Proceedings of the Edinburgh Mathematical Society (1993) 37, 13–23 ©

A UNIQUENESS PROBLEM IN SIMPLE TRANSCENDENTAL EXTENSIONS OF VALUED FIELDS

by SUDESH K. KHANDUJA*

(Received 26th February 1992)

Let v_0 be a valuation of a field K_0 with value group G_0 and v be an extension of v_0 to a simple transcendental extension $K_0(x)$ having value group G such that G/G_0 is not a torsion group. In this paper we investigate whether there exists $t \in K_0(x) \backslash K_0$ with $v(t)$ non-torsion mod G_0 such that v is the unique extension to $K_0(x)$ of its restriction to the subfield $K_0(t)$. It is proved that the answer to this question is "yes" if v_0 is henselian or if v_0 is of rank 1 with G_0 a cofinal subset of the value group of v in the latter case, and that it is "no" in general. It is also shown that the affirmative answer to this problem is equivalent to a fundamental equality which relates some important numerical invariants of the extension $(K, v)/(K_0, v_0)$.

1991 *Mathematics subject classification*: 12F20, 13A18.

1. Introduction

Throughout $K_0(x)$ is a simple transcendental extension of a field K_0 and v_0 is a Krull valuation of K_0 with value group G_0 and residue field k_0. In this paper we investigate the following **uniqueness problem**.

Suppose that v is a valuation of $K_0(x)$ which extends v_0 and has value group G such that G/G_0 is not a torsion group. Does there exist $t \in K_0(x)$ with $v(t)$ non-torsion mod G_0 such that v is the unique extension to $K_0(x)$ of the valuation obtained by restricting v to $K_0(t)$?

It is proved in this paper that the answer to this question is "yes" if v_0 is henselian or if v_0 is of rank one with G_0 a cofinal subset of the value group of v in the latter case. It is also shown that the affirmative answer to the uniqueness problem is equivalent to a fundamental equality which relates some important numerical invariants of the extension $(K_0(x), v)/(K_0, v_0)$. Using this equality, an example has been given to show that the answer to the problem is "no" in general.

It may be remarked that the corresponding problem for an extension $(K_0(x), v)/(K_0, v_0)$, where the residue field of v is a transcendental extension of the residue field of v_0, has already been dealt with by Matignon and Ohm in [7] and [8]. Polzin has also considered the analogous problem for a residually transcendental

*The research was partially supported by CSIR, New Delhi, vide grant No. 25(53)/90:EMR II.

extension $(K, v)/(K_0, v_0)$ where K is a function field of transcendence degree one over K_0 and v is of rank one in [9].

2. Additional notation and statements of results

For any $F = F(x) \in K_0(x) \backslash K_0$ and any element δ in a totally ordered abelian group containing G_0 as an ordered subgroup, we shall denote by $v_0^{(F,\delta)}$ the valuation of the field $K_0(F) \subseteq K_0(x)$ defined on $K_0[F]$ by

$$v_0^{(F,\delta)}\left(\sum_{i=0}^{m} a_i F^i\right) = \min_i (v_0(a_i) + i\delta).$$

In what follows v is an extension of v_0 to $K_0(x)$ whose value group will be denoted by G and residue field by k. We shall assume throughout that G/G_0 is not a torsion group. If $F \in K_0(x)$ is such that $v(F) = \delta$ is not torsion mod G_0, then in view of the strong triangle law, the restriction of v to the field $K_0(F)$ is $v_0^{(F,\delta)}$. Clearly the value group of $v_0^{(F,\delta)}$ is $G_0 + Z\delta$; its residue field is k_0 by [2, § 10.1, Prop. 1]. Since $[K_0(x): K_0(F)] < \infty$, k is a finite extension of k_0 and the group G/G_0 is finitely generated. Let G_1 denote the subgroup of G defined by

$$G_1 = \{g \in G \mid g \text{ is torsion mod } G_0\}.$$

Then G_1/G_0 being a finitely generated abelian torsion group is finite. We shall denote by N, S and I (to be more precise by $N(v/v_0)$ etc.) the natural numbers defined by

$$N = \min\{\deg f \mid f \in K_0[x], v(f) \text{ is not torsion mod } G_0\},$$

$$S = [k : k_0],$$

$$I = [G_1 : G_0].$$

In some cases, an affirmative answer to the "uniqueness problem" is given by:

