THE GROUP CONFIGURATION THEOREM FOR GENERICALLY STABLE TYPES

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Abstract. We generalize Hrushovski's group configuration theorem to the case where the type of the configuration is generically stable, without assuming tameness of the ambient theory. The properties of generically stable types, which we recall in the second section, enable us to adapt the proof known in the stable context.

§1. Introduction. In his thesis [9], Ehud Hrushovski proved a group configuration theorem, building a type-definable group from combinatorial data, in a stable setting. The aim of this paper is to generalize the theorem, using only hypotheses on the type of the configuration, without assuming tameness of the theory.

First, we shall introduce generically stable types, and state some of their known properties. Then, we will define some notions of genericity in definable groups and definable homogeneous spaces, and show a couple of results regarding groups with generically stable generics. Having done that, we shall state and prove a group configuration theorem (Theorem 3.37) for generically stable types. The proofs will be similar to the stable case, although a bit trickier. In passing, we also write down a uniqueness result (Proposition 3.24), recovering a group with generically stable generics from its configuration, up to some notion of equivalence. That result is not new, and has actually been improved substantially, for instance, in [12, Theorem 2.15] (see Remark 3.30).

From now on, we fix a complete theory T, in a language $\mathcal{L}$, and work inside T^{eq} to ensure elimination of imaginaries. We let acl, resp. dcl, denote the algebraic closure, resp. definable closure, in T^{eq}. Moreover, we let $\mathbb{U}$ denote a very saturated and strongly homogeneous model of T. A subset A of a model M is called *small*, with respect to M, if M is $|A|^+$-saturated and $|A|^+$-strongly homogeneous. Note that we might consider models $M \subset \mathbb{U}$ and sets $A \subset M$ such that A is small with respect to M, and M itself is small with respect to $\mathbb{U}$. By default, the sets of parameters we consider are small with respect to $\mathbb{U}$. If a, b are small tuples, we may write ab or $a^\frown b$ for the concatenation.

As far as groups are concerned, we shall only need basic notions for group actions, such as stabilizers, orbits, faithfulness, transitivity, equivariant maps. It will be useful

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to know that any transitive action of a group G is isomorphic[1] to the action of G on G/H, for some subgroup $H \leq G$, and that the stabilizers are then conjugates of H.

§2. Generically stable types.

2.1. Forking, invariant, and definable types.

Definition 2.1. Let $A \subseteq \mathbb{U}$ be a set of parameters and $\phi(x, y)$ be a formula over A. Let b be a tuple.

1. The formula $\phi(x, b)$ divides over A if there is an A-indiscernible sequence $(b_i)_{i<\omega}$, with $b_0 = b$, such that the partial type $\{\phi(x, b_i) \mid i < \omega\}$ is inconsistent.
2. The formula $\phi(x, b)$ forks over A if $\phi(x, b)$ implies a finite disjunction of formulas, possibly with additional parameters, all of which divide over A.

For any natural number k, a partial type π is k-inconsistent if, for any choice of pairwise non-equivalent formulas $\phi_1, \dots, \phi_k$ in π, the conjunction $\bigwedge_i \phi_i$ is not satisfiable.

Then, by compactness and indiscernibility, one may replace "inconsistent" with "k-inconsistent for some k" in the above definition of dividing.

Definition 2.2.

1. A partial type $\pi(x)$ divides (resp. forks) over a set A if there exists a formula $\phi(x)$ (possibly with parameters outside of A) such that $\pi(x) \models \phi(x)$ and $\phi(x)$ divides (resp. forks) over A.
2. Let a, b be tuples, and C be a set. The tuple a is independent from b over C, which we denote $a \mathop{\smile\!\!\!\!\!\mid}_C b$, if $tp(a/Cb)$ does not fork over C.
3. Let $p \in S(A)$. The type p is extensible (resp. stationary) if, for any $B \supseteq A$, there exists a (resp. a unique) $q \in S(B)$ such that $q|_A = p$ and q does not fork over A. Given a stationary type $p \in S(A)$ and $B \supseteq A$, we let $p|_B \in S(B)$ denote its unique nonforking extension.

Definition 2.3. Let $(a_i)_{i \in I}$ be a family of elements. Let A be a set of parameters. We say that (a_i) is an independent family over A if, for all $i \in I$, we have $a_i \mathop{\smile\!\!\!\!\!\mid}_A (a_j)_{j \in I, j \neq i}$.

Notation 2.4. *To simplify notations, if A is a small set of parameters, we write $|A|$ instead of $|A| + |\mathcal{L}| + \aleph_0$.*

Definition 2.5. Let M be a model of T, let $p \in S(M)$, and $A \subseteq M$.

1. We say that p is A-definable if, for all formulas without parameters $\phi(x, y)$, there exists a formula $d_p x \phi(x, y)$ with parameters in A such that, for all $b \in M$, we have $M \models d_p x\, \phi(x, b)$ if and only if $\phi(x, b) \in p(x)$. We say that d_p is "the" defining scheme for p. Indeed, since M is a model, the defining scheme is unique up to equivalence.
2. We say that p is definable if it is M-definable.
3. If p is definable, the canonical basis of p is the smallest dcl-closed set $A \subseteq M$ such that p is A-definable. By elimination of imaginaries, this set is well-defined.

[1] An isomorphism of actions is an equivariant bijection; its inverse is automatically equivariant.

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4. In the case where M is $|A|^+$-saturated, we say that p is A-invariant if, for any formula without parameters $\phi(x, y)$, for all $b_1, b_2 \in M$, if $b_1 \equiv_A b_2$, then $\phi(x, b_1) \in p(x)$ if and only if $\phi(x, b_2) \in p(x)$. In other words, the formula $\phi(x, b)$ being in the type p depends only on the type of b over A.
5. If $B \supseteq A$ is a set of parameters (not necessarily a model), and $q \in S(B)$, we say that q is A-invariant (resp. A-definable, resp. definable) if it admits *some*, not necessarily unique, A-invariant (resp. A-definable, resp. B-definable) extension q_1 to an $|A|^+$-saturated model $N \supseteq B$.

FACT 2.6 (See [13, Proposition 1.9] and [16, Section 2.2, discussion before Lemma 2.18]). *Let A be a small subset of a model M. Let $p \in S(M)$ be A-invariant. Then, for any model $N \supseteq M$, the type p has a unique extension $p|_N \in S(N)$ which is A-invariant.*

Moreover, if p is A-definable, then $p|_N$ is A-definable, using the same defining scheme as p, and the whole conclusion holds even if A is not small with respect to M.

REMARK 2.7. Thanks to this fact, if $p \in S(M)$ is A-invariant, *where M is $|A|^+$-saturated*, and if $B \supseteq A$, we can write $p|_B \in S(B)$ for the restriction to B of $p|_N$, where N is a model containing MB. By uniqueness, the type $p|_B$ is well-defined, for it does not depend on the choice of the model N.

Because of this fact, we consider it useful to view invariant types as families of types, or as processes which construct complete types in a coherent way, and to identify two invariant types if they admit a common invariant extension to a sufficiently saturated and sufficiently strongly homogeneous model.

DEFINITION 2.8. Let A be a small subset of a model M. Let $p, q \in S(M)$, where p is A-invariant. Let us assume that p is in the variable x, and q in the variable y. We define the tensor product $p \otimes q \in S(M)$ as follows:

If $\phi(x, y, z)$ is a formula without parameters, and if $c \in M$, then $\phi(x, y, c) \in p \otimes q$ if and only if, for some (equivalently, for every) element $b \in M$ realizing $q|_{Ac}$, we have $\phi(x, b, c) \in p|_{Abc}$.

The tensor product is well-defined, and is a complete type over M. Indeed, we can check that the realizations of $p \otimes q$ are exactly the tuples of the form ab, where b realizes q, and a realizes $p|_{Mb}$. Note that if q is also A-invariant, then $p \otimes q$ is A-invariant. If p and q are A-definable, then so is $p \otimes q$.

DEFINITION 2.9. Let A be a small subset of a model M. Let $p \in S(M)$ be an A-invariant type. Let α be an ordinal. A sequence $(a_i)_{i<\alpha}$ is a *Morley sequence* of p over A if, for every $i < \alpha$, the element a_i realizes the type $p|_{A \cup (a_j)_{j<i}}$.

We now give a few facts on forking. For more details, see, for instance, [3], [8, Section 4], or [17, Section 7.1].

PROPOSITION 2.10. *Let $A \subseteq B \subseteq C$ be small sets of parameters, and a, b, c be small tuples.*

1. *Let $p \in S(C)$. Assume that p does not fork over A. Then p does not fork over B and $p|_B$ does not fork over A.*
2. *Let $p \in S(B)$ be a type which does not fork over A. Then, there exists $q \in S(C)$ extending p such that q does not fork over A.*
3. *Assume $acl(A) \subseteq B$. Then:*

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(a) *$tp(a/B)$ forks over A if and only if $tp(a/B)$ forks over $acl(A)$.*
(b) *$a \mathop{\smile\!\!\!\!\!\!|}_A b$ if and only if $acl(Aa) \mathop{\smile\!\!\!\!\!\!|}_A acl(Ab)$.*

4. *Assume $a \in acl(Ab) \cap acl(Ac)$ and $b \mathop{\smile\!\!\!\!\!\!|}_A c$. Then $a \in acl(A)$.*
5. *Let $M \supset A$ be an $|A|^+$-saturated and $|A|^+$-strongly homogeneous model. Let $p \in S(M)$ be an A-invariant type. Then p does not fork over A.*
6. *Assume that $a \mathop{\smile\!\!\!\!\!\!|}_A B$ and $b \mathop{\smile\!\!\!\!\!\!|}_{Aa} B$. Then $ab \mathop{\smile\!\!\!\!\!\!|}_A B$.*

Proof. The first point is a consequence of the definition of forking, and the fact that B-indiscernible sequences are A-indiscernible. For the second point, see [17, Lemma 7.1.11].

For the third point, see [5, Proposition 2.12]. Let us prove the fourth point. By the third point, we deduce that $a \mathop{\smile\!\!\!\!\!\!|}_A a$. This implies that the formula $x = a$ does not fork over A. In particular, it does not divide over A. Then, one can check that no infinite A-indiscernible sequence containing a is injective. This implies that $a \in acl(A)$, as required. For the fifth point, see [17, Exercise 7.1.4]. For the sixth point, see [16, Lemma 5.18]. ⊣

Lemma 2.11. *Let a be an element, $C \subseteq D = acl(D)$, such that $tp(a/D)$ is definable over C. Then, for every $b \in acl(Ca)$, the type $tp(b/D)$ is definable over $acl(C)$. Similarly, if $b \in dcl(Ca)$, then $tp(b/D)$ is definable over C.*

Proof. Let us prove the first point, the second one being easier. If $b \in acl(Ca)$, let $k < \omega$, and $\phi(x, y)$ be a formula over C such that $\models \phi(a, b)$ and $\models \forall x \, \exists^{\leq k} y \, \phi(x, y)$. By Definition 2.5(5), let $M \supset D$ be a sufficiently saturated model and a C-definable extension $p \in S(M)$ of $tp(a/D)$. Note that p is not necessarily unique, but we only want to find *some* $acl(C)$-definable extension of $tp(b/D)$. To simplify notations, let us assume that a realizes p. Let $\psi(y, z)$ be a formula without parameters. Let q denote $tp(b/M)$. Let us consider the following C-definable binary relation: $d_1 E d_2$ if and only if $\models d_p x[\forall y \, \phi(x, y) \to [\psi(y, d_1) \leftrightarrow \psi(y, d_2)]]$. Then, for $d_1, d_2 \in M$, if $d_1 E d_2$, then $\models \psi(b, d_1) \leftrightarrow \psi(b, d_2)$. Indeed, in this context, the element a realizes $p|_{Cd_1 d_2}$.

Claim 2.12. *The relation E is a C-definable finite equivalence relation.*

Proof. Reflexivity and symmetry are clear. Let us prove transitivity. Let $d_1 E d_2$ and $d_2 E d_3$. Let α realize $p|_{Cd_1 d_2 d_3}$. Let β be such that $\models \phi(\alpha, \beta)$. Then, we have $\models [\psi(\beta, d_1) \leftrightarrow \psi(\beta, d_2)] \wedge [\psi(\beta, d_2) \leftrightarrow \psi(\beta, d_3)]$. So $\models [\psi(\beta, d_1) \leftrightarrow \psi(\beta, d_3)]$. Since $\alpha \models p|_{Cd_1 d_3}$, we have indeed $d_1 E d_3$.

Finally, we shall prove that E has only finitely many classes. As E is C-definable, and $C \subseteq M$, it is enough to prove that $E(M)$ has only finitely many classes. Since a realizes $p|_M$, we know that, for $d_1, d_2 \in M$, we have $d_1 E d_2$ if and only if $\models \forall y \, (\phi(a, y) \to [\psi(y, d_1) \leftrightarrow \psi(y, d_2)])$. But $\models \exists^{\leq k} y \, \phi(a, y)$, therefore $E(M)$ has at most 2^k classes. So E has at most 2^k classes, so it is a finite equivalence relation. ⊣

Then, by elimination of imaginaries, let $c_1/E, \dots, c_r/E$ be the codes of the classes modulo E, where the c_i are in M. These codes are in $acl(C)$. To construct the definition $d_q y \, \psi(y, z)$, let $I \subseteq \{1, \dots, r\}$ be the set of indices i such that $\models \psi(b, c_i)$. Using the definition of the relation E, one can check that the formula $\bigvee_{i \in I} z E c_i$ is an appropriate definition of q for the formula $\psi(y, z)$. Moreover, since the c_i/E are in $acl(C)$, this formula is equivalent to a formula defined over $acl(C)$. ⊣

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2.2. General properties of generically stable types. The definition of generically stable types below is from [15, Definition 2.1]. Most of the properties in this subsection come from [7, Appendix A], [15, Proposition 2.1], and [1, Fact 1.9, Lemma 2.1, Theorem 2.2].

