the set*

$$L = \bigcap \{L(a){:}a \in A\}$$

has a unit.

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Proof. From **(6**, Theorem 1**)** and the semi-lattice ordering of X, it follows that X has a zero and hence that L is not empty. Again by **(6**, Theorem 1**)** L has a maximal element, x_1. If there exists $x \in L - L(x_1)$ then it may be shown that $M(x) \cap M(x_1)$ is a semi-lattice containing A, and consequently that $M(x) \cap M(x_1)$ has a zero, x_0. It follows that $x_1 < x_0 \leqslant a$ for all $a \in A$, contradicting the maximality of x_1 in L. Therefore, $L \subset L(x_1)$, which is to say that x_1 is a unit for L.

THEOREM 1. *For the semi-lattice* $(X, \leqslant)$ *to be complete it is necessary and sufficient that, for each* $x \in X$, $L(x)$ *be compact in the interval topology.*

Proof. Suppose $(X, \leqslant)$ is complete. In view of Alexander's lemma **(1)** it suffices, in order to show $L(x)$ compact, to prove that if $\{x_\alpha : \alpha \in A\}$ and $\{x_\beta : \beta \in B\}$ are subsets of $L(x)$ such that

$$\mathfrak{F} = \{M(x_\alpha):\alpha \in A\} \cup \{L(x_\beta):\beta \in B\}$$

is a non-empty collection with finite intersection property, then $\mathfrak{F}$ has a non-empty intersection. We consider two alternatives: either B is empty or it is not. If B is empty, then $x \in \bigcap \mathfrak{F}$; if B is not empty, then by the finite intersection property, $x_\alpha \leqslant x_\beta$ for each $\alpha \in A$ and $\beta \in B$. Therefore, since X is complete,

$$x_\alpha \leqslant \wedge \{x_\beta:\beta \in B\} = x_0$$

for each $\alpha \in A$. Clearly, $x_0 \in \bigcap \mathfrak{F}$.

Conversely, suppose that $L(x)$ is compact for each $x \in X$ and that A is a non-empty subset of X. By Lemma 1,

$$L = \bigcap \{L(a):a \in A\}$$

has a unit, x_1, and it is clear that $x_1 = \wedge A$.

COROLLARY 1.1 (Frink). *For the lattice* $(X, \leqslant)$ *to be complete it is necessary and sufficient that* X *be compact in the interval topology.*

Proof. If $(X, \leqslant)$ is complete as a lattice, then both $(X, \leqslant)$ and $(X, \geqslant)$ are complete as semi-lattices. Therefore, X has a unit, x_1, and $L(x_1) = X$ is compact, by Theorem 1. Conversely, the compactness of X implies the completeness of $(X, \leqslant)$ and $(X, \geqslant)$ as semi-lattices, which is equivalent to the lattice-completeness of $(X, \leqslant)$.

COROLLARY 1.2. *For the semi-lattice* $(X, \leqslant)$ *to be complete it is necessary and sufficient that* $(L(x), \leqslant)$ *be a complete lattice, for each* $x \in X$.

Proof. The sufficiency is immediate from Corollary 1.1 and Theorem 1. To prove the necessity, let $x \in X$ where $(X, \leqslant)$ is a complete semi-lattice. By Theorem 1, $L(x)$ is compact. If a and b are elements of $L(x)$, then (see the argument of Lemma 1) $M(a) \cap M(b)$ has a zero, and that zero is $a \vee b$. Thus, $(L(x), \leqslant)$ is a compact (and hence complete) lattice.

3. A theorem of Tarski. If A and B are partially ordered sets, a function $f:A \to B$ is *isotone* if $a_1 \leqslant a_2$ implies $f(a_1) \leqslant f(a_2)$. A *chain* of a partially ordered set is a simply ordered subset. A chain is *maximal* if it is properly contained in no other chain.

The following theorem is due to Tarski **(5)**.

THEOREM T. *Let $(X, \leqslant)$ be a complete lattice. If $f:X \to X$ is an isotone function then the set P of fixed points of f is non-empty; further, $(P, \leqslant)$ is a complete lattice.*

Theorem T fails if the word "lattice" is everywhere replaced by "semi-lattice" (see §4). However, we have

THEOREM 2. *Let $(X, \leqslant)$ be a semi-lattice and let $f:X \to X$ be an isotone function. If X is compact in the interval topology, then the set P of fixed points of f is non-empty. If X is a complete semi-lattice and P is non-empty, then $(P, \leqslant)$ is a complete semi-lattice.*

Proof. If X is compact, it has a zero which precedes its f-image; thus, the set

$$U = \{x : x \leqslant f(x)\}$$

is not empty and contains a maximal chain, C. By the compactness of X, C has a least upper bound u. Since f is isotone, we have $x \leqslant f(x) \leqslant f(u)$ for all $x \in C$, and therefore

$$u \leqslant f(u) \leqslant f(f(u)) \leqslant \ldots .$$

If $u \neq f(u)$ then the maximality of C is contradicted, so that P is non-empty. Now if X is complete as a semi-lattice (and not necessarily compact) and P is non-empty, then by Corollary 1.2, $(L(p), \leqslant)$ is a complete lattice for each $p \in P$. Readily $f(L(p)) \subset L(p)$, so that Theorem T implies that $(P \cap L(p), \leqslant)$ is a complete lattice. By Corollary 1.2 the theorem follows at once.