Theorem 2.1. *Let v_0 be a valuation of K_0 with value group G_0 and v be an extension of v_0 to $K_0(x)$ with value group G such that G/G_0 is not a torsion group. Suppose that*

(i) *either (K_0, v_0) is henselian,*
(ii) *or (K_0, v_0) has henselian completion (in fact any rank-1 valued field satisfies this property) and G_0 is a confined subset of G.*

Then for any $P \in K_0[x]$ of minimum degree N such that $v(P)$ is not torsion mod G_0, v is the unique extension (upto equivalence) of its restriction $v_0^{(P,\gamma)}$ to the subfield $K_0(P)$, where $\gamma = v(P)$.

For a prolongation v of v_0 to $K_0(x)$ having value group G such that G/G_0 is not a torsion group, we shall denote by Δ (more precisely by $\Delta(v/v_0)$) the non-empty subset of $K_0(x)\backslash K_0$ defined by

$$\Delta = \{F \in K_0(x) \mid v(F) \text{ is not torsion mod } G_0\}.$$

In view of the Lüroth Lemma [**10**, p. 197], it is clear that

$$N = \min \{[K_0(x): K_0(F)] \mid F \in \Delta\}.$$

Corresponding to an element F of Δ, we define a natural number $I(F)$ and a rational number $D^h(F)$ by

$$I(F) = \text{ramification index of } v/v_0^{(F,\delta)} = [G: G_0 + Z\delta],$$

$$D^h(F) = \text{henselian defect of } v/v_0^{(F,\delta)} = [K_0(x)^h: K_0(F)^h]/SI(F)$$

where L^h denotes henselisation of a valued field L with respect to the underlying valuation.

We assume the following Theorem A which has been proved by Kuhlmann and also jointly by Khanduja and Garg (cf. [**6**, Thm. 5.4] or [**5**, Thm. 0.2]); Kuhlmann's proof is to appear in the series "Algebra, Logic and Applications" edited by MacIntyre and Göbel.

Theorem A. *$D^h(F)$ is independent of the choice of F in $\Delta(v/v_0)$.*

For F in $\Delta(v/v_0)$, $D^h(F)$ will be denoted by D^h or sometimes by $D^h(v/v_0)$.

We say that an element F of $K_0(x)$ satisfies the uniqueness property for v/v_0 if

(i) $v(F)$ is not torsion mod G_0;

(ii) v is the unique extension to $K_0(x)$ of the valuation obtained by restricting v to $K_0(F)$.

The relation between the "uniqueness problem" and a fundamental equality involving the constants N, I, S and D^h is established by:

Theorem 2.2. *Let v_0 be a valuation of a field K_0, v be a prolongation of v_0 to $K_0(x)$ and let $k_0 \subseteq k$, $G_0 \subseteq G$ be their respective residue fields and value groups. Assume that G/G_0 is not a torsion group. There exists an element of $K_0(x)$ which satisfies the uniqueness property, if and only if, $N = ISD^h$ holds for v/v_0.*

In the last section, we construct an example of an extension $(K_0(x), v)/(K_0, v_0)$ for which $N > ISD^h$ holds.

3. Proof of Theorem 2.2

The theorem will be deduced from a couple of lemmas.