DEFINITION 2.13.

1. Let $(a_i)_{i \in I}$ be a sequence of elements of the same sort. Let B be a set. The mean, or average, of the types of the a_i over B is a partial (possibly complete) type, containing the formulas $\phi(x, b)$ over B such that, for cofinitely many indices $i \in I$, we have $\models \phi(a_i, b)$.
2. Let A be a set, and $p \in S(M)$, where $M \supset A$ is a sufficiently saturated model. The type p is *generically stable* over A if p is A-invariant and, for all ordinals $\alpha \geq \omega$, for all Morley sequences $(a_i)_{i<\alpha}$ of p over A, the mean of the types of the a_i over M is a complete type over M.

REMARK 2.14. The property for infinite ordinals in the definition above is equivalent to that for countably infinite ordinals. Indeed, a mean over an infinite index set is always a consistent partial type. If it is not complete, there exists a formula witnessing incompleteness. Then, countably many indices are enough to witness incompleteness of the mean for this formula. Thus, if the model M is sufficiently saturated, it is enough to check the property for Morley sequences made of elements of M.

PROPOSITION 2.15. *Let $p \in S(M)$ be a complete type, generically stable over a small set $A \subset M$. Then*:

1. *For any infinite Morley sequence $(a_i)_i$ of p over A, the mean of the types of the a_i is the type p itself.*
2. *For any $\phi(x, y)$ over A, there exists a natural number n_ϕ such that, for any infinite Morley sequence $(a_i)_i$ of p over A, for any b, we have $\phi(x, b) \in p$ if and only if the set of indices i such that $\models \neg\phi(a_i, b)$ contains at most n_ϕ elements.*
3. *The type p is definable over A.*
4. *Any Morley sequence of p over A is an indiscernible set over A.*
5. *If $B \subset M$ is a small set such that p is B-invariant, then p is generically stable over B.*
6. *The type $p|_A$ has a unique nonforking extension to M, which is p.*

PROOF. For points 2, 3, 4, and 6, see Proposition 2.1 in [15]. For point 1, see the proof of [15, Proposition 2.1.i]. Let us now prove point 5. Let $B \subset M$ be a small set such that p is B-invariant. Then, by point 3, p is A-definable, so B-definable as well. By elimination if imaginaries, the type p is thus $dcl(A) \cap dcl(B)$-definable. Let $C = dcl(A) \cap dcl(B)$.

CLAIM 2.16. *The type p is generically stable over C.*

PROOF. By Remark 2.14, let α be a countable infinite ordinal, let $(a_i)_{i<\alpha}$ be a Morley sequence of p over C, made of elements of M. By contradiction, assume that the mean of the types of the a_i over M is not a complete type. Let $(a'_i)_{i<\alpha}$ be a Morley sequence of p over A, made of elements of M as well. Then, the infinite tuples $(a_i)_{i<\alpha}$ and $(a'_i)_{i<\alpha}$ have the same type over C. So, by strong homogeneity of M, there exists $\sigma \in Aut(M/C)$ such that $\sigma(a_i) = a_i'$ for every $i < \alpha$. Since the mean of the types of the a_i over M is not a complete type, we deduce that the mean of the types of the a_i' over M is not a complete type either, which contradicts the assumption of generic stability of p. ⊣

Then, unfolding the definition of generic stability, we deduce that p is generically stable over $dcl(B) \supseteq C$, so over B as well. ⊣

DEFINITION 2.17.

1. Because of these properties, we may call a type $p \in S(A)$ generically stable if, for some, equivalently for every, sufficiently saturated model $M \supset A$, the type p has a (necessarily unique) nonforking extension $q \in S(M)$ which is generically stable over A.
2. If $B \supseteq A$, we may also say that a type $q \in S(B)$ is generically stable over A if $q|_A$ is generically stable in the above sense, and q does not fork over A.

The following Fact is a consequence of stationarity.

FACT 2.18. *If $q \in S(M)$ does not fork over A, where $M \supset A$ is a (not necessarily sufficiently saturated) model, then $q|_A$ is generically stable, in the sense of Definition 2.17(1), if and only if q is generically stable over A in the sense of Definition 2.17(2). If M is sufficiently saturated, this is also equivalent to q being generically stable over A, in the sense of Definition 2.13(2).*

REMARK 2.19. The definition of generically stable types above is stronger than that of [7, Definition 1.8]. More precisely, a type $p \in S(A)$ is generically stable, in the sense of [7], if and only if all its extensions to $acl(A)$ are generically stable, in the sense of Definition 2.17. This is why our definition implies stationarity, whereas that of [7] does not. However, it is the only difference.

PROPOSITION 2.20. *Let $p \in S(A)$ be a generically stable type. Let B be a set of parameters containing A. Let $q \in S(B)$ be the unique nonforking extension of p. Then, q is still generically stable and, for any $C \supseteq B$, we have $q|_C = p|_C$.*

PROOF. Let $M \supset C$ be a sufficiently saturated model, and let $p' \in S(M)$ be the nonforking extension of the type $p \in S(A)$. We know that q does not fork over A. So, by Proposition 2.10(2), q has an extension $q' \in S(M)$ which does not fork over A. Then, $q'|_A = q|_A = p$. So, by stationarity of p, we have $q' = p'$. So q' is generically stable over A, so a fortiori over B. Then, by Proposition 2.15, $q'|_B = q$ is stationary. Therefore, q' is indeed the unique nonforking extension of q, and q' is generically stable over B. Since we have also proved the equality $q' = p'$, we are done. ⊣

PROPOSITION 2.21 (Transitivity). *Let $p \in S(M)$ be a type generically stable over A. Let B, C be sets of parameters such that $A \subseteq B \subseteq C \subset M$. Let $a \in M$ be a realization of $p|_A$, such that $a \mathop{\smile\!\!\!\!\!\!|}_A B$ and $a \mathop{\smile\!\!\!\!\!\!|}_B C$. Then $a \mathop{\smile\!\!\!\!\!\!|}_A C$.*

PROOF. By stationarity and Proposition 2.20, one can check that $tp(a/C) = p|_C$. ⊣

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PROPOSITION 2.22 (*Symmetry* [7, Theorem A.2, Lemma A.5]).] *Let* $p \in S(A)$ *be generically stable. Let* $q \in S(A)$ *be a type which does not fork over* A*. Let* a, b *be such that* $a \models p|_A$ *and* $b \models q$*. Then* $a \mathop{\smile\!\!\!\!\!\!|}_A b$ *if and only if* $b \mathop{\smile\!\!\!\!\!\!|}_A a$*.*

The following lemma will be used repeatedly throughout the proof of the group configuration theorem.

LEMMA 2.23 (*Swap*). *Let* A *be a set of parameters. Let* b, c, d *be elements such that the types* $tp(b/A)$*,* $tp(c/A)$*, and* $tp(d/A)$ *are generically stable.*

1. *If* $c \mathop{\smile\!\!\!\!\!\!|}_A d$ *and* $b \mathop{\smile\!\!\!\!\!\!|}_A cd$*, then* $bc \mathop{\smile\!\!\!\!\!\!|}_A d$*.*
2. *If* $b \mathop{\smile\!\!\!\!\!\!|}_A c$ *and* $bc \mathop{\smile\!\!\!\!\!\!|}_A d$*, then* $b \mathop{\smile\!\!\!\!\!\!|}_A cd$*.*

PROOF. The first point is actually a consequence of Proposition 2.10(1) and (6), and holds in general. Let us prove the second point. We know that $bc \mathop{\smile\!\!\!\!\!\!|}_A d$, in particular $tp(bc/A)$ does not fork over A. Since $tp(d/A)$ is generically stable, we can apply symmetry, to deduce $d \mathop{\smile\!\!\!\!\!\!|}_A bc$. Also by symmetry, we have $c \mathop{\smile\!\!\!\!\!\!|}_A b$. So, by the first point applied to (d, c, b), we have $dc \mathop{\smile\!\!\!\!\!\!|}_A b$. In particular, $dc \mathop{\smile\!\!\!\!\!\!|}_A A$. So, by symmetry again, we get $b \mathop{\smile\!\!\!\!\!\!|}_A cd$, as required. ⊣

The following lemma can be useful in several contexts.

LEMMA 2.24. *Let* $p \in S(A)$ *be a generically stable type, and let* a *realize* p*. Then, for any infinite cardinal* κ*, there exists a* κ*-saturated and* κ*-strongly homogeneous model* M *containing* A *such that* a *realizes* $p|_M$*.*

PROOF. Let κ be an infinite cardinal. Let $N \supseteq A$ be κ-saturated and κ-strongly homogeneous. Let α realize $p|_N$. Then, we have $\alpha \equiv_A a$, so there exists an automorphism $\sigma \in Aut(\mathbb{U}/A)$ such that $\sigma(a) = \alpha$. Let $M = \sigma(N)$. Then, the model M is κ-saturated and κ-strongly homogeneous, since it is isomorphic to N. Also, we have $\alpha \models p|_N$, and the type $p|_{\mathbb{U}}$ is A-invariant. Since $\sigma \in Aut(\mathbb{U}/A)$, this implies that the element $\sigma(\alpha) = a$ realizes the type $p|_{\sigma(N)} = p|_M$, as required. ⊣

PROPOSITION 2.25. *Let* $p = tp(a/M)$ *be a type generically stable over* A*, where* $A \subseteq M$*.*

1. *If* $b \in dcl(Aa)$*, then the type* $tp(b/M)$ *is generically stable over* A*.*
2. *If* $b \in acl(Aa)$*, then the type* $tp(b/M)$ *is generically stable over* $acl(A)$*.*

PROOF. For the first point, see [6, Proposition 1.2]. Although the statement there only deals with the case $b \in dcl(a)$, the proof can easily be adapted to the hypothesis $b \in dcl(Aa)$.

Let us then prove the second point. Note that both the hypotheses and the conclusion only depend on $tp(ab/M)$. By Lemma 2.24, let $N \succeq M$ be a very[2] saturated and very strongly homogeneous model such that a realizes $p|_N$. By Lemma 2.11, we know that $q = tp(b/N)$ is definable over $acl(A)$, so is a fortiori $acl(A)$-invariant. Let $\phi(y, x, n)$ be a formula with parameters in N such that $\phi(y, a, n)$ isolates the type of b over Na. Let $M_1 \preceq N$ be a $|M|^+$ saturated model containing Mn, and which is small with respect to N.

[2] Finding a cardinal κ such that κ-saturation and κ-strong homogeneity of N imply the existence of a suitable model M_1, is left to the reader.

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We will show that q is generically stable over M_1. Let $r = tp(ab/M_1)$. Then, by construction, we have $p(x) \cup r(x, y) \models q(y)$. So, by Theorem 3.5(3) in [11], the type q is generically stable over M_1. Since it is $acl(A)$-invariant, we conclude by point 5 of Proposition 2.15 that it is generically stable over $acl(A)$, as required. ⊣

2.3. Strong germs. In this subsection, we state useful results on germs of definable maps at generically stable types.

REMARK 2.26. Let M be a model, and $A \subseteq M$ be a small parameter set. Let $p \in S(M)$ be an A-definable type. Let X be an A-definable set, and $(f_b)_{b \in X}$ be an A-definable family of definable maps, such that f_b is defined on $p|_{Ab}$, for every $b \in X$. Then, the equivalence relation on X defined by $b_1 \sim b_2$ if and only if $p|_{Ab_1b_2} \models f_{b_1}(x) = f_{b_2}(x)$ is A-definable, since the type $p|_{Ab_1b_2}$ is definable by the defining scheme of p.

DEFINITION 2.27. In the above context, if b is an element of X, we shall let $[f_b]_p$, or $[f_b]$ if the context is clear, denote the code of the class of the element b for the equivalence relation $\sim$ defined above. We call this code the *germ* of the function f_b at the type p.

In general, the germ of a definable map at a given definable type encodes less information than the code of said definable map. In some sense, it only captures the "local" (for the Stone topology of the type space) behavior of the map.

NOTATION 2.28. *If a type p is definable and admits a unique definable extension to a model, we let $Cb(p)$ denote its canonical basis, i.e., the definable closure of the codes of the formulas in its defining scheme. Similarly, if $tp(a/B)$ is definable and admits a unique definable extension to a model, we write $Cb(a/B)$ for the canonical basis of $tp(a/B)$.*

DEFINITION 2.29. Let $p \in S(A)$ be a definable type which admits a unique A-definable extension to a model, and let f be a definable map, possibly using parameters outside of A. We say that f is defined at p, or well-defined at p, if, for some/any $B \supseteq A$ such that f is B-definable, the function f is defined at $p|_B$.

DEFINITION 2.30. If $p = tp(a/A)$ is a complete type, and h is an A-definable map defined at p, we let h_*p, or $h(p)$, denote the type $h_*p = tp(h(a)/A)$. It is called the image of p under h. Note that this does not depend on the choice of the realization a.

REMARK 2.31. In the definition above, if p is A-invariant (resp. A-definable, resp. generically stable over A), then so is h_*p. Moreover, if p admits a unique A-invariant extension to a sufficiently saturated model, we have $h_*(p|_B) = (h_*p)|_B$ for all $B \supseteq A$.

PROPOSITION 2.32. *Let $p \in S(B)$ be an A-definable type, where $A \subseteq B$, such that, for some/any model $M \supseteq B$, the type p admits a unique A-definable extension $p|_M$. Let a be a realization of p, let $c \in B$ and f_c be an Ac-definable map such that f_c is defined at p. Then, the canonical basis of $tp(af_c(a)/B)$ is interdefinable over A with the definable closure of the set $[f_c]$. In other words, we have the following equality: $dcl(ACb(af_c(a)/B)) = dcl(A[f_c])$.*

PROOF. To simplify notations, let $C = Cb(af_c(a)/B)$. First note that the result does not depend on the choice of the realization a. So, we may assume that a realizes $p|_M$, for some model M containing B.

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Claim 2.33. *We have* $C \subseteq dcl(A[f_c])$.