4. A theorem of Davis. Recently **(3)** Anne C. Davis proved

THEOREM D. *For a lattice $(X, \leqslant)$ to be complete it is necessary and sufficient that every isotone function $f:X \to X$ have a fixed point.*

There exist complete semi-lattices which do not have the fixed point property for isotone functions. The interval $0 \leqslant t < 1$ of real numbers is a simple example. The natural semi-lattice analogue to Theorem D is

THEOREM 3. *For a semi-lattice $(X, \leqslant)$ to be compact in its interval topology it is necessary and sufficient that every isotone function $f:X \to X$ have a fixed point.*

LEMMA 2. *If $(X, \leqslant)$ is a semi-lattice and if every isotone function $f:X \to X$ has a fixed point, then, for each $x \in X$, $(L(x), \leqslant)$ is a lattice.*

Proof. If not, there are elements a, b, and x of X such that a and b precede x and $M(a) \cap M(b)$ has no zero. Let C be a maximal chain in the non-empty set

$$(M(a) \cap M(b)) \cup \bigcap \{L(z):z \in M(a) \cap M(b)\}$$

and let

$$C^+ = C \cap M(a) \cap M(b),$$
$$C^- = C - C^+.$$

Now C^+ and C^- are non-empty chains, C^+ has no g.l.b., and C^- has no l.u.b. One can show that there exist (generalized) sequences x_α in C^+ and x_β in C^- such that (a) x_α is monotone decreasing and, for each $t \in C^+$, there exists $\alpha(t)$ such that $\alpha > \alpha(t)$ implies $x_\alpha \leqslant t$, and (b) x_β is monotone increasing and, for each $t \in C^-$, there exists $\beta(t)$ such that $\beta > \beta(t)$ implies $x_\beta \geqslant t$. Define $f:X \to C$ as follows: if $x \in \bigcap \{L(x_\alpha)\}$ then

$$f(x) = \min \{x_\beta : x_\beta \leqslant\!|\!\lessgtr x\},$$

and if $x \in X - \bigcap \{L(x_\alpha)\}$ then

$$f(x) = \min \{x_\alpha : x \leqslant\!|\!\lessgtr x_\alpha\}.$$

It is easy to verify that f is well defined and isotone. Further, $f(x_\alpha) < x_\alpha$ and $f(x_\beta) > x_\beta$, so that f is without fixed points. This is a contradiction, whence we infer that $(L(x), \leqslant)$ is a lattice.

LEMMA 3. *Under the hypotheses of Lemma* 2, *if* $x \in X$, *then* $(L(x), \leqslant)$ *is a complete lattice.*

Proof. Let $f:L(x) \to L(x)$ be isotone. Then f can be extended in an isotone manner to $\bar{f}:X \to L(x)$ where

$$\bar{f}(a) = f(a \wedge x).$$

By hypothesis the function $\bar{f}$ has a fixed point which must also be a fixed point of f. By Lemma 2 and Theorem D, $(L(x), \leqslant)$ is a complete lattice.

LEMMA 4. *Under the hypotheses of Lemma* 2, *every maximal chain of* X *is a complete lattice.*

Proof. Let C be a maximal chain of X, and define $f:X \to C$ by

$$f(x) = \sup L(x) \cap C.$$

By Lemma 3, $L(x)$ is a complete lattice for each $x \in X$, and since C meets each $L(x)$, this mapping is well defined, isotone, and $f(x) = x$ if, and only if, $x \in C$. Now if C is incomplete as a lattice then by Theorem D there is an isotone function $g:C \to C$ without fixed points. The composition $gf:X \to C$ is therefore without fixed points, which is a contradiction. Hence $(C, \leqslant)$ is complete as a lattice.

Proof of Theorem 3. The necessity was established in Theorem 2. For the sufficiency, let $(X, \leqslant)$ be a semi-lattice in which every isotone $f:X \to X$ has a fixed point. To prove that X is compact in the interval topology it is sufficient (see the argument of Theorem 1) to prove that if $\mathfrak{F}$ is any non-empty collection of subbasic closed sets with finite intersection property, then $\bigcap \mathfrak{F}$ is non-empty. Now $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2$ where

$$\mathfrak{F}_1 = \{M(x_\alpha):\alpha \in A\},$$
$$\mathfrak{F}_2 = \{L(x_\beta):\beta \in B\}.$$

If $\mathfrak{F}_2$ is non-empty then from Lemma 3 and Corollary 1.1 each $L(x_\beta)$ is compact and hence $\bigcap \mathfrak{F}$ is non-empty. If $\mathfrak{F}_2$ is empty, then $\mathfrak{F}_1$ is not and we may assume that $A = \{\alpha_1, \alpha_2, \ldots\}$ is well ordered. Let

$$y_{\alpha_1} = x_{\alpha_1}$$

and, for $\gamma > \alpha_1$,

$$y_\gamma = \inf \bigcap \{M(y_\alpha) : \alpha < \gamma\} \cap M(x_\gamma).$$

To see that y_γ exists, suppose y_α is defined for all $\alpha < \gamma$. Now $\{y_\alpha : \alpha < \gamma\}$ is a chain and hence the set

$$\{z_\alpha : z_\alpha = \inf M(y_\alpha) \cap M(x_\gamma)\}$$

is a chain. By Lemma 4, $z_\gamma = \sup \{z_\alpha : \alpha < \gamma\}$ exists and by Lemma 3, $(L(z_\gamma), \leqslant)$ is a complete lattice so that y_γ exists. Applying Lemma 4 again, $y_0 = \sup \{y_\alpha : \alpha \in A\}$ exists and, clearly, $y_0 \in \cap \mathfrak{F}$.

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