Lemma 3.1. *Let v_0, v and $G_0 \subseteq G$ be as in Theorem 2.2. If $P = P(x) \in K_0[x]$ is a polynomial of degree N with $v(P)$ non-torsion* mod G_0, *then $G = G_1 + Zv(P)$ and $I = I(P)$.*

Proof. Recall that

$$I = [G_1 : G_0], \text{ where } G_1 = \{g \in G \mid g \text{ is torsion mod } G_0\},$$

and

$$I(P) = [G : G_0 + Z\gamma], \text{ where } \gamma = v(P).$$

Clearly the lemma is proved as soon as it is shown that $G = G_1 + Z\gamma$.

After successive division by powers of $P(x)$ any non-zero polynomial $f(x) \in K_0[x]$ can be uniquely written in the form

$$f(x) = \sum_{i=0}^{m} f_i(x) P(x)^i$$

where $f_i(x) \in K_0[x]$ is either zero or has degree less than N. Since $v(P)$ is non-torsion $\operatorname{mod} G_0$ and since $v(f_i)$ is torsion $\operatorname{mod} G_0$ for $f_i \neq 0$, no two non-zero terms in the sum for $f(x)$ have the same v-valuation, and hence by the strong triangle law

$$v(f) = \min_i (v(f_i) + i\gamma).$$

This proves that $G = G_1 + Z\gamma$.

Recall that for a finite extension $(L, w)/(L_0, w_0)$ of valued fields, w is the only extension (up to equivalence) of w_0 to L, if and only if $[L : L_0] = [L^h : L_0^h]$, where L^h, L_0^h denote the henselisations of L, L_0 with respect to w, w_0 (cf. [3, p. 125 (17.3)] or [8, 1.1]). The following lemma is an immediate consequence of Lemma 3.1 and the result quoted above. We omit its proof.

Lemma 3.2. *Let v_0, v, $G_0 \subseteq G$ and $P(x)$ be as in the above lemma. Suppose that $N = ISD^h$ holds for v/v_0. Then $P(x)$ satisfies the uniqueness property for v/v_0.*

Next we prove a lemma which together with Lemma 3.2 immediately yields Theorem 2.2.

Lemma 3.3. *Let v_0, v and $G_0 \subseteq G$ be as in the above lemmas. If there exists $F(x) \in K_0(x) \backslash K_0$ which satisfies the uniqueness property for v/v_0, then $N = ISD^h$ holds for v/v_0.*

Proof. Let $P(x)$ be a polynomial of degree N over K_0 having $v(P)$ non-torsion $\bmod G_0$, so that $I(P)=I$ by Lemma 3.1; consequently

$$N=[K_0(x):K_0(P)]\geqq[K_0(x)^h:K_0(P)^h]=ISD^h.$$

It only remains to be shown that $(N/IS)\leqq D^h$. Since F satisfies the uniqueness property for v/v_0, keeping in view the result quoted just before Lemma 3.2, we have

$$[K_0(x):K_0(F)]=[K_0(x)^h:K_0(F)^h]=I(F)SD^h,$$

i.e.,

$$D^h=\frac{[K_0(x):K_0(F)]}{I(F)S}.$$

So the proof of the lemma is complete as soon as we prove:

Lemma 3.4. *Let v_0, v and $G_0\subseteq G$ be as in Theorem 2.2. For any non-zero element $F\in K_0(x)$ with $v(F)$ non-torsion* $\bmod G_0$, *the inequality*

$$\frac{N}{I}\leqq\frac{\deg F}{I(F)}$$

holds, where $\deg F$ *stands for* $[K_0(x):K_0(F)]$.

Proof. Fix a polynomial $P(x)\in K_0[x]$ of degree N with $v(P)=\gamma$(say) non-torsion $\bmod G_0$, so that $G=G_1+Z\gamma$ by Lemma 3.1. Let $F(x)$ be any non-zero element of $K_0(x)$ with $v(F)$ non-torsion $\bmod G_0$. There exists λ in G_1 and a non-zero integer n such that $v(F)=\lambda+n\gamma$. Therefore

$$\begin{aligned}I(F)&=[G_1+Z\gamma:G_0+Z(\lambda+n\gamma)]\\&=[G_1+Z\gamma:G_1+Z(\lambda+n\gamma)][G_1+Z(\lambda+n\gamma):G_0+Z(\lambda+n\gamma)]\\&=|n|[G_1:G_0]\end{aligned}$$

which shows that

$$I(F)/I=|n|=m(\text{say}).$$

So the desired inequality can be rerwritten as $Nm\leqq\deg F$.