Proof. Let $b = f_c(a)$. It suffices to show that $tp(ab/M)$ is definable over $A[f_c]$. We know that $tp(a/M)$ is definable over A, so a fortiori over Ac. Also, we have $ab \in dcl(Ac, a)$. Thus, by Lemma 2.11, applied to $Ac \subseteq M = acl(M)$, we know that $q = tp(ab/M)$ is definable over Ac. Let us show that q is invariant over $A[f_c]$, which will be enough to conclude. Let $\sigma \in Aut(M/A[f_c])$. Let us show that $\sigma(q) = q$. By hypothesis on σ, we then have $[f_{\sigma(c)}] = \sigma([f_c]) = [f_c]$. Thus, $p(x) \models f_c(x) = f_{\sigma(c)}(x)$.

Besides, we have $p(x) \cup \{f_c(x) = y\} \models q(x, y)$. Therefore $\sigma(p)(x) \cup \{f_{\sigma(c)}(x) = y\} \models \sigma(q)(x, y)$. But p is A-invariant, and we proved that $p(x) \models f_c(x) = f_{\sigma(c)}(x)$. So $p(x) \cup \{f_c(x) = y\} \models \sigma(q)(x, y)$. Since we also know that $p(x) \cup \{f_c(x) = y\} \models q(x, y)$, we deduce that $\sigma(q)(x, y) = q(x, y)$. Thus, the type $tp(ab/M)$ is definable over $A[f_c]$, as desired. ⊣

Claim 2.34. *We have* $[f_c] \in dcl(AC)$.

Proof. By definition of C, the type q defined by

$$q(x, y) = tp(af_c(a)/M)(x, y),$$

is C-definable. We use compactness. Let d be such that $d \equiv_{AC} c$. It suffices to show that $[f_d] = [f_c]$. We know, by choice of the type q, that $q(x, y) \models f_c(x) = y$. In other words, $\models d_q xy\ (f_c(x) = y)$. Note that the formula $\phi(z) = d_q xy\ (f_z(x) = y)$ is over AC, since q is C-definable and f is A-definable. By choice of d, we have $d \equiv_{AC} c$. Thus $\models d_q xy\ (f_d(x) = y)$. So, if q' is the C-definable extension of q to Md, we have

$$q'(x, y) \models f_d(x) = y \ \wedge \ f_c(x) = y.$$

Finally, $q'(x, y) \models f_d(x) = f_c(x)$, so $p(x)|_{Md} \models f_d(x) = f_c(x)$, i.e., $[f_c] = [f_d]$, as desired. ⊣

So, we have indeed proved that $[f_c]$ and C are interdefinable over A. ⊣

Proposition 2.35 [1, Lemma 2.1]. *Let* $q(x, y) \in S(M)$ *be a type generically stable over* $A \subseteq M$. *Let* a, b *be a realization of* $q(x, y)$.

1. *If* $b \in acl(Ma)$, *then* $b \in acl(Aa)$.
2. *If* $b \in dcl(Ma)$, *then* $b \in dcl(Aa)$.

One then gets the following statement:

Corollary 2.36 (*Strong germs* [1, Theorem 2.2]). *Let* $p \in S(M)$ *be a type generically stable over a set* $A \subseteq M$, *let* X *be an* A*-definable set, and let* $(f_c)_{c\in X}$ *be an* A*-definable family of definable maps, such that, for all* $c \in X$, *the map* f_c *is defined at* p. *Then, there exists an* A*-definable family of definable maps* F *such that, for all* $c \in X$, *we have* $p(x)|_{Ac} \models f_c(x) = F_{[f_c]}(x)$.

Proof. Let $c \in X$, and $a \models p|_{Ac}$. Then, by [1, Theorem 2.2], we have $f_c(a) \in dcl(Aa[f_c])$. We now wish to make this fact uniform in c, i.e., we look for a suitable A-definable family F of definable maps. By compactness, there is a finite collection of A-definable families of definable maps $F_1, \dots, F_k$ such that, for all $c \in X$, we

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have $\models d_p x\, f_c(x) = (F_i)_{[f_c]}(x)$ for *some* $i = 1, \ldots, k$. It then suffices to glue these families of maps together into a single F to conclude the proof. ⊣

REMARK 2.37.

1. In fact, using Proposition 2.32, one can show that Corollary 2.36 is essentially the same as Proposition 2.35. Thus, both statements could be seen as "the strong germs property".
2. In Corollary 2.36, the choice of the definable family F does not matter, as long as it is A-definable and satisfies $p(x)|_{Ac} \models f_c(x) = F_{[f_c]}(x)$ for all $c \in X$. Specifically, the constructions we shall carry out in Sections 3 and 4 do not depend on that choice.
3. The strong germs property will be crucial in the proof of Theorem 3.37. In fact, most of Section 4 will be devoted to the study of the action of germs of definable maps on certain generically stable types. In some sense, considering germs of definable maps enables us to build a type-definable group, instead of an ind-type-definable one. However, defining the group operation, and the action on the space, relies heavily on the strong germs property.

Let us also give two useful facts about definable bijections.

LEMMA 2.38. *Let A be a parameter set, $M \supseteq A$ a model.*

1. *Let $a, b \in M$. Then a and b are interdefinable over A if and only if there exists an A-definable bijection σ such that $\sigma(a) = b$.*
2. *Let $p \in S(M)$ be an A-definable type. Let f, g be M-definable bijections, defined at p. Assume that $[f] = [g]$. Then, the germs $[f^{-1}]$ and $[g^{-1}]$, at the type $f_*(p) = g_*(p)$, are equal.*

PROOF. Let us prove the first point. One direction is straightforward. Assume that $dcl(Aa) = dcl(Ab)$. Let f be an A-definable map sending a to b, and g an A-definable map sending b to a. So, we have $g \circ f(a) = a$ (and $f \circ g(b) = b$). Then, let X be the A-definable set $\{\alpha \mid g \circ f(\alpha) = \alpha\}$. By definition, we have $a \in X$. Hence, the A-definable map $f|_X$ is injective, and sends a to b. One may thus pick $\sigma = f|_X$.

Let us now prove the second point. Let a realize p. So, by hypothesis, we have $f(a) = g(a) =: b$. By construction, the element b realizes the definable type $f_*(p) = g_*(p) \in S(M)$, and we have $f^{-1}(b) = g^{-1}(b) = a$. Thus, b is a witness for the equality of the germs $[f^{-1}]$ and $[g^{-1}]$. ⊣

2.4. Commutativity.

FACT 2.39 (*Commutativity* see [4, Remark 5.18]). *Let $p, q \in S(M)$ be A-invariant types, where A is a small set contained in M. Assume that p is generically stable over A. Then, $p(x) \otimes q(y) = q(y) \otimes p(x)$, this equality being between A-invariant types (see Remark 2.7 for an explanation of this idea).*

DEFINITION 2.40. Let p be a definable type which admits a unique definable extension to a model, and f a definable family of definable maps. We say that an element a acts generically on p via f, if the definable map f_a is well-defined at p, in the sense of Definition 2.29. If the definable family of definable maps f is implicit, we just say that a acts generically on p.

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We say that a type-definable set X acts generically on p if, for some implicitly given f, all elements $a \in X$ act generically on p via f.

Definition 2.41. Let $p(x) \in S(M)$ be an A-invariant type, where $A \subset M$ is a small set, and let $\mathcal{F}$ be a set of invariant types. We say that p *commutes with* $\mathcal{F}$ if, for all invariant types $q(y)$ in $\mathcal{F}$, we have $p(x) \otimes q(y) = q(y) \otimes p(x)$, this being an equality of invariant types. See Remark 2.7 for an explanation of this idea.

We say that $\mathcal{F}$ is a commutative family of types if, for all p in $\mathcal{F}$, the type p commutes with $\mathcal{F}$.

Corollary 2.42. *Let $p \in S(M)$ be a B-invariant type, where $M \supseteq B$ is a model, and B is small with respect to M. Let h be a B-definable map such that h is defined at p, and $\mathcal{F}$ a family of invariant types, in the sense of Remark 2.7. If p commutes with $\mathcal{F}$, then $h_* p$ commutes with $\mathcal{F}$.*

Proof. Let $q(y)$ be an element of $\mathcal{F}$. By hypothesis on $\mathcal{F}$, there exists a small set $C \supseteq B$ such that $q(y)$ is C-invariant. Let us show that $h_* p(x) \otimes q(y) = q(y) \otimes h_* p(x)$. Let $D \supseteq C$, and let (k, b) realize $h_* p(x) \otimes q(y)|_D$.

Then, k realizes $h_* p|_{Db}$. So, by Remark 2.31 applied to Db, there exists a realizing $p|_{Db}$ such that $h(a) = k$. So (a, b) realizes $p \otimes q|_D$. Since p commutes with $\mathcal{F}$, the pair (b, a) realizes $q \otimes p|_D$, so $b \models q|_{Da}$, a fortiori $b \models q|_{Dh(a)} = q|_{Dk}$. Therefore, (b, k) realizes $q \otimes h_* p|_D$. We have proved that $h_* p$ commutes with $\mathcal{F}$. ⊣

Remark 2.43. These notions give us some form of symmetry for tensor products of generically stable types; see Lemma 4.14 for an example of how this commutativity is used. However, we do not know if the class of generically stable types is closed under tensor products, outside well-behaved theories, e.g., NIP.

§3. The group configuration theorem.

3.1. Genericity and group configurations. Here, we define a notion of genericity for definable types concentrating on a type-definable group G, or G-space X. We then define group configurations, and explain how to build such using generic types. Few of the results are new, except maybe Propositions 3.10 and 3.14 in the case of G-spaces, which are well-known for stable theories. For more results, and a more general framework allowing definable *partial* types to be generic, see Section 3 in [10].

Definition 3.1.

1. A type-definable group Γ is given by a type-definable set, along with a relatively definable map $m : \Gamma \times \Gamma \to \Gamma$ which defines a group operation.
2. Let G be a type-definable group, and X a type-definable space on which G acts definably. Assume everything is defined over some set A. Let $\cdot$ denote both the group operation of G, and the action of G on X.

   Let $B \supseteq A$, and $p, q \in S(B)$ be definable types concentrating on X. We define $Stab_\phi(p, q)$ by the formula $\forall y\, [d_q x\, \phi(g \cdot x, y) \leftrightarrow d_p x\, \phi(x, y)]$. We then define $Stab(p, q)$ as the intersection of all the $Stab_\phi(p, q)$ with G. If $p = q$, we write $Stab(p)$ instead of $Stab(p, p)$. In the case where $p, q \in G$, we also define the right stabilizer $Stab^r(p, q)$ by considering the right action by translations, and similarly for $Stab^r(p)$.

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Remark 3.2. Let $p, q \in S(B)$ be as in the definition above. Let M be a model over which everything is defined. Then $Stab(p, q)(M)$ is precisely the set of elements $g \in G(M)$ such that $g \cdot p|_M = q|_M$. Also, $Stab(p)$ is a type-definable subgroup of G.

Definition 3.3. Let G be a type-definable group acting definably on a type-definable space X. Let M be a sufficiently saturated model over which everything is defined.

1. Let $H \leq G$ be a type-definable subgroup. We say that H is of bounded index in G if the cardinality of G/H is bounded, i.e., does not grow beyond a fixed cardinal, regardless of the size of the model.
2. Let $p \in S(M)$ be a definable type. We say that p is a definable generic of the G-space X if $p(x) \models$ "$x \in X$", and $Stab(p)$ is of bounded index in G. Letting G act definably and regularly on itself by left translations, we can also speak of definable generic types in G.
3. We say that the space X is type-connected if it has a definable generic type over M whose stabilizer is G itself. It is generically stable if it has a generically stable generic. The group G is type-connected (resp. generically stable) if it is type-connected (resp. generically stable) for the left regular action by translations.

Remark 3.4.

1. Other, weaker notions of genericity have been developed. For instance, there is a notion of f-genericity, which relies on forking rather than definable types (see [12, Definition 3.3]). However, in this paper, we will only be interested in definable generics. Thus, we shall call them "generics".
2. In this paper, we shall only use the notion of type-connectedness defined above. There also exists a notion of connectedness, where the emphasis is on relatively definable subgroups of finite index, instead of type-definable subgroups of bounded index.

Lemma 3.5. *Let G be a type-definable group. Let X, Y be type-definable G-spaces, and $f : X \to Y$ be a definable G-equivariant map. Let $p \in S_X(M)$ be a definable type, where M is a model containing all the parameters involved. Then, $Stab(f_*(p)) \geq Stab(p)$. In particular, if p is generic in X, then $f_*(p)$ is generic in Y.*

Proof. We may assume that M is sufficiently saturated. Let $c \in Stab(p)(M)$. Then, we compute $c \cdot f_*(p) = f_*(c \cdot p) = f_*(p)$, so $c \in Stab(f_*(p))$, as required. ⊣

Proposition 3.6. *Let G be a type-definable group with a definable generic type. The following are equivalent:*

1. *The group G is type-connected.*
2. *The group G has no type-definable proper subgroup of bounded index.*

It these hold, then, for any definable generic type p, we have $Stab(p) = G$.

