A REDUCTION THEOREM FOR PERFECT LOCALLY FINITE MINIMAL NON-FC GROUPS

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A group $G$ is said to be a minimal non-FC group, if $G$ contains an infinite conjugacy class, while every proper subgroup of $G$ merely has finite conjugacy classes. The structure of imperfect minimal non-FC groups is quite well-understood \cite{3} (see also \cite{14}, Section 8). These groups are in particular locally finite. At the other end of the spectrum, a perfect locally finite minimal non-FC group must be a $p$-group \cite{2}, \cite{9}. And it has been an open question for quite a while now, whether such groups exist or not. In \cite{10}, Theorem 2.4, it was shown that such $p$-groups have a non-trivial representation as subgroups of the McLain group $\text{M}(Q, F_p)$, that is, as groups of infinite upper unitriangular matrices of order type $Q$ over the field $F_p$ with $p$ elements, in which all but finitely many non-diagonal entries are zero. The purpose of this note is to obtain the following considerable improvement, which should provide a major step in the discussion of existence of perfect minimal non-FC $p$-groups.

**Theorem.** Every perfect locally finite minimal non-FC group has a quotient, which acts as a barely transitive $p$-group of finitary permutations on some infinite set.

Recall, that finitary permutations of the set $\Omega$ fix all but finitely many elements in $\Omega$. The structure of groups of finitary permutations has been studied intensely in the seventies and again during the last ten years (see \cite{13} for references). A subgroup of the symmetric group $\text{Sym}(\Omega)$ on an infinite set $\Omega$ is said to be barely transitive, if it acts transitively on $\Omega$, while each of its proper subgroups has finite orbits. Barely transitive groups were brought up by B. Hartley \cite{4}, \cite{5} in connection with groups of Heineken-Mohamed type, and have been investigated during the last years mainly by M. Kuzucuoğlu \cite{7}, \cite{8}. Obviously every barely transitive group without proper finite quotients is a minimal non-FC group. In particular, the question about existence of perfect locally finite minimal non-FC $p$-groups turns out now to be equivalent to the question about existence of perfect barely transitive $p$-groups, which in addition act finitarily on the underlying set.

**Proof of the Theorem.** Let $G$ be a perfect locally finite minimal non-FC group. Recall that $G$ is a $p$-group. Since $G$ is perfect, the centre $\zeta_1(G)$ is the highest term of the upper central series in $G$. From passing to $G/\zeta_1(G)$ we may assume that $G$ has trivial centre. Consider a non-trivial normal subgroup $N$ of $G$. The socle $S$ of the FC- and $p$-group $N$ is an elementary-abelian normal subgroup in $G$ (\cite{14}, p. 10). Consider a fixed non-trivial element $x \in S$, and let $\Omega = \{x^g | g \in G\}$ and $V = \langle \Omega \rangle \leq N$. Since $G$ has no proper finite image and trivial centre, the set $\Omega$ must be infinite. Since $G$ is a minimal non-FC group without maximal subgroups, it acts barely transitively on $\Omega$ via conjugation. Moreover, $G$ acts finitarily linearly on the $F_p$-vector space $V$: For every $g \in G$, the proper subgroup $V(g)$ of $G$ is an FC-group, whence
$|V : C_1(g)| \leq |V(g) : C_{V(g)}(g)| < \infty$. It remains to show, that $G$ acts as a finitary permutation group on $\Omega$.

To this end, we assume that some $g \in G$ has infinite support on $\Omega$. Let $M = \langle g^G \rangle$. Since $G$ is a locally finite $p$-group, $g \notin M'$, and so $M/M' \neq 1$. Since $G$ is perfect, $M$ is a proper normal subgroup of $G$. Since $G$ acts barely transitively on $\Omega$, the $M$-orbits $\Omega_i$ ($i \in \omega$) are finite and form a system of imprimitivity. Let $V_i = \langle \Omega_i \rangle \leq N$. Since $g$ has infinite support on $\Omega$, we have $|V_i, g| \neq 1$ for infinitely many $i \in \omega$. However, $|V, g|$ is a finite-dimensional $F_\rho$-vector space, hence finite. Thus there is a one-dimensional subspace $U$ in $[V, g]$ such that $U \subseteq [V_i, g]$ for infinitely many $i \in \omega$. Let $I$ be the set of all such $i \in \omega$. Fix $i_0 \in I$, and choose $g_i \in G$ ($i \in I$) satisfying $\Omega_i^{g_i} = \Omega_{i_0}$. Since $V_{i_0}$ is finite, there is an infinite set $I_0 \subseteq I$ such that $U^{g_i} = U^{g_j}$ for all $i, j \in I_0$. Consider the normalizer $H = N_G(U)$. Fix $\omega_0 \in \Omega_{i_0}$. Since $g_i g_i^{-1} \in H$ for all $i \in I_0$, the elements $\omega_0 g_i^{-1} (i \in I_0)$ are contained in an infinite $H$-orbit on $\Omega$. Hence $H = G$, and $U$ is a normal subgroup of order $p$ in $G$. But then $1 \neq U \subseteq \xi_1(G)$, a contradiction. The proof of the Theorem is complete.

A group $G$ is said to be a minimal non-CC group, if $U/C_1(x^U)$ is a Černikov group for all $x \in U < G$, while this property fails for $G$ in place of $U$. Obviously, every perfect locally finite minimal non-FC group is a minimal non-CC group. Many results about minimal non-FC groups have been transferred to minimal non-CC groups [12], [6]. The following is an immediate consequence of [6], [1], and of our Theorem above.

**Corollary 1.** Every locally graded minimal non-CC group has a quotient, which acts as a barely transitive $p$-group of finitary permutations on some infinite set.

We also obtain a generalization of [12].

**Corollary 2.** No non-trivial quotient of a locally graded minimal non-CC group lies in a proper variety.

**Proof.** Let $G$ be a locally graded minimal non-CC group. Every quotient of $G$ is also such a group [12]. Consider $N \triangleleft G$ and assume, that $G/N$ lies in a proper variety. From Corollary 1 we may assume that $G/N$ is a transitive group of finitary permutations of an infinite set $\Omega$. But this contradicts [11, Theorem 1].

**REFERENCES**

2. V. V. Belyaev, Minimal non-FC groups, in *All Union Symposium on Group Theory* (Kiev 1980), 97–108.