As in the proof of Lemma 3.1, on representing $F(x)$ as

$$F(x)=\sum_{i\geqq 0}F_i(x)P(x)^i$$

where $F_i(x) \in K_0[x]$ is either 0 or has degree less than $N = \deg P(x)$, and using the fact that $P(x)$ is a polynomial of minimum degree such that $v(P(x)) = \gamma$ is non-torsion mod G_0, we see that

$$\lambda + n\gamma = v(F(x)) = \min_i (v(F_i(x)) + i\gamma).$$

This shows that the index i for which the above minimum is attained is n. In particular n is positive, i.e. $n = m$, and the term $F_n(x)P(x)^n$ occurs in the representation of $F(x)$ with $F_n(x) \neq 0$; consequently

$$\deg F(x) \geqq \deg(F_n(x)P(x)^n) \geqq Nn$$

as desired.

4. Proof of Theorem 2.1

Let $P(x) \in K_0[x]$ be a polynomial of smallest degree N such that $v(P) = \gamma$(say) is not torsion mod G_0. We fix an algebraic closure $\bar{K}_0$ of K_0, a divisible closure $\overline{G_0 + Z\gamma}$ of the group $G_0 + Z\gamma$ and a prolongation w of v to $\bar{K}_0(x)$ with value group contained in $\overline{G_0 + Z\gamma}$. Since $w(P)$ is not torsion mod G_0, there exists a linear factor $x - \beta$ (say) *of* $P(x)$ such that $w(x-\beta)$ is not torsion mod G_0; set $w(x-\beta) = \delta$.

Let v' be any prolongation of $v_0^{(P,\gamma)}$ to $K_0(x)$. On replacing v' by an equivalent valuation, we can assume that the value group of v' is contained in $\overline{G_0 + Z\gamma}$. Let w' be a prolongation of v' to $\bar{K}_0(x)$ whose value group is also contained in $\overline{G_0 + Z\gamma}$. For some root β' of $P(x)$, $w'(x - \beta')$ must be non-torsion mod G_0, say $w'(x - \beta') = \delta'$. Let $\bar{v}_0$ and $\bar{v}_0'$ denote respectively the restrictions of w, w' to $\bar{K}_0$. Let σ be an automorphism of $\bar{K}_0/K_0$ which maps β to β'; such an automorphism exists because $P(x)$ is irreducible over K_0.

Any polynomial $f(x) \in K_0[x]$ can be uniquely written as a finite sum

$$f(x) = c_0 + c_1(x - \beta) + \dots, c_i \in K_0[\beta].$$

On taking the image of coefficients under σ, we can write

$$f(x) = \sigma(c_0) + \sigma(c_1)(x - \beta') + \dots$$

In view of the fact that δ, δ' are non-torsion mod G_0, we have by the strong triangle law

$$v(f) = w(f) = \min_i (\bar{v}_0(c_i) + i\delta),$$

$$v'(f) = w'(f) = \min_i (\bar{v}_0'(\sigma(c_i)) + i\delta').$$

Assume first that v_0 is henselian, then $\bar{v}'_0 \circ \sigma = \bar{v}_0$, for the value group of both these valuations is contained in the same divisible closure of G_0. If we write

$$P(x) = a_1(x-\beta) + \cdots + a_N(x-\beta)^N, a_i \in K_0[\beta],$$

then by what has been proved above

$$\gamma = v(P) = \min_i (\bar{v}_0(a_i) + i\delta),$$

and

$$\gamma = v'(P) = \min_i (\bar{v}'_0(\sigma(a_i)) + i\delta') = \min_i (\bar{v}_0(a_i) + i\delta'),$$

so that

$$\delta = \max_{1 \leqq i \leqq N} ((\gamma - \bar{v}_0(a_i))/i) = \delta'.$$

Hence $v(f) = v'(f)$ for all f in $K_0[x]$ and the theorem is proved in the first case.