Proof. The implication 2. $\implies$ 1. is straightforward: by definition, the stabilizer of any generic type is of bounded index. Let us prove 1. $\implies$ 2. Let p be a generic type for G, whose stabilizer is G itself. Let $H \leq G$ be a type-definable subgroup

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of bounded index. Then, if M is a sufficiently saturated model containing all the parameters involved, it represents every coset of H. Then, $p|_M$ concentrates on a coset of H. So, the stabilizer $Stab(p)$ is contained in a conjugate of H. As $Stab(p) = G$, we deduce that $H = G$, as desired. ⊣

LEMMA 3.7. *Let G be a type-definable group, defined over a set of parameters A. Let $p \in S(A)$ be a generically stable generic type for G, such that $Stab(p) = G$.*

1. *Let $B \supseteq A$ and a realizing $p|_B$. Then, the element a^{-1} realizes $p|_B$. In other words, we have $p^{-1} = p$.*
2. *Let $g \in G$ and a realizing $p|_{Ag}$. Then, the element $a \cdot g$ realizes $p|_{Ag}$. In other words, the right stabilizer of p is also equal to the whole group G.*
3. *The type p is the unique generic type of the group G.*
4. *Any element of G is the product of two realizations of p.*

PROOF. Let α, β realize $(p \otimes p)|_B$. Then, since $\beta^{-1} \in Stab(p) = G$ and $\alpha \models p|_{B\beta}$, we know that $\beta^{-1} \cdot \alpha$ realizes $p|_B$. Then, by Fact 2.39, we know that $\alpha\beta \equiv_B \beta\alpha$. So $\alpha^{-1} \cdot \beta$ realizes $p|_B$. Then, since $p|_B$ is a complete type, we have shown that, for all elements c realizing $p|_B$, the element c^{-1} realizes $p|_B$, as desired.

Let us then prove the second point. If g is in G and a realizes $p|_{Ag}$, then, by the first point, we know that a^{-1} realizes $p|_{Ag}$. So, by hypothesis on the stabilizer of p, the element $g^{-1} \cdot a^{-1}$ realizes $p|_{Ag}$. Then, again by the first point, the element $a \cdot g = (g^{-1} \cdot a^{-1})^{-1}$ realizes $p|_{Ag}$, as stated.

For the third point, see [15, Lemma 2.1]. Finally, let us prove the fourth point. Let $g \in G$, and $a \models p|_{Ag}$. Then, we have $g = (g \cdot a) \cdot a^{-1}$,where $g \cdot a$ realizes p because $Stab(p) = G$, and a^{-1} realizes p by the first point. ⊣

REMARK 3.8. Without generic stability, there may be several generics whose stabilizers are equal to G itself. For instance, in DOAG or RCF, the definable types at $+\infty$ and $-\infty$ both satisfy $Stab(p) = G$, where G is the additive group.

COROLLARY 3.9 [10, Lemma 3.9]. *Let G be a generically stable type-definable group. Then, the type-connected component G^{00} of G, i.e., the smallest type-definable subgroup of bounded index, exists. The group G^{00} is the stabilizer of any generic type of G, and has a unique generic type.*

PROOF. Let us first prove the existence of the type-connected component G^{00}. Let $p \in S(N)$ be a generically stable generic type for G, where N is sufficiently saturated. Then, as $H = Stab(p)$ is of bounded index, every left coset and every right coset of $Stab(p)$ is represented in N. So p concentrates on some left coset of H, and on some right coset of H as well. Let $g \in G(N)$ be such that p concentrates on $H \cdot g$. Now, let q be the translate $p \cdot g^{-1}$. Then, the type q concentrates on H. Also, by Proposition 2.25, it is generically stable.

Now, by Lemma 3.5, we have $Stab(q) \geq Stab(p) = H$. Then, since q concentrates on H, a fortiori it concentrates on $H_1 := Stab(q)$. So, by Proposition 3.6, the type-definable group H_1 has no proper type-definable subgroup of bounded index. Moreover, as $H \leq H_1 \leq G$, we know that H_1 is of bounded index in G. Hence, H_1 is indeed the smallest type-definable subgroup of bounded index of G. This implies

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that $H_1 = G^{00}$ is normal, and even invariant under definable automorphisms, in G, and the inclusion $H \le H_1$ implies that $H_1 = H = G^{00}$.

Since q is generically stable, concentrates on G^{00}, and is stabilized by G^{00}, we may apply Lemma 3.7, to deduce that the group G^{00} has a unique generic type, which is q.

Now, let p_1 be some generic of G. By definition, $Stab(p_1)$ is of bounded index, so it contains G^{00}. On the other hand, the complete type p_1 concentrates on some coset of G^{00}, so $Stab(p_1)$ is contained in some conjugate of G^{00}. Since the latter is normal, we have in fact $Stab(p_1) = G^{00}$, as desired. ⊣

Proposition 3.10. *Let G be a generically stable type-definable group acting definably and transitively on a type-definable space X.*

1. *If G is type-connected, then X has a unique generic type, whose stabilizer is G.*
2. *In general, the space X has generically stable generics, they are left translates of each other, and all definable generics are generically stable.*

Proof. Let us show that the second point follows from the first. We know that the type-connected component G^{00} of G exists: it is the stabilizer of any generic of G. Then, we consider the action of G^{00} on X. By the first point, each G^{00}-orbit contains a unique generic type, whose stabilizer is G^{00}. Note that, since G^{00} is of bounded index in G and the action is transitive, there are only boundedly many G^{00}-orbits. Now, let M be a sufficiently saturated model over which everything is defined. So M contains a point in each G^{00}-orbit. Let $q_1, q_2 \in S(M)$ be two generic types of X. Let $x_1, x_2 \in X(M)$ be in the G^{00}-orbits of (the realizations of) q_1 and q_2 respectively, and let $g \in G(M)$ be such that $g \cdot x_1 = x_2$. We shall prove that g sends the type q_1 to q_2. Since G^{00} is normal, we have $g \cdot G^{00} = G^{00} \cdot g$. Thus, we can compute $G^{00}(M) \cdot g \cdot q_1 = g \cdot G^{00}(M) \cdot q_1 = g \cdot q_1$, so the type $g \cdot q_1$ is generic. Also, since g sends x_1 to x_2, the type $g \cdot q_1$ concentrates on the same G^{00}-orbit as q_2. Hence, by the first point, we have $g \cdot q_1 = q_2$, as desired.

Let us now prove the first point. Assume that G is type-connected. By Lemma 3.7, let p be the unique generic type of G. We know that p is generically stable. Let M be a big enough model containing all the parameters involved, and let $x_0 \in X(M)$. Let $f : G \to X$ be the definable map $g \mapsto g \cdot x_0$. By transitivity of the action, it is onto. Let q be the type $f_*(p)$. By Proposition 2.25, it is generically stable. We know that $Stab(q)(M) = G(M)$, because $Stab(p) = G$. So, since M is sufficiently saturated, we have $Stab(q) = G$, so q is generic in X.

Now, let q_1 be another generic type in X. Without loss of generality, we may assume that q_1 is M-definable. We want to show that $q_1 = q$. Let $x_1 \models q_1|_M$, and $g_1 \in G$ such that $f(g_1) = x_1$. Let $g \models p|_{Mg_1}$. Then, by Lemma 3.7, we have $g \cdot g_1 \models p|_{Mg_1}$, so $f(g \cdot g_1) \models f_*(p)|_{Mg_1} = q|_{Mg_1}$. In particular, we have $f(g \cdot g_1) = g \cdot x_1 \models q|_M$. On the other hand, since q_1 is generic in X, and G is type-connected, we have $Stab(q_1) = G$. Moreover, by Fact 2.39, we have $(x_1, g) \models (q_1 \otimes p)|_M$. So $g \cdot x_1 \models q_1|_{Mg}$. Therefore, $g \cdot x_1$ realizes both $q|_M$ and $q_1|_M$. So $q = q_1$, as desired. ⊣

Definition 3.11. Let A be a set of parameters. A *regular group configuration* over A is a tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ of elements satisfying the following properties:

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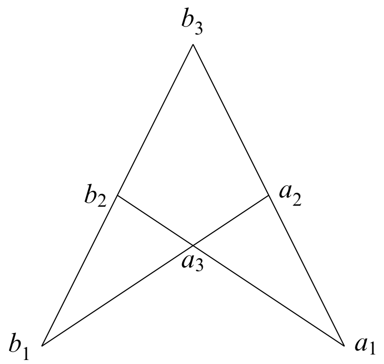

1. If α, β, γ are three non-colinear points in the diagram above, then the triplet (α, β, γ) is an independent family over A.
2. If α, β, γ are three colinear points in the diagram above, then $\alpha \in \mathrm{acl}(A\beta\gamma)$.

A *definable group configuration* over A is a tuple of elements $(a_1, a_2, a_3, b_1, b_2, b_3)$ satisfying the following properties:

1. The type $tp(a_1a_2a_3b_1b_2b_3/A)$ is definable.
2. If α, β, γ are three non-colinear points in the diagram above, then the types $tp(\alpha\beta/A\gamma)$ and $tp(\alpha/A\beta\gamma)$ are A-definable.
3. The equalities $acl(Ab_1b_2) = acl(Ab_1b_3) = acl(Ab_2b_3)$ hold.
4. For all natural numbers i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, the elements a_j and a_k are interalgebraic over Ab_i, and the element b_i is interalgebraic over A with the canonical basis $Cb(a_ja_k/acl(Ab_i))$.

A *generically stable (resp. generically stable regular) group configuration* over A is a definable (resp. regular) group configuration $(a_1, a_2, a_3, b_1, b_2, b_3)$ over A, such that the type $tp(a_1a_2a_3b_1b_2b_3/A)$ is generically stable.

We might call "quadrangle" a 6-tuple of elements which has not been proven to be a definable or regular group configuration (yet).

Remark 3.12.

1. Recall that, by Definition 2.5(5), a type $tp(a/Ab)$ is A-definable if and only if it admits an A-definable extension to a model. This implies that $a \mathop{\smile\!\!\!\!\!\!\;|}_A b$. In particular, this implies that, in definable group configurations, non-colinear triples are independent families.
2. In the case of a generically stable 6-tuple, independence over M of the non-colinear triplets can be checked more easily, using Lemma 2.23 and symmetry. For instance, the set $\{a_1, a_2, a_3\}$ being independent over M is equivalent to having $a_1 \mathop{\smile\!\!\!\!\!\!\;|}_M a_2$ and $a_3 \mathop{\smile\!\!\!\!\!\!\;|}_M a_1a_2$.
3. In Proposition 3.14, we will show how one can construct such group configurations, if given generically stable generics of type-definable groups, resp. spaces. The converse, i.e., recovery of a group from a group configuration, is the point of group configuration theorems. In this paper, we shall prove Theorem 3.37.

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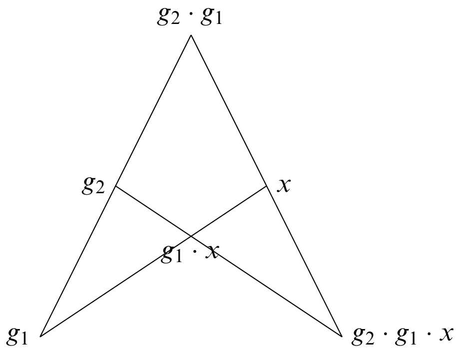
4. Naturally, the fact that a quadrangle is a (regular, or definable, etc.) group configuration over a set A only depends on its type over A. In fact, we shall often consider several copies of the same configuration.

DEFINITION 3.13 (Equivalent quadrangles). Let A be a set of parameters. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ and $(a'_1, a'_2, a'_3, b'_1, b'_2, b'_3)$ be quadrangles. We say that these quadrangles are *equivalent over* A, or *interalgebraic over* A if, for $i = 1, 2, 3$, we have $acl(Aa_i) = acl(Aa'_i)$ and $acl(Ab_i) = acl(Ab'_i)$.

PROPOSITION 3.14. *Let G be a type-definable type-connected group acting definably on a type-definable type-connected space X. Assume that the action is free (resp. faithful) and transitive. Let p, q be the generics of G and X respectively. Assume that p and q are generically stable. Let (g_1, g_2, x) be a triplet realizing $p^{\otimes 2} \otimes q|_M$, where M is a sufficiently saturated model over which everything is defined. Then, the following family is a regular group configuration (resp. a definable group configuration) over M:*



DEFINITION 3.15. Such a quadrangle is called a "group configuration for (G, X) over M".

REMARK 3.16. Note that, by Proposition 2.25, a quadrangle such as above is generically stable over M if and only if the tensor product $p^{\otimes 2} \otimes q$ is generically stable over M.

PROOF OF PROPOSITION 3.14. The algebraicity relations are clear in the case of a free action. In fact, let us deal with the more subtle case of a faithful transitive action. Since any free action is in particular faithful, and using Remark 3.12(1), our proof will also include a proof of the independence relations for the case of a free action.

First, note that the type $tp(g_1, g_2, x/M)$ is definable, since it is the tensor product $p \otimes p \otimes q$. Then, by Lemma 2.11, the type over M of the sextuple is definable. This is the first point of the definition. The third point is easier to check, since $b_1 = g_1$, $b_2 = g_2$ and $b_3 = g_2 \cdot g_1$.

Let us now prove the second point of the definition. First, note that $tp(g_1 \cdot x/M) = tp(g_2 \cdot g_1 \cdot x/M) = tp(x/M) = q$, because $Stab(q) = G$, and $x \models q|_{Mg_1g_2}$. Similarly, $tp(g_2 \cdot g_1/M) = p$. So, to prove the second point, we may use stationarity and commutativity for generically stable types. By saturation of M, let $x_0 \in X(M)$, and let $f : G \to X$ be the definable map $g \mapsto g \cdot x_0$.

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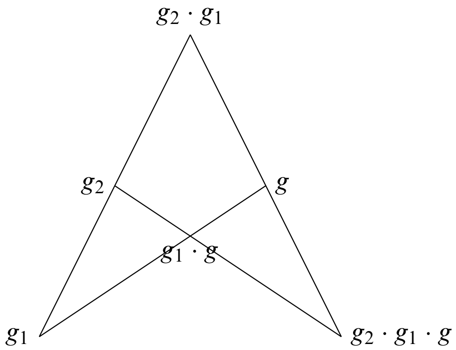
Claim 3.17. *We have $f_*(p) = q$. In particular, there exists $g \models p|_{Mg_1g_2}$ such that $f(g) = x$.*

Proof. By transitivity of the action, this map is onto. Since p is generic in G, by Lemma 3.5, the type $f_*(p)$ is generic in X. Thus, uniqueness of the generic of X (Proposition 3.10(1)) implies that $f_*(p) = q$. The existence of g then follows from the fact that x realizes $q|_{Mg_1g_2}$. ⊣

Claim 3.18. *To prove the second point of the definition, it suffices to do it for the following quadrangle*:



Proof. This follows from the fact that f is M-definable, with $f(g) = x$, and Proposition 2.10(3)(b). ⊣

Note that, by choice of g, we have $(g_1, g_2, g) \models p^{\otimes 3}|_M$. In other words, by equivariance of f, we reduced to the case $X = G$, with the action by left translation. Recall that, by Lemma 3.7, we have $Stab^r(p) = G = Stab(p)$. Also note that the setting is more symmetric now: we only have elements in the group, they are all generic, and they come from (products of) realizations of a generically stable type, so that we may use commutativity of the tensor product. The proof below relies on this symmetry.

Claim 3.19. *It suffices to show that, for any point a in the diagram, and any line l which avoids a, we have $a \perp\!\!\!\perp_M l$, i.e., a is independent from the triple of elements in l over M.*

Proof. This follows from commutativity, stationarity, and the fact that any point on a line l is definable over M and the other two points of l. ⊣

Now, consider the alphabet $\Sigma = \{g_1, g_2, g\}$. Elements in the diagram above can be represented as words in that alphabet, where the letters only come in increasing order, if we set $g_2 < g_1 < g$.

Claim 3.20. *For any point a, for any line l avoiding a, there are two points b, c on the line l such that either the first letters of the words representing a, b, c are pairwise distinct, or the last letters are.*

Proof. For each element, there are two lines to check. The twelve verifications are left to the reader. ⊣

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CLAIM 3.21. *Let a, b, c be points in the diagram. Then*:

- *If a begins with g_2, b begins with g_1, and c begins with g, then we have $a \mathop{\smile\!\!\!\!\!\!\;|}_M bc$ and $b \mathop{\smile\!\!\!\!\!\!\;|}_M c$.*
- *Similarly, if a ends with g, b ends with g_1 and c ends with g_2, then we have $a \mathop{\smile\!\!\!\!\!\!\;|}_M bc$ and $b \mathop{\smile\!\!\!\!\!\!\;|}_M c$.*

PROOF. The first result follows from genericity of g_2 over Mg_1g, genericity of g_1 over Mg, and the fact that letters only appear in increasing order, with $g_2 < g_1 < g$. Similarly, the second result follows from genericity of g over Mg_1g_2, and genericity of g_1 over Mg_2. ⊣

To conclude the proof of the second point of the definition of generically stable group configurations, it suffices to combine Claims 3.19, 3.20, and 3.21, keeping in mind Proposition 2.10(3)(b), and the fact that any element on a line is definable over M and the other two.

Finally, let us prove the fourth point of the definition of a definable group configuration. The first part follows from definability over M of the group action. Let us now prove the statements dealing with the canonical bases. We follow [14, Chapter 5, Remark 4.1]. Since the action is faithful, we may harmlessly identify any element of G with the permutation of X it defines. Then, by Proposition 2.32, to prove that, say g_1 is interalgebraic over M with $Cb(x, g_1 \cdot x/acl(Mg_1))$, it suffices to prove that g_1 is interalgebraic over M with its germ at q. Let us show that, in fact, for any $h \in G$, the element h is definable over $M^\frown[h]$.

CLAIM 3.22. *Let $\gamma_1, \beta_1, \gamma_2, \beta_2 \in G$ be such that $[\gamma_1] = [\gamma_2]$ and $[\beta_1] = [\beta_2]$. Then, we have $[\gamma_1 \cdot \beta_1] = [\gamma_2 \cdot \beta_2]$.*

PROOF. Let a be a realization of $q|_{M\gamma_1\gamma_2\beta_1\beta_2}$. Then, since $Stab(q) = G$ and $[\beta_1] = [\beta_2]$, we have $\beta_1 \cdot a = \beta_2 \cdot a \models q|_{M\gamma_1\gamma_2\beta_1\beta_2}$. Since $[\gamma_1] = [\gamma_2]$, this implies that $\gamma_1 \cdot \beta_1 \cdot a = \gamma_2 \cdot \beta_2 \cdot a$, as required. ⊣

Thus, the type-definable set Γ, whose elements are the germs at q of the form $[g]$, for $g \in G$, can be equipped with a definable group operation in a natural way, such that the map $g \in G \mapsto [g] \in \Gamma$ is an M-definable surjective group homomorphism. Moreover, by the strong germs property, i.e., Corollary 2.36, the generic action of G on q induces a generic action of Γ on q.

CLAIM 3.23. *If $g \in G$ is such that $[g] = 1$, then $g = 1$.*

PROOF. Let $g \in G$ be such that $[g] = 1$. Let $a \in X$ be arbitrary. We want to show that $g \cdot a = a$, which will imply that $g = 1$. Let $b \models q|_{Mg}$. By transitivity, let $h \in G$ be such that $h \cdot b = a$. Let $\gamma \models p|_{Mg^\frown h^\frown a^\frown b}$. Then, by Lemma 3.7(2), the elements $\gamma \cdot h$ and $\gamma \cdot g \cdot h$ are generic over Mb. So, by commutativity for generically stable types, we have $b \models q|_{M\gamma \cdot h}$ and $b \models q|_{M\gamma \cdot g \cdot h}$. Now, since germs can be composed, and $[g] = 1$, we have $[\gamma \cdot g \cdot h] = [\gamma \cdot h]$. Also, since $b \models q|_{M\gamma \cdot h}$ and $b \models q|_{M\gamma \cdot g \cdot h}$, we have $[\gamma \cdot h] \cdot b = (\gamma \cdot h) \cdot b$ and $[\gamma \cdot g \cdot h] \cdot b = (\gamma \cdot g \cdot h) \cdot b$. As $[\gamma \cdot g \cdot h] = [\gamma \cdot h]$, we deduce that $\gamma \cdot g \cdot h \cdot b = \gamma \cdot h \cdot b$, i.e., $\gamma \cdot g \cdot a = \gamma \cdot a$. So $g \cdot a = a$. As a was arbitrary, faithfulness implies that $g = 1$, as desired. ⊣

Thus, the M-definable group homomorphism $g \in G \mapsto [g] \in \Gamma$ is an isomorphism. This implies in particular that, for any $h \in G$, we have $dcl(Mh) = dcl(M[h])$. ⊣

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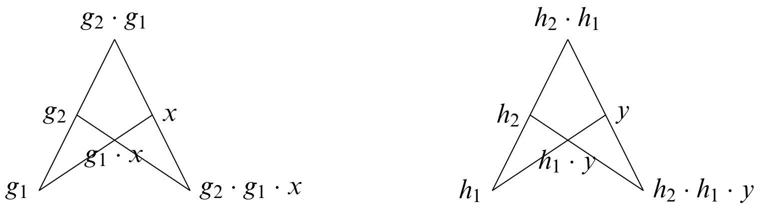
In fact, such a configuration captures the structure of the group and its action, up to some notion of correspondence.

PROPOSITION 3.24. *Let (G, X) and (H, Y) be type-definable transitive faithful actions, where G, H and X, Y are type-connected, type-definable with generically stable generics. Let M be a sufficiently saturated model over which everything is defined. Let $(g_1, g_2, g_2 \cdot g_1, g_2 \cdot g_1 \cdot x, x, g_1 \cdot x)$ and $(h_1, h_2, h_2 \cdot h_1, h_2 \cdot h_1 \cdot y, y, h_1 \cdot y)$ be configurations built as in Proposition 3.14, for (G, X) and (H, Y) respectively. Assume that these configurations are equivalent over M. Then, there exist type-definable sets $S \leq G \times H$ and $T \subseteq X \times Y$, and finite normal subgroups $N_1 \lhd G$, $N_2 \lhd H$ such that:*

1. *The projection of the subgroup S to $G/N_1 \times H/N_2$ is the graph of a group isomorphism $G/N_1 \simeq H/N_2$.*
2. *The set T is an S-invariant finite-to-finite surjective correspondence between X and Y.*

PROOF. Let $C \subset M$ be a small algebraically closed set of parameters over which everything is defined, and which captures the interalgebraicities. Thus, we have the following configurations, which are equivalent over C:



For $1 \leq i \leq 2$, let $c_i = (g_i, h_i) \in G \times H$. Also, let $c_3 = c_2 \cdot c_1 = (g_2 \cdot g_1, h_2 \cdot h_1) \in G \times H$. Let $p_i = tp(c_i/M)$, for $1 \leq i \leq 3$.

CLAIM 3.25. *The p_i are generically stable over $acl(C) = C$.*

PROOF. Recall that, by construction, we have $tp(g_1/M) = tp(g_2/M) = tp(g_2 \cdot g_1/M)$, and this type is the unique generically stable generic of G. The last equality follows from the fact $g_2 \in G = Stab(tp(g_1/M))$ and $tp(g_1/Mg_2)$ is the generic of G. So, this type is generically stable over $C = acl(C)$. Then, by interalgebraicity and Proposition 2.25, the types p_i are generically stable over $acl(C) = C$. ⊣

By assumption and interalgebraicity, we have $c_2 \mathop{\smile\!\!\!\!\!\!\mid}_M c_1$ and $c_3 \mathop{\smile\!\!\!\!\!\!\mid}_M c_1$. Moreover, by definition, we have $c_2 \cdot c_1 = c_3$. Thus, we have $c_1 = (g_1, h_1) \in Stab^r(p_2, p_3)$. Let us also define $S = Stab^r(p_2)$ and $Z = Stab^r(p_2, p_3)$. So S is a type-definable subgroup of $G \times H$. Let $c_1' \in M$ be such that $c_1' \equiv_C c_1$. So, there exist $g \in G(M)$ and $h \in H(M)$ such that $c_1' = (g, h) \in G \times H$.

CLAIM 3.26. *The following equality of type-definable sets holds: $S \cdot c_1' = Z$.*

PROOF. Let $s \in S$. Let γ realize $p_2|_{Cc_1's}$. Then, by definition of S as a stabilizer, we have $\gamma \cdot s \models p_2|_{Cc_1's}$. Moreover, we know that $c_1 \in Stab^r(p_2, p_3)$, so c_1' is in

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$Stab^r(p_2, p_3)$ as well. Thus, as $\gamma \cdot s$ realizes $p_2|_{Cc_1' s}$, we deduce that $\gamma \cdot s \cdot c_1'$ realizes $p_3|_{Cc_1' s}$. Since we chose γ realizing $p_2|_{Cc_1' s}$, we can conclude that $s \cdot c_1' \in Stab^r(p_2, p_3) = Z$. As $s \in S$ was arbitrary, we have just proved that $S \cdot c_1' \subseteq Z$. The other inclusion is proved similarly, by picking an arbitrary element α in Z, and letting the product $\alpha \cdot c_1'^{-1}$ act by right-translation on (a realization of) p_2. ⊣

Let $\pi : G \times H \to G$ be the canonical projection. Since π is a group morphism, the claim implies that $\pi(S) \cdot \pi(c_1') = \pi(Z)$. In particular, as $c_1 = (g_1, h_1) \in Z$, we get $g^{-1} \cdot g_1 \in \pi(S)$. However, by construction, g_1 is, in the group G, generic over M. Since g is in M, this implies that $g^{-1} \cdot g_1$ is also generic over M. Therefore, the M-type-definable subgroup $\pi(S) \leq G$ contains an element generic over M. So, the generic of G concentrates on $\pi(S)$. Since, by Lemma 3.7(4), any element of G is the product of two realizations of this generic type, we deduce that $\pi(S)$ is equal to the whole group G. Symmetrically, the projection of S to the second coordinate is equal to H.

Let $N_1 = \{n_1 \in G \mid (n_1, 1) \in S\}$ and $N_2 = \{n_2 \in H \mid (1, n_2) \in S\}$. We then have $N_1 \times \{1\} \lhd S$, so $N_1 = \pi(N_1 \times \{1\}) \lhd \pi(S) = G$. Similarly, we have $N_2 \lhd H$. Thus, projecting the subgroup $S \leq G \times H$, we get a subgroup $\Sigma \leq G/N_1 \times H/N_2$, which is the graph of a definable group isomorphism $G/N_1 \to H/N_2$. Indeed, since S projects surjectively onto G and H, the subgroup Σ projects surjectively onto G/N_1 and H/N_2.

Claim 3.27. *The groups N_1 and N_2 are finite.*

Proof. Let us show that N_1 is finite; the proof for N_2 will be symmetric. Let $n_1 \in N_1(M)$. Then, we know that $c_2 = (g_2, h_2)$ realizes $p_2|_{Cn_1}$. Then by definition of S, the element $(g_2 \cdot n_1, h_2)$ realizes $p_2|_{Cn_1}$. Recall that the configurations $(g_1, g_2, g_2 \cdot g_1, g_2 \cdot g_1 \cdot x, x, g_1 \cdot x)$ and $(h_1, h_2, h_2 \cdot h_1, h_2 \cdot h_1 \cdot y, y, h_1 \cdot y)$ are equivalent over C. In particular, the elements g_2 and h_2 are interalgebraic over C, so we also have $g_2 \cdot n_1 \in acl(Ch_2)$. Thus, since we are in a group, we deduce $n_1 \in acl(Cc_2)$. However, we know that $c_2 \mathop{\smile\!\!\!\!\!\!|}_C n_1$. So, by Proposition 2.10(4), we have $n_1 \in acl(C)$. As $n_1 \in N_1$ was arbitrary, we deduce by compactness that N_1 is finite. ⊣

Now, by saturation of M, let $(x_0, y_0) \in X(M) \times Y(M)$ be such that $(x_0, y_0) \equiv_C (g_1 \cdot x, h_1 \cdot y)$. Let us consider the following type-definable set: $T = S \cdot (x_0, y_0) \subseteq X \times Y$. Let $q_3 = tp(g_1 \cdot x, h_1 \cdot y/M)$ and $q_1 = tp(g_2 \cdot g_1 \cdot x, h_2 \cdot h_1 \cdot y/M)$.

Claim 3.28. *The types $tp(g_1 \cdot x/M)$ and $tp(g_2 \cdot g_1 \cdot x/M)$ are generically stable over C.*

Proof. Let η_X denote the generically stable type which is the unique (by the first point of Proposition 3.10) generic of the space X. By assumption, and symmetry of the tensor product for generically stable types, the element x realizes $\eta_X|_{Mg_1g_2}$. Then, by Lemma 2.24, there exists a model N containing Mg_1g_2 such that x realizes $\eta_X|_N$. Then, by the first point of Proposition 3.10, the stabilizer of η_X is G itself, so the elements $g_1 \cdot x$ and $g_2 \cdot g_1 \cdot x$ also realize $\eta_X|_N$. Then, the types $tp(g_1 \cdot x/N)$ and $tp(g_2 \cdot g_1 \cdot x/N)$ are generically stable over C. In particular, their restrictions to C are generically stable, and we have $g_1 \cdot x \mathop{\smile\!\!\!\!\!\!|}_C N$ and $g_2 \cdot g_1 \cdot x \mathop{\smile\!\!\!\!\!\!|}_C N$. A fortiori, this implies $g_1 \cdot x \mathop{\smile\!\!\!\!\!\!|}_C M$ and $g_2 \cdot g_1 \cdot x \mathop{\smile\!\!\!\!\!\!|}_C M$. We conclude using Proposition 2.20. ⊣

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Then, by equivalence over C of the configurations, Proposition 2.25 implies that q_1 and q_3 are generically stable over $C = acl(C)$. Also, by hypothesis, we have $(g_2, h_2) \in Stab(q_3, q_1)$.

Claim 3.29. *The set T is closed under the action of $S \leq G \times H$. Moreover, for all $x_1 \in X$, $y_1 \in Y$, the sets $T \cap (\{x_1\} \times Y)$ and $T \cap (X \times \{y_1\})$ are finite and nonempty.*

Proof. Since T is the S-orbit of a point in the space $X \times Y$, it is closed under the action of S. Moreover, we have proved that S projects onto G and onto H. Therefore, by transitivity of the actions of G on X, and of H on Y, the projections $T \to X$ and $T \to Y$ are onto.

Thus, it remains to prove finiteness of the fibers, so to speak. Let S act on Y via the formula $\gamma \cdot y = h \cdot y$, for $\gamma = (g, h) \in S$ and $y \in Y$. Also, for any $y \in Y$, let T_y denote $T \cap (X \times \{y\})$. By symmetry, it suffices to prove that, for any $y_1 \in Y$, the type-definable set T_{y_1} is finite. Note that, if $y_1, y_2 \in Y$, if $\gamma \in S$ is an element such that $\gamma \cdot y_1 = y_2$, then $\gamma \cdot T_{y_1} \subseteq T_{y_2}$. Then, since we also have $\gamma^{-1} \cdot y_2 = y_1$, the equality $\gamma \cdot T_{y_1} = T_{y_2}$ holds. Note that, since S projects onto H, and the latter acts transitively on Y, so does S. Thus, it suffices to prove that T_{y_1} is finite, for *some* $y_1 \in Y$. We shall prove that T_{y_0} is finite.

By compactness and saturation of M, it suffices to prove that, if $(a, y_0) \in T_{y_0}(M)$, then $a \in acl(Cy_0)$. So, let $a \in X(M)$ be such that $(a, y_0) \in T_{y_0}$. Let us prove that $a \in acl(Cy_0)$. By saturation of M, and definition of T, there exists $(g, h) \in S(M)$ such that $(a, y_0) = (g \cdot x_0, h \cdot y_0)$. In particular, we have $h \cdot y_0 = y_0$. Now, recall that $Stab^r(p_2) = S$, and $(g_2, h_2) \models p_2|_M$. Therefore, we have $(g_2 \cdot g, h_2 \cdot h) \models p_2|_M$. In particular, as $(g_2, h_2) \in Stab(q_3, q_1)$, the element $c := (g_2 \cdot g, h_2 \cdot h)$ is also in $Stab(q_3, q_1)$. Also, by commutativity for the generically stable types $q_3|_C$ and $p_2|_C$, we have $(x_0, y_0) \models q_3|_{Cc}$. Thus, $c \cdot (x_0, y_0) \models q_1|_{Cc}$. In particular, by equivalence over C of the configurations, we have $g_2 \cdot g \cdot x_0 \in acl(Ch_2 \cdot h \cdot y_0)$. So $g \cdot x_0 \in acl(Cg_2, h_2 \cdot h \cdot y_0) \subseteq acl(Cg_2, h_2, h \cdot y_0)$. Now, recall that $acl(Cg_2) = acl(Ch_2)$, that $h \cdot y_0 = y_0$, and $a = g \cdot x_0$. Thus, we have $a \in acl(Cg_2, y_0)$. Since $a, y_0 \in M$ and g_2 is generic over M, we have $g_2 \mathop{\smile\!\!\!\!\!\mid}_C a \frown y_0$, so $g_2 \mathop{\smile\!\!\!\!\!\mid}_{Cy_0} a$. Therefore, by Proposition 2.10(4), we have $a \in acl(Cy_0)$, as desired. ⊣

Thus, we can use the set T to define an S-equivariant finite correspondence between X and Y. ⊣

Remark 3.30. The result above, which relies on generically stable generics, is far from optimal. In fact, [12, Theorem 2.15] is a recent technical result which can be used to build definable group homomorphisms, with much weaker hypotheses. See the end of the proof of [12, Theorem 2.19] for an application of this tool.

Proposition 3.31. *Any generically stable regular group configuration over A is a generically stable group configuration over A.*

Proof. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a generically stable regular group configuration over A. Let us show that it is a generically stable group configuration over A. The second point of the definition of definable group configuration, i.e., definability of relative types, follows from independence, i.e., the first point of the definition of a regular group configuration, and stationarity of generically stable types. Then, the

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only properties to check are those regarding the canonical bases $Cb(a_j a_k/acl(Ab_i))$, for i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$. By symmetry of the context, it suffices to show that b_1 and $Cb(a_2a_3/acl(Ab_1))$ are interalgebraic over A.

First, let N be a sufficiently saturated model containing Ab_1, such that $a_2 \downarrow_{Ab_1} N$, which exists by extensibility of $tp(a_2/Ab_1)$. Then, by Fact 2.18, the type $tp(a_2/N)$ is generically stable over Ab_1: it is in fact the unique nonforking extension of $tp(a_2/Ab_1)$. Then, by Proposition 2.25, the type $tp(a_2a_3/N)$ is generically stable over $acl(Ab_1)$, and stationarity yields the equality: $Cb(a_2a_3/acl(Ab_1)) = Cb(a_2a_3/N)$.

On the one hand, it is clear that $Cb(a_2a_3/acl(Ab_1)) \subseteq acl(Ab_1)$. Conversely, let us show that b_1 is algebraic over $A \cup Cb(a_2a_3/acl(Ab_1))$. Let $C = A \cup Cb(a_2a_3/acl(Ab_1))$. By assumption, we know that $b_1 \in acl(Aa_2a_3) \subseteq acl(Ca_2a_3)$. Moreover, since the type $tp(a_2a_3/N)$ is generically stable over $acl(Ab_1)$, and $C \subset N$ contains $Cb(a_2a_3/acl(Ab_1))$, the type $tp(a_2a_3/N)$ is definable over C. Thus, by Proposition 2.15(5), this type is generically stable over C. In particular, $a_2a_3 \downarrow_C N$. Thus, since $b_1 \in N$, we have $a_2a_3 \downarrow_C b_1$. Finally, since $b_1 \in acl(Ca_2a_3)$, we can apply Proposition 2.10(4), which implies that $b_1 \in acl(C)$, as desired. ⊣

PROPOSITION 3.32. *If a quadrangle is equivalent over A to a generically stable (regular) group configuration over A, then it is itself a generically stable (regular) group configuration over $acl(A)$.*

PROOF. Indeed, Proposition 2.25 implies that generic stability is preserved. Moreover, the algebraicity relations are preserved, and, in the case of generically stable regular group configurations, so are the independence relations, thanks to Proposition 2.10(3). In the case of generically stable group configurations, the second point of the definition (i.e., definability of the relative types) is preserved thanks to Lemma 2.11: let α, β be elements such that $tp(\alpha/A\beta)$ is A-definable, and let α', β' be such that $acl(A\alpha) = acl(A\alpha')$ and $acl(A\beta) = acl(A\beta')$. Then, by Definition 2.5(5), there exists a model $M \supset A\beta$ and a type $p \in S(M)$ extending $tp(\alpha/A\beta)$ such that p is A-definable. We may assume that α realizes p. Then, $tp(\alpha/M)$ is $acl(A)$-definable, so $tp(\alpha'/M)$ is also definable over $acl(A)$. Since $M \supset acl(A\beta) = acl(A\beta')$, we have indeed shown that $tp(\alpha'/acl(A)\beta')$ is definable over $acl(A)$.

Lemma 2.11 also implies the required properties for the canonical bases: let (b_1, a_2, a_3) and (b_1', a_2', a_3') be such that $acl(Ab_1) = acl(Ab_1')$, $acl(Aa_2) = acl(Aa_2')$ and $acl(Aa_3) = acl(Aa_3')$. Assume that $tp(a_2a_3/acl(Ab_1))$ is definable, and b_1 is interalgebraic over A with the canonical basis of this type. Then, applying Lemma 2.11 once, we deduce that the type $tp(a_2'a_3'/acl(Ab_1'))$ is definable. Let $C \subseteq acl(Ab_1') = acl(Ab_1)$ denote its canonical basis.

CLAIM 3.33. *We have $acl(AC) = acl(Ab_1')$.*

PROOF. Applying Lemma 2.11 again, the type $tp(a_2a_3/acl(Ab_1))$ is definable over $acl(AC)$. So, by the property of $Cb(a_2a_3/acl(Ab_1))$, we have $b_1 \in acl(AC)$. So $b_1' \in acl(AC)$. Since $C \subseteq acl(Ab_1')$, we are done. ⊣

This concludes the proof. ⊣

NOTATION 3.34. *If $p(x, y)$ is a complete type in several variables, where x, y are tuples of variables, we let p_x denote the restriction of p to the tuple of variables x.*

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PROPOSITION 3.35. *Let* $\alpha = (a_1, a_2, a_3, b_1, b_2, b_3)$ *be a tuple, and* $A_0 \subseteq A$ *be sets of parameters. Assume that* $tp(\alpha/A)$ *is generically stable over* A_0.

Then, the sextuple α *is a generically stable (regular) group configuration over* A *if and only if it is a generically stable (regular) group configuration over* A_0.

PROOF. First, let us assume that α is a generically stable (regular) group configuration over A, and prove that it is a generically stable (regular) group configuration over A_0.

The interalgebraicity relations are proved using Proposition 2.35. Let us prove one of the independence relations, say $a_1a_2 \perp_{A_0} b_1$. We know that $a_1a_2 \perp_A b_1$, that $a_1a_2 \perp_{A_0} A$ and that $tp(a_1a_2/A_0)$ is generically stable. So, by Proposition 2.21, we have $a_1a_2 \perp_{A_0} Ab_1$, which implies that $a_1a_2 \perp_{A_0} b_1$, as desired. Note that this independence relation, along with stationarity of generically stable types, imply that $tp(a_1a_2/A_0b_1)$ is A_0-definable.

To conclude, let us now prove the statement regarding the canonical bases, in the case of generically stable group configurations. Let c denote $Cb(a_1a_2/acl(A_0b_3))$. We wish to prove that $b_3 \in acl(A_0c)$.

CLAIM 3.36. *We have* $a_1a_2 \perp_{A_0b_3} A$, *thus* $Cb(a_1a_2/acl(Ab_3)) = Cb(a_1a_2/acl(A_0b_3)) = c$.

PROOF. By hypothesis, we have $a_2 \perp_A b_3$ and $a_2 \perp_{A_0} A$. So, by transitivity for generically stable types, we have $a_2 \perp_{A_0} Ab_3$. Thus, by points 1 and 3 of Proposition 2.10, we have $a_1a_2 \perp_{A_0b_3} A$, which proves the claim. ⊣

Then, by definition, we have $c \subseteq acl(A_0b_3)$. By Lemma 2.24 applied to b_3 and A_0, let M be a model containing A_0, such that the type of b_3 over M is the canonical extension of $tp(b_3/A_0)$, which is also the canonical extension of $tp(b_3/A)$. Then, since $c \subseteq acl(A_0b_3)$, Proposition 2.25 implies that the type $tp(b_3c/M)$ is generically stable over $acl(A_0)$. We also know that $b_3 \in acl(Ac)$, in particular $b_3 \in acl(Mc)$. Thus, by Proposition 2.35, we have $b_3 \in acl(A_0c)$, as required.

Let us now prove the converse: assume that α is a generically stable (regular) group configuration over A_0, and prove that it is a generically stable (regular) group configuration over A. The algebraicity relations are clear, because $A \supseteq A_0$. For the independence relations, we use stationarity: for instance, let us prove $a_1a_2 \perp_A b_1$. Since α is a generically stable group configuration over A_0, we know that $tp(a_1a_2b_1/A_0)$ is the tensor product $tp(a_1/A_0) \otimes tp(a_2/A_0) \otimes tp(b_1/A_0)$, and is generically stable. Then, by stationarity, the nonforking extension $tp(a_1a_2b_1/A)$ is the tensor product $tp(a_1/A) \otimes tp(a_2/A) \otimes tp(b_1/A)$. This implies the independence $a_1a_2 \perp_A b_1$.

The statement regarding canonical bases follows from equalities of the form $Cb(a_1a_2/acl(Ab_3)) = Cb(a_1a_2/acl(A_0b_3))$, which were proved above. ⊣

3.2. Statement of the theorem.

THEOREM 3.37. *Let* M *be a* $|\mathcal{L}|^+$*-saturated model. Let* $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$ *be a generically stable group configuration over* M. *Let* $p_0(x_1, x_2, x_3, y_1, y_2, y_3) = tp(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0/M)$. *Let* $C_0 \subset M$ *be the canonical basis of the type* p_0. *Then,*

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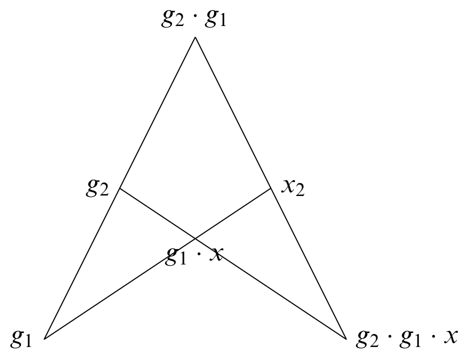
there exists a type-definable generically stable group Γ *acting transitively, faithfully and definably on a type-definable set* X, *elements* $b_1', b_2'', b_3''' \in M$, *and elements* $g_1, g_2 \in \Gamma$, $x \in X$, *whose types over* M *are generic, such that*:

1. *The tuple* $b_1'b_2''b_3'''$ *realizes* $(p_{0_{y_1}} \otimes p_{0_{y_2}} \otimes p_{0_{y_3}})|_{C_0}$.
2. *The group* Γ *is type-connected and type-definable over* $acl(C_0b_1'b_2''b_3''')$. *The space* X *is type-connected and type-definable over* $acl(C_0b_1'b_2''b_3''')$.
3. *The following is a generically stable group configuration over* M *which is equivalent, over* M, *to the quadrangle* $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$:



Moreover, let (R) *denote the property "*p_0 *is a generically stable regular group configuration over* M*". Then, if* (R) *holds, we may assume that* $X = \Gamma$, *and that the action of* Γ *on itself is by left translations.*

Note that, in particular, Γ and X are type-definable over M, which is enough information if one does not need to control parameters. Also, recall that any transitive action of a group Γ is isomorphic, in an explicit way, to the action of Γ on some Γ/H, for $H \leq \Gamma$, and that the stabilizers are then conjugates of H.

The following proof is adapted from that of Elisabeth Bouscaren [2], with ideas from [14, Chapter 5, Remark 1.10, Theorem 4.5] for the general case of a faithful transitive action.

The proof is divided into two steps: first, we find a group configuration which is equivalent to the original one, where some algebraicity relations have been replaced by (inter)definability. This is the content of Proposition 3.38. Then, using this stronger property, we consider some definable bijections permuting elements in (copies of) the new group configuration, and build a type-definable group from (the germs of) such maps. This is done in Section 4.

3.3. Replacing algebraicity with definability. The goal of this subsection is the following proposition, which enables us, at the cost of enlarging the basis, to replace some of the algebraicity in the configuration by definability, while keeping a generically stable type. This will then enable us to consider definable bijections that permute elements of the configuration.

PROPOSITION 3.38. *Under the hypotheses of Theorem 3.37, there exist elements* $b_1', b_2'', b_3''' \in M$ *and a configuration* $(a_1, a_2, a_3, b_1, b_2, b_3)$ *equivalent over* M *to* $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$, *such that*:

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1. *The tuple* $b_1'b_2''b_3'''$ *realizes* $(p_{0_{y_1}} \otimes p_{0_{y_2}} \otimes p_{0_{y_3}})|_{C_0}$.
2. *The type* $tp(a_1, a_2, a_3, b_1, b_2, b_3/M)$ *is generically stable over* $C := acl(C_0b_1'b_2''b_3''')$.
3. *We have* $a_1 \in dcl(Ca_3b_2)$, $a_2 \in dcl(Ca_3b_1)$, *and* $a_3 \in dcl(Ca_1b_2) \cap dcl(Ca_2b_1)$.

Remark 3.39.

1. Note that, if the property (R) in Theorem 3.37 held, the proposition would yield a configuration equivalent over M to a generically stable *regular* group configuration, so, by Proposition 3.32, still a *regular* group configuration over M, generically stable over $acl(C_0b_1'b_2''b_3''')$. Then, by Proposition 3.35, the new configuration would be a generically stable *regular* group configuration over $acl(C_0b_1'b_2''b_3''')$.
2. During the proof of this proposition, and later on as well, we will be considering several copies of configurations, or fragments of such. *The reader is advised to draw them*, to keep track of which tuples are actually configurations, and which of these have the same type. It can also be useful to mark the elements that are in M.

The following result will be useful in this subsection.

Lemma 3.40. *Let* A, B, B' *be parameter sets, and* a *be an element such that* $a \in acl(B) \cap acl(B')$. *Let* α *be the code of the set of conjugates of* a *over* ABB'. *Assume that* $B \mathop{\smile\!\!\!\!\!\!\mid}_{Aa} B'$. *Then,* a *and* α *are interalgebraic over* A.

Proof. First, note that a belongs to the finite set coded by α, so $a \in acl(A\alpha)$. Let us prove that α is algebraic over Aa. We know that α codes a subset of the *finite* set of conjugates of a over B, so $\alpha \in acl(Ba)$. Similarly, $\alpha \in acl(B'a)$. Thus, by Proposition 2.10(4) and the hypothesis $B \mathop{\smile\!\!\!\!\!\!\mid}_{Aa} B'$, we have $\alpha \in acl(Aa)$, as required. ⊣

Let us now prove the proposition.

Proof of Proposition 3.38. By saturation of M, let $b_1', b_2'' \in M$ be such that $b_1'b_2'' \equiv_{C_0} b_1^0b_2^0$.

Claim 3.41. *We have* $a_1^0b_2^0a_3^0b_1' \equiv_{C_0} a_1^0b_2^0a_3^0b_1^0$ *and* $b_1^0a_2^0a_3^0b_2'' \equiv_{C_0} b_1^0a_2^0a_3^0b_2^0$.

Proof. The type $tp(a_1^0b_2^0a_3^0/Mb_1^0)$ is the unique nonforking extension of the generically stable type $tp(a_1^0b_2^0a_3^0/C_0)$. Indeed, we have $a_1^0b_2^0 \mathop{\smile\!\!\!\!\!\!\mid}_M b_1^0$, and $a_1^0b_2^0 \mathop{\smile\!\!\!\!\!\!\mid}_{C_0} M$, so by transitivity for generically stable types (Proposition 2.21), we have $a_1^0b_2^0 \mathop{\smile\!\!\!\!\!\!\mid}_{C_0} Mb_1^0$. Since $a_3^0 \in acl(C_0a_1^0b_2^0)$, we have $a_1^0b_2^0a_3^0 \mathop{\smile\!\!\!\!\!\!\mid}_{C_0} Mb_1^0$. Thus, the type $tp(a_1^0b_2^0a_3^0/Mb_1^0)$ is generically stable over C_0, in particular it is C_0-invariant. As $b_1' \in M$ has the same type over C_0 as b_1^0, we have indeed $a_1^0b_2^0a_3^0b_1' \equiv_{C_0} a_1^0b_2^0a_3^0b_1^0$.

The other result is proved similarly, using the fact that $tp(b_1^0a_2^0a_3^0/Mb_2^0)$ is generically stable over C_0, so C_0-invariant. ⊣

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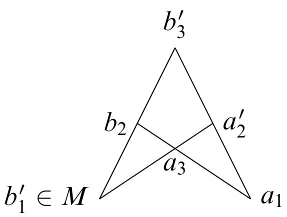
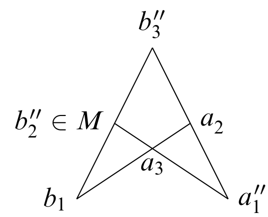
Then, let b_3', a_2' and b_3'', a_1'' be such that $a_1^0 b_2^0 a_3^0 b_1' a_2' b_3' \equiv_{C_0} a_1^0 b_2^0 a_3^0 b_1^0 a_2^0 b_3^0$ and $b_1^0 a_2^0 a_3^0 a_1'' b_2'' b_3'' \equiv_{C_0} b_1^0 a_2^0 a_3^0 a_1^0 b_2^0 b_3^0$. In other words, the following are copies of the original configuration:





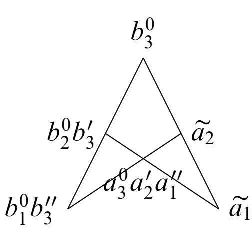
Then, let $\widetilde{a}_1$ be the code of the set of conjugates of a_1^0 over $C_0 b_2^0 a_3^0 a_2' b_3'$. Similarly, let $\widetilde{a}_2$ be the code of the set of conjugates of a_2^0 over $C_0 b_1^0 a_3^0 a_1'' b_3''$.

Claim 3.42. *We have $acl(C_0\widetilde{a}_1) = acl(C_0 a_1^0)$ and $acl(C_0\widetilde{a}_2) = acl(C_0 a_2^0)$.*

Proof. For the first interalgebraicity, we wish to apply Lemma 3.40 to $a = a_1^0$, $A = C_0$, $B = C_0 b_2^0 a_3^0$, and $B' = C_0 b_3' a_2'$. For the second one, we apply the same lemma to $a = a_2^0$, $A = C_0$, $B = C_0 b_1^0 a_3^0$, and $B' = C_0 b_3'' a_1''$. The only hypotheses that do not follow immediately from the constructions are the independence properties. To prove $b_2^0 a_3^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0 a_1^0} b_3' a_2'$ and $b_1^0 a_3^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0 a_2^0} b_3'' a_1''$, we use the fact that the copies have the same type over C_0 as the original configuration. So, it suffices to prove $b_2^0 a_3^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0 a_1^0} b_3^0 a_2^0$ and $b_1^0 a_3^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0 a_2^0} b_3^0 a_1^0$. Using the algebraicity properties of the configuration, along with Proposition 2.10(3)(b), it suffices to prove $b_2^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0 a_1^0} b_3^0$ and $b_1^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0 a_2^0} b_3^0$. By Proposition 2.10(1), these follow from $b_2^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0} a_1^0 b_3^0$ and $b_1^0 \mathop{\smile\!\!\!\!\!\mid}_{C_0} a_2^0 b_3^0$, which hold by Remark 3.12(1). ⊣

The motivation for building these copies is that $\widetilde{a}_1 \in dcl(C_0 b_2^0 a_3^0 b_3' a_2')$ and $\widetilde{a}_2 \in dcl(C_0 b_1^0 a_3^0 b_3'' a_1'')$.

Then, let $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ denote the following configuration:



Let $C_1 = acl(C_0 b_1' b_2'') \subset M$.

Claim 3.43. *The quadrangle $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ is equivalent over C_1 to $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$.*

Proof. The only things left to check are $b_3' \in acl(C_1 b_2^0)$, $b_3'' \in acl(C_1 b_1^0)$, and that $a_2' a_1''$ is in $acl(C_1 a_3^0)$. These can be checked by inspecting the above copies of the configuration, keeping in mind that C_1 contains $C_0 b_1' b_2''$. ⊣

Thus, by Proposition 3.32, the tuple $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ is a generically stable group configuration over C_1, and over M. *What we gained is that* $\alpha_1 \in dcl(C_1\alpha_3\beta_2)$ *and* $\alpha_2 \in dcl(C_1\alpha_3\beta_1)$. Also note that $\beta_3 = b_3^0$.

Now, let $\widetilde{\alpha_3}$ be the code of the set of conjugates of α_3 over $C_1\beta_1\alpha_2\beta_2\alpha_1$. As in Claim 3.42, by Lemma 3.40, we have $acl(C_1\alpha_3) = acl(C_1\widetilde{\alpha_3})$. So, by Proposition 3.32, the quadrangle $(\alpha_1, \alpha_2, \widetilde{\alpha_3}, \beta_1, \beta_2, \beta_3)$ is a generically stable group configuration over C_1, and over M. By construction, we get $\widetilde{\alpha_3} \in dcl(C_1\beta_1\alpha_2\beta_2\alpha_1)$.

Claim 3.44. *We still have* $\alpha_1 \in dcl(C_1\widetilde{\alpha_3}\beta_2)$ *and* $\alpha_2 \in dcl(C_1\widetilde{\alpha_3}\beta_1)$.

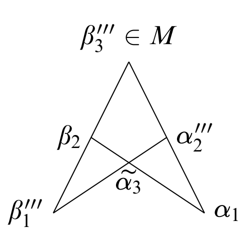
Proof. Recall that $\alpha_1 \in dcl(C_1\alpha_3\beta_2)$ and $\alpha_2 \in dcl(C_1\alpha_3\beta_1)$. So, let j and l be some C_1-definable maps such that $j(\alpha_3, \beta_2) = \alpha_1$ and $l(\alpha_3, \beta_1) = \alpha_2$. Then, for any element α which is a conjugate of α_3 over $C_1\beta_1\alpha_2\beta_2\alpha_1$, we have $j(\alpha, \beta_2) = \alpha_1$ and $l(\alpha, \beta_1) = \alpha_2$. As $\widetilde{\alpha_3}$ is the code of that set of conjugates, we are done. ⊣

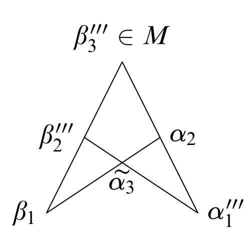
Claim 3.45. *The types* $tp(\widetilde{\alpha_3}\beta_2\alpha_1/M\beta_3)$ *and* $tp(\widetilde{\alpha_3}\alpha_2\beta_1/M\beta_3)$ *are generically stable over* C_1*, in particular, they admit* C_1*-invariant extensions.*

Proof. Recall that, since the new sextuple is a group configuration over M, we have $\widetilde{\alpha_3}\beta_2\alpha_1 \perp_M \beta_3$. Also, we proved above (right before Claim 3.44) that $tp(\widetilde{\alpha_3}\beta_2\alpha_1/M)$ is generically stable over C_1. Thus, by stationarity for generically stable types, the type $tp(\widetilde{\alpha_3}\beta_2\alpha_1/M\beta_3)$ is generically stable over C_1, as required. The other proof is similar. ⊣

Now, let $\beta_3''' \in M$ realize $tp(\beta_3/C_1) = tp(b_3^0/C_1)$. Then, by C_1-invariance of the types $tp(\widetilde{\alpha_3}\beta_2\alpha_1/M\beta_3)$ and $tp(\widetilde{\alpha_3}\alpha_2\beta_1/M\beta_3)$, we have $\widetilde{\alpha_3}\beta_2\alpha_1\beta_3 \equiv_{C_1} \widetilde{\alpha_3}\beta_2\alpha_1\beta_3'''$ and $\widetilde{\alpha_3}\alpha_2\beta_1\beta_3 \equiv_{C_1} \widetilde{\alpha_3}\alpha_2\beta_1\beta_3'''$. So, let β_1''', α_2''' and β_2''', α_1''' be such that $\widetilde{\alpha_3}\beta_2\alpha_1\beta_3\beta_1\alpha_2 \equiv_{C_1} \widetilde{\alpha_3}\beta_2\alpha_1\beta_3'''\beta_1'''\alpha_2'''$ and $\widetilde{\alpha_3}\alpha_2\beta_1\beta_3\beta_2\alpha_1 \equiv_{C_1} \widetilde{\alpha_3}\alpha_2\beta_1\beta_3'''\beta_2'''\alpha_1'''$.

In other words, we have two copies of the configuration $(\alpha_1, \alpha_2, \widetilde{\alpha_3}, \beta_1, \beta_2, \beta_3)$:





Finally, the configuration $(a_1, a_2, a_3, b_1, b_2, b_3)$ we wish to consider is the following: $(\alpha_1\alpha_2''', \alpha_2\alpha_1''', \widetilde{\alpha_3}, \beta_1\beta_2''', \beta_2\beta_1''', \beta_3)$. We let C denote $acl(C_1\beta_3''') = acl(C_0b_1'b_2''\beta_3''')$.

Claim 3.46. *We have* $b_1'b_2''\beta_3''' \models (p_{0_{y_1}} \otimes p_{0_{y_2}} \otimes p_{0_{y_3}})|_{C_0}$.

Proof. We already know that $b_1'b_2'' \models ((p_0)_{y_1,y_2})|_{C_0} = (p_{0_{y_1}} \otimes p_{0_{y_2}})|_{C_0}$, by independence. Then, by stationarity and by Remark 3.12(1), the type $tp(\beta_3/C_0b_1'b_2'')$ is equal to $(p_0)_{y_3}|_{C_0b_1'b_2''}$. Also recall that β_3''' realizes the same type as β_3 over $C_1 = acl(C_0b_1'b_2'')$, in particular we have $\beta_3''' \models (p_0)_{y_3}|_{C_0b_1'b_2''}$, as required. ⊣

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CLAIM 3.47. *We have* $\widetilde{\alpha}_3 \in dcl(C\alpha_2\alpha_1'''\beta_1\beta_2''') \cap dcl(C\alpha_1\alpha_2'''\beta_2\beta_1''')$.

PROOF. Recall that $\widetilde{\alpha}_3 \in dcl(C_1\beta_1\alpha_2\beta_2\alpha_1)$. So, in the copies above, we have indeed $\widetilde{\alpha}_3 \in dcl(C_1\beta_1'''\alpha_2'''\beta_2\alpha_1)$ and $\widetilde{\alpha}_3 \in dcl(C_1\beta_1\alpha_2\beta_2'''\alpha_1''')$. ⊣

CLAIM 3.48. *We have* $\alpha_1\alpha_2''' \in dcl(C\widetilde{\alpha}_3\beta_2\beta_1''')$ *and* $\alpha_2\alpha_1''' \in dcl(C\widetilde{\alpha}_3\beta_1\beta_2''')$.

PROOF. We proved in Claim 3.44 that $\alpha_1 \in dcl(C_1\widetilde{\alpha}_3\beta_2)$ and $\alpha_2 \in dcl(C_1\widetilde{\alpha}_3\beta_1)$. So, in the first copy (the one on the left in the diagrams above), we also have $\alpha_2''' \in dcl(C_1\widetilde{\alpha}_3\beta_1''')$. Thus, we get $\alpha_1\alpha_2''' \in dcl(C_1\widetilde{\alpha}_3\beta_2\beta_1''')$.

The other result is proved similarly: we have $\alpha_1''' \in dcl(C_1\widetilde{\alpha}_3\beta_2''')$ and $\alpha_2 \in dcl(C_1\widetilde{\alpha}_3\beta_1)$, so $\alpha_2\alpha_1''' \in dcl(C_1\widetilde{\alpha}_3\beta_2'''\beta_1)$. ⊣

Thus, the tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ having been defined as $(\alpha_1\alpha_2''', \alpha_2\alpha_1''', \widetilde{\alpha}_3, \beta_1\beta_2''', \beta_2\beta_1''', \beta_3)$, Claims 3.47 and 3.48 correspond to the third point of Proposition 3.38.

CLAIM 3.49. *The quadrangle* $(a_1, a_2, a_3, b_1, b_2, b_3)$ *is equivalent over* C *to* $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$.

PROOF. This amounts to checking that $(a_1, a_2, a_3, b_1, b_2, b_3)$ is equivalent over C to the configuration $(\alpha_1, \alpha_2, \widetilde{\alpha}_3, \beta_1, \beta_2, \beta_3)$. It relies on the following algebraicity properties: $\alpha_2''' \in acl(C_1\beta_3'''\alpha_1) = acl(C\alpha_1)$, $\alpha_1''' \in acl(C\alpha_2)$, $\beta_2''' \in acl(C\beta_1)$, and $\beta_1''' \in acl(C\beta_2)$. ⊣

Then, by Proposition 2.25, the type $tp(a_1a_2a_3b_1b_2b_3/M)$ is generically stable over C. This finishes the proof, by setting $b_3''' = \beta_3'''$. ⊣

§4. Constructing a group using germs of definable bijections. In this section, we construct an appropriate type-definable group, and finish the proof of the theorem.

4.1. Composition of germs. The aim of this subsection is to build an appropriate group from germs of definable bijections. *In this subsection and the next, the context is that of the conclusion of Proposition 3.38.*

DEFINITION 4.1. By Lemma 2.38, let f_{b_1} and g_{b_2} be definable bijections sending respectively a_2 to a_3 and a_3 to a_1, where f and g are C-definable families of definable bijections. Let $h_{b_1b_2}$ be the composite $g_{b_2} \circ f_{b_1}$.

Then, the independence hypotheses on the configuration imply that, in the sense of Definition 2.29, the functions f_{b_1} and $h_{b_1b_2}$ are well-defined at $tp(a_2/C)$, and that the function g_{b_2} is well-defined at $tp(a_3/C)$. Moreover, we will show that the germs of these functions can be composed.

PROPOSITION 4.2. *Let* $A, D \supseteq C$ *be sets of parameters. Let* $\beta_1, \beta_2, \alpha_1, \alpha_2, \alpha_3$ *be realizations of* $tp(b_1/M)|_D$, $tp(b_2/M)|_D$, $tp(a_1/M)|_D$, $tp(a_2/M)|_D$, *and* $tp(a_3/M)|_D$ *respectively.*

1. *If* $\alpha_2 \mathop{\smile\!\!\!\!\!\!|}_D A\beta_1$, *then* $f_{\beta_1}(\alpha_2) \mathop{\smile\!\!\!\!\!\!|}_D A\beta_1$.
2. *If* $\alpha_3 \mathop{\smile\!\!\!\!\!\!|}_D A\beta_2$, *then* $g_{\beta_2}(\alpha_3) \mathop{\smile\!\!\!\!\!\!|}_D A\beta_2$.

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3. *If* $\alpha_3 \downarrow_D A\beta_1$, *then* $(f_{\beta_1})^{-1}(\alpha_3) \downarrow_D A\beta_1$.
4. *If* $\alpha_1 \downarrow_D A\beta_2$, *then* $(g_{\beta_2})^{-1}(\alpha_1) \downarrow_D A\beta_2$.

Proof. Let us prove the first implication, the other ones being similar. Assume that $\alpha_2 \downarrow_D A\beta_1$. We know, by hypothesis, that $\alpha_2 \downarrow_C D$. Moreover, the type $tp(\alpha_2/C)$ is generically stable. So, by transitivity, $\alpha_2 \downarrow_C AD\beta_1$. Then, $\alpha_2 \downarrow_{C\beta_1} AD$, so

$$f_{\beta_1}(\alpha_2) \underset{C\beta_1}{\downarrow} AD. \tag{1}$$

On the other hand, since $\alpha_2 \downarrow_C AD\beta_1$, we have $\alpha_2 \downarrow_C \beta_1$. So, by stationarity, $\alpha_2\beta_1 \equiv_C a_2b_1$, so $f_{\beta_1}(\alpha_2)\alpha_2\beta_1 \equiv_C f_{b_1}(a_2)a_2b_1 = a_3a_2b_1$. Since $a_3 \downarrow_C b_1$, we deduce that $f_{\beta_1}(\alpha_2) \downarrow_C \beta_1$.

Recall that the type $tp(f_{\beta_1}(\alpha_2)/C) = tp(a_3/C)$ is generically stable. So, by (1) and transitivity, we have $f_{\beta_1}(\alpha_2) \downarrow_C \beta_1 AD$. So, by monotonicity, we have indeed the independence $f_{\beta_1}(\alpha_2) \downarrow_D \beta_1 A$. ⊣

Corollary 4.3. *Let* $\beta_1, \beta_1', \beta_1'', \beta_1'''$ *be realizations of* $tp(b_1/C)$, β_2, β_2' *be realizations of* $tp(b_2/C)$.

Then, the following germs are well-defined, i.e., only depend on the germs of the functions involved: $[g_{\beta_2}]^{-1} \circ [g_{\beta_2'}]$, $[g_{\beta_2}] \circ [f_{\beta_1}]$, $[f_{\beta_1}]^{-1} \circ [f_{\beta_1'}]$, $[f_{\beta_1}] \circ [f_{\beta_1'}]^{-1}$, *and*

$[f_{\beta_1}]^{-1} \circ [f_{\beta_1'}] \circ [f_{\beta_1''}]^{-1} \circ [f_{\beta_1'''}]$.

Then, these germs coincide with the germs of the composite functions: *for instance, we have* $[g_{\beta_2} \circ f_{\beta_1}] = [g_{\beta_2}] \circ [f_{\beta_1}]$.

Proof. We shall use Proposition 4.2, taking realizations of the generically stable types involved, independent from all the parameters that appear. Recall that, by Lemma 2.38, the germ of the inverse of a definable bijection f only depends on the germ of f.

Let us prove for instance that the germ of $(g_{\beta_2})^{-1} \circ g_{\beta_2'}$ only depends on the germs $[g_{\beta_2}]$ and $[g_{\beta_2'}]$. Let b_2^1, b_2^2 be realizations of $tp(b_2/C)$ such that $[g_{\beta_2}] = [g_{b_2^1}]$ and $[g_{\beta_2'}] = [g_{b_2^2}]$. Let us show that the germs of the functions $(g_{\beta_2})^{-1} \circ g_{\beta_2'}$ and $(g_{b_2^1})^{-1} \circ g_{b_2^2}$ are equal.

Let α_3 realize the type of a_3 over C, such that $\alpha_3 \downarrow_C b_2^1 b_2^2 \beta_2 \beta_2'$. Then, since $[g_{\beta_2'}] = [g_{b_2^2}]$, we have $g_{\beta_2'}(\alpha_3) = g_{b_2^2}(\alpha_3)$. Moreover, by Proposition 4.2 applied to $D = C$ and $A = b_2^1\beta_2$, we know that $g_{b_2^2}(\alpha_3) \downarrow_C b_2^1\beta_2$. Since we have assumed that $[g_{\beta_2}] = [g_{b_2^1}]$, we can deduce, using Lemma 2.38, that $(g_{b_2^1}^{-1} \circ g_{b_2^2})(\alpha_3) = (g_{\beta_2}^{-1} \circ g_{b_2^2})(\alpha_3) = (g_{\beta_2}^{-1} \circ g_{\beta_2'})(\alpha_3)$. As $\alpha_3 \downarrow_C b_2^1 b_2^2 \beta_2 \beta_2'$, we have shown that the germs of the composites $(g_{\beta_2})^{-1} \circ g_{\beta_2'}$ and $(g_{b_2^1})^{-1} \circ g_{b_2^2}$ are equal. ⊣

Definition 4.4. Let F, resp. G, be the type-definable set of the germs of functions of the form f_{β_1}, resp. g_{β_2}, where β_1 realizes $tp(b_1/C)$, resp. β_2 realizes $tp(b_2/C)$.

Remark 4.5. By completeness of $tp(b_1/C)$ and $tp(b_2/C)$, the partial types over C defining F and G are in fact complete. By Proposition 2.25 and Fact 2.18, these

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types are generically stable. Moreover, since the type $r = tp(a_1a_2a_3b_1b_2b_3/M)$ is generically stable over C, it commutes with itself. So, by Corollary 2.42, the definable types F and G commute with r. Then, applying this corollary again, we deduce, by associativity, that any tensor product whose factors are among F, G, or r, is commutative. In other words, the family $\{F, G, r\}$ is commutative.

Recall that $h_{b_1b_2}$ denotes the composite $g_{b_2} \circ f_{b_1}$.

LEMMA 4.6. *The germ of the definable map $h_{b_1b_2}$ is interalgebraic over C with b_3.*

PROOF. Recall that, by stationarity, the type $tp(a_2/acl(Cb_3))$ is generically stable over C, thus over $acl(C)$. Since $a_1 \in acl(Ca_2b_3)$, Proposition 2.25 implies that $tp(a_1a_2/acl(Cb_3))$ is generically stable over $acl(C)$. Also recall that we have a definable group configuration over C. Thus, the element b_3 is interalgebraