concepts which are of use in ordinary mathematics under different names—like the set of Cauchy sequences in $\mathbb{Q}$.

At a more advanced level, the discussion of measure theory is impressive for the fact that the elementary theory of the Lebesgue integral can be done at all; but it is even more apparent than elsewhere that this is paraplegic mathematics, struggling desperately to match the achievements of its un-handicapped ideal. A point which strikes me here and in the chapters on normed spaces is that the constructivist approach seems to lead to theorems similar to those which can be proved in ordinary mathematics with countable choice alone; for instance, the Hahn–Banach theorem here is restricted to separable spaces.

I am inclined to think that the author's wish to avoid the paradoxical aspects of constructivism has in fact deprived him of the only inspiration likely to come from this approach. Constructivism, as presented here, merely leads to enormous technical difficulties without shedding much light on the real questions of mathematics. Of course there are many useful flashes of ingenuity which are struck by the effort of thinking the basic theory out anew. But I should like to suggest that, if there is anything of substantial value in constructivism, it is more likely to come from a rigorous formulation, within Aristotelian logic of the acceptable rules of proof—requiring, perhaps, a proof of $\forall x P(x)$ to be a proof of $P(x)$ which is a recursive function of x—followed by a systematic analysis of the ways in which the theory differs from real mathematics. As an example of a topic in which such an approach has had great success I offer effective descriptive set theory.

In conclusion: No doubt it is good for constructivists to learn some functional analysis. I do not think that there is yet much reason for analysts to learn constructivism.

D. H. FREMLIN

In 1967 Errett Bishop's book 'Foundations of constructive analysis' appeared. Since then there has been a steady flow of work within the framework set out in Bishop's book. The book under review partially replaces Bishop's book, which has been out of print for some time. The two books do not have quite the same range of subjects but the present book contains improvements and developments since 1967, many of them due to Bridges himself.

The book gives a clear self-contained introduction to constructive analysis. Readers willing to restrict their methods of proof to meet constructive requirements should be able to pick up the perhaps unfamiliar pattern of thought without unnecessary effort. As in Bishop's book, the logic and philosophy of constructive mathematics is treated very briefly. Just as with classical analysis constructive analysis can be learnt and used without undue reflection on the fundamental notions.

The ideas motivating this book have their origin in Brouwer's intuitionistic criticism of non-constructive methods. But the rather extreme subjective aspects of Brouwer's thought have been avoided and the presentation is fairly straight forward. Bridges follows Bishop in the following respects:

(i) Mathematical objects are kept concrete, in the sense that they are always in principle arithmetically representable.

(ii) All operations on these objects are intended to be computable.

(iii) The language is kept as close as possible to the standard set theoretical one.

This entails a systematic avoidance of abstract objects obtained when taking the quotient of a set by an equivalence relation. Instead, each set has to carry with it the equivalence relation which holds between two concrete objects when they represent the same abstract object of conventional mathematics. This systematic departure from the conventional presentation can be irksome at first, but it is easy to adapt to it in practice. In fact it would be possible to give a standard set theoretical presentation of constructive analysis that *did* allow the quotient construction, but then the constructive computational character of the mathematics would no longer be explicit and it would be necessary to give a separate account of the procedure for making it explicit.

In constructive mathematics the meaning of a mathematical statement is given by specifying the mathematical constructions that are to count as proofs of the statement. The truth of a

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mathematical statement is then identified with its proveability in this informal sense. This account leads to an explanation of the logical operators in terms of the informal notion of 'proof', in contrast to the classical explanation in terms of 'truth'. Not surprisingly, some classically accepted methods of reasoning are no longer acceptable. What must be surprising, at first sight, is the remarkable coherence of the intuitionistic logic that results when the unacceptable methods are dropped.

Brouwer felt it necessary to use the conceptually rather exotic notion of a free choice sequence in developing his intuitionistic analysis. While this notion continues to fascinate logicians and may have interesting sheaf theoretic models Bishop showed quite convincingly that Brouwer's free choice sequences are unnecessary for analysis. The apparent difficulties are overcome by suitable redefinitions of the basic notions. For example, if the standard definitions are used, the proof that every continuous function $f: [0, 1] \rightarrow \mathbb{R}$ is uniformly continuous is non-constructive. The tactic is to redefine continuity so that the result does follow constructively. This turns out to be possible without losing the continuity of the standard continuous functions. Of course, indiscriminate use of this sort of tactic of redefinition would make the whole subject worthless. Bishop's achievement was to demonstrate that an intelligent limited use of this tactic suffices to allow the constructivisation of a great deal of classical analysis.

All the mathematical results obtained in the book under review are valid from the classical standpoint. but some classical results do not have constructive proofs, and there is an obvious question to be faced. Why should any mathematician restrict his methods to those allowed in this book? The devoted constructivist would say that constructive methods are correct, non-constructive methods are meaningless and one ought not to use meaningless methods. Bridges' view is that constructive proofs give more information—e.g. a constructive existence proof gives a procedure for constructing the object proved to exist. But the availability of such extra information has not yet had any significant impact on classical mathematics. I would answer the question first by giving the conventional answer that the restriction to constructive methods leads to a discipline having its own intrinsic interest, with a distinctive range of problems. The present book amply demonstrates this. But it fails to give any indication of how the subject may come to interact fruitfully with other disciplines in the future.

To my mind we will not have to wait long before there will be a significant interaction between constructive mathematics and theoretical computer science. Also, the fact that the internal logic of a topos is intuitionistic means that many of the proofs of constructive mathematics can be carried over to the development of mathematics inside many topoi. So topos theory is another area which one can expect to interact significantly with constructive mathematics.

In conclusion, for those mathematicians who wish to gain a working knowledge of constructive analysis following the Bishop school this is a useful book that has no rival (except for Bishop's out of print book). But those who seek logical analysis or philosophical explanation of the ideas of constructive mathematics will have to look elsewhere.

P. ACZEL

Collected Papers of G. H. Hardy, edited by a committee appointed by the London Mathematical Society, Vol. VII (Clarendon Press, Oxford, 1979), 897 pp., £30·00.

It is an honour to be asked to review the seventh and last volume of the Collected Papers of the late Professor G. H. Hardy, ably edited by Professor Rankin and Dr. Busbridge. But it is an impossible task to give a brief appreciation of a volume of 897 pages.

The first half consists of research papers, some written in collaboration with Bochner, Littlewood or Titchmarsh; this part is for the specialist. Any mathematician will find much to enjoy in the rest of the book—elementary notes, addresses, reviews, obituary notices, and problems from the *Educational Times*. Who would not be amused at the complex curve

$$(x+iy)^2 = \lambda(x-iy)$$

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