Assume now that v_0 is of rank 1 and that G_0 is a confinal subset of G. Let $(K_0(x)^h, v^h)$ be a henselisation of $(K_0(x), v)$ and let $(K_0^h, v_0^h) \subseteq (K_0(x)^h, v^h)$ be the henselisation of (K_0, v_0). We shall denote by v_1 the restriction of v^h to $K_0^h(x)$. Observe that the residue field and value group of v_1 are the same as those of v, and that

$$I(v_1/v_0^h) = I(v/v_0), S(v_1/v_0^h) = S(v/v_0), D^h(v_1/v_0^h) = D^h(v/v_0).$$

So by the first case above and Theorem 2.2, the proof in this case is complete as soon as we show that $N(v_1/v_0^h) = N(v/v_0)$. By definition $N(v_1/v_0^h) \leqq N(v/v_0)$. To prove equality, let $f(x) = \sum_{i=0}^r a_i x^i$ be any non-zero polynomial over K_0^h of degree r(say). It is enough to show the existence of a polynomial $g(x)$ over K_0 of degree r with $v_1(f) = v_1(g)$. Since G_0 is cofinal in G, there exists λ_0 in G_0 such that $\lambda_0 > v_1(fx^{-i})$ for $0 \leqq i \leqq r$. In the rank 1 case, K_0 being dense in K_0^h, we can choose b_i in K_0, $b_r \neq 0$ satisfying

$$v_0(a_i - b_i) \geqq \lambda_0, 0 \leqq i \leqq r;$$

this implies that

$$v_1(a_i x^i - b_i x^i) > v_1(f), 0 \leqq i \leqq r,$$

which show that $v_1(f-g) > v_1(f)$, where $g = \sum_{i=0}^r b_i x^i$. Therefore $v_1(f) = v_1(g)$ and the theorem is proved.

5. An example

We shall construct rank 2 valuations v_0, v of certain fields $K_0 \subseteq K_0(x)$ satisfying

(i) $S(v/v_0) = I(v/v_0) = 1$;

(ii) $N(v/v_0) \geqq 2$;

(iii) the residue field of v_0 has characteristic 0, so that by a well-known result (cf. [**1**, Prop. 15]) $D^h(v/v_0) = 1$.

We first introduce some notations and definitions.

Let K be a field and y an indeterminate. By the y-adic valuation u of $K(y)$, we mean the valuation which is defined for any $f(y)$ in $K[y]$ by $u(f(y))$ = the highest power of the monomial y dividing $f(y)$.

Let w be a valuation of a field K and γ be an element of a totally ordered abelian group containing the value group of w. Let $w^{(y,\gamma)}$ denote the valuation of $K(y)$ defined on $K[y]$ by

$$w^{(y,\gamma)}\left(\sum_{i=0}^{m} a_i y^i\right) = \min_i \,(w(a_i) + i\gamma);$$

the valuation $w^{(y,\gamma)}$ will be referred to as the valuation defined by min, w, γ and y.

Let w be a valuation of a field K having valuation ring R_w and residue field L_w. Let $\bar{w}$ be a valuation of L_w. As in [**11**, p. 43] pr [**3**, p. 58, Thm. 8.7] by the composite valuation $w \circ \bar{w}$, we mean a valuation of K (determined uniquely up to equivalence) with valuation ring R given by

$$R = \{\alpha \in R_w \mid \bar{w}(\bar{\alpha}) \geqq 0\},$$

where $\xi \to \bar{\xi}$ denotes the canonical homomorphism from R_w onto L_w.

We shall use the following result proved in [**4**, Lemma 9].

Lemma B. *Let w be a valuation of a field K having value group $\mathbb{Z}$ (the group of rational integers) and let $\bar{w}$ be a finite rank valuation of the residue field of w with value group G. Let π be an element of K satisfying $w(\pi) = 1$. For $\beta \neq 0$ in K, if β^* denotes the class of $\beta/\pi^{w(\beta)}$ in the residue field of w, then*

$$u(\beta) = (w(\beta), \bar{w}(\beta^*)) \quad \beta \neq 0 \text{ in } K$$

defines a valuation of K with value group $Z \times G$ (lexicographically ordered); also u is equivalent to a composite valuation $w \circ \bar{w}$.

We now begin with the construction of v_0 and v. Let s, z be complex numbers algebraically independent over $\mathbb{Q}$, the field of rational numbers and let y be an indeterminate over the field $\mathbb{C}$ of complex numbers. Define

$$K_0=Q(s,y), K_1=K_0(\sqrt{s+1}) \quad \text{and} \quad x=\sqrt{s+1}+yz.$$

Then $K_1(x)=K_1(z)$. Let w_0, w_1, w and w' denote respectively the restrictions of the y-adic valuation of $\mathbb{C}(y)$ to the fields K_0, K_1, $K_0(x)$ and $K_1(x)$; let $L_0 \subseteq L_1 \subseteq L \subseteq L'$ denote their respective residue fields. We shall regard L' to be a subfield of $\mathbb{C}$. Clearly $L_0=\mathbb{Q}(s)$, $L_1=\mathbb{Q}(\sqrt{s+1})$; we show that

$$L=L'=\mathbb{Q}(\sqrt{s+1},z).$$

Since $x-\sqrt{s+1}=yz$, so the w'-residue of $(x-\sqrt{s+1})/y$ is (identified with) z which is transcendental over the residue field L_1 of w_1. As in [2, §10.1, Prop. 2] it can be easily shown that w' is in fact the valuation $w_1^{(z,0)}$ of the field $K_1(x)=K_1(z)$ (defined by min, w_1, 0 and z) and that its residue field L' is given by

$$L'=L_1(z)=\mathbb{Q}(\sqrt{s+1},z).$$

So to prove that $L=\mathbb{Q}(\sqrt{s+1},z)$ it is enough to show that $\sqrt{s+1}$ and z are in L. Since the w-residue of x is (identified with) $\sqrt{s+1}$ and that of $(x^2-(s+1))/y$ is $2z\sqrt{s+1}$, the desired assertion is proved.

Let u_0 be the s-adic valuation of $L_0=\mathbb{Q}(s)$ and u_1 be an extension of u_0 to $L_1=\mathbb{Q}(\sqrt{s+1})$. We fix an irrational number γ. Let $u=u_1^{(z,\gamma)}$ be the valuation of $L=L_1(z)$ defined by min, u_1, γ and z.

In view of the fact that u_0 has two extensions to L_1 (because if $\xi=\sqrt{s+1}$, then $s=\xi^2-1$ implies that u_0 extends to $\mathbb{Q}(\xi)$ either by $u_1(\xi-1)=1$, $u_1(\xi+1)=0$ or the reverse), both u_1 and u_0 have the same residue field, i.e. $\mathbb{Q}$. Since γ is an irrational number, the residue field of u is again $\mathbb{Q}$ (cf. [2, § 10.1, Prop. 1]). Clearly the value group of u is $\mathbb{Z}+\mathbb{Z}\gamma$.

We take v as the composite valuation $w \circ u$ (defined by the formula given in Lemma B) with value group $\mathbb{Z}\times(\mathbb{Z}+\mathbb{Z}\gamma)$ lexicographically ordered, and denote the restriction of v to K_0 by v_0. Then the value group of v_0 is $\mathbb{Z}\times\mathbb{Z}$, so that $I(v/v_0)=1$.

Since the residue field of the composite $w \circ u$ of two valuations equals (up to isomorphism) the residue field of the valuation u, (see [11, Chap. VI, Thm. 2]) both v and v_0 have $\mathbb{Q}$ as the residue field:, therefore $S(v/v_0)=1$.

It only remains to be shown that if $x-\alpha$ is any linear polynomial over K_0, then $v(x-\alpha)$ is torsion $\bmod \mathbb{Z}\times\mathbb{Z}$, (in fact $v(x-\alpha)$ will be in $\mathbb{Z}\times\mathbb{Z}$. As in Lemma B, let $(x-\alpha)^*$ denote the w-residue of $(x-\alpha)/y^n$ in L, where $n=w(x-\alpha)$. It will be shown that $(x-\alpha)^*$ is in $\mathbb{Q}(\sqrt{s+1})$; consequently $u((x-\alpha)^*)$ will be in $\mathbb{Z}$ as desired.

Observe that for any α in $K_0=\mathbb{Q}(s,y)$, $w'(\sqrt{s+1}-\alpha)\leqq 0$, because the rational function $\sqrt{s+1}-\alpha$ cannot vanish at $y=0$. It follows that

$$w'\left(\frac{\sqrt{s+1}-\alpha}{y^n}\right)\leqq -n.$$

Also by definition of x,

$$w'\left(\frac{x-\sqrt{s+1}}{y^n}\right)=w'\left(\frac{yz}{y^n}\right)=1-n,$$

therefore keeping in view that $w'((x-\alpha)/y^n)=0$, we have by the strong triangle law

$$0=w'\left(\frac{x-\alpha}{y^n}\right)=w'\left(\frac{\sqrt{s+1}-\alpha}{y^n}\right)\leqq -n,$$

and hence

$$w'\left(\frac{x-\sqrt{s+1}}{y^n}\right)>0.$$

If $\beta\to\bar{\beta}$ denotes the canonical homomorphism from the valuation ring of w onto the residue field of w, then it is clear that

$$(x-\alpha)^*=\left(\frac{x-\sqrt{s+1}}{y^n}\right)^- +\left(\frac{\sqrt{s+1}-\alpha}{y^n}\right)^- =\left(\frac{\sqrt{s+1}-\alpha}{y^n}\right)^-$$

which is in L_1 as desired.

REFERENCES

1. J. Ax, A metamathematical approach to some problems in number theory, 1969 Number Theory Institute, *Proc. Sympos. Pure Math.* **XX** (1971), 161–190.

2. N. Bourbaki, *Commutative Algebra, Chapter 6, Valuations* (Hermann Publishers in Arts and Science, 1972).

3. O. Endler, *Valuation Theory* (Springer-Verlag, New York, 1972).

4. S. K. Khanduja and U. Garg, Rank 2 valuations of $K(x)$, *Mathematika* **37** (1990), 97–105.

5. S. K. Khanduja and U. Garg, Stability theorem for valued function fields, submitted for publication.

6. F. V. Kuhlmann, Henselian function fields and tame fields, Manuscript 1990.

7. M. Matignon and J. Ohm, A structure theorem for simple transcendental extensions of valued fields, *Proc. Amer. Math. Soc.* **104** (1988), 392–402.

8. M. Matignon and J. Ohm, Simple transcendental extensions of valued fields III: The uniquenesss property, *J. Math. Kyoto Univ.* **30** (1990), 347–366.

9. M. Polzin, Prolongement de la valeur absolue de Gauss et probleme de Skolem, *Bull. Soc. Math. France* **116** (1988), 103–132.

10. B. L. van der Waerden, *Modern Algebra I* (Ungar, New York, 1964).

11. O. Zariski and P. Samuel, *Commutative Algebra, Vol II* (Van Nostrand, Princeton, 1960).

Centre for Advanced Study in Mathematics
Panjab University
Chandigarh 160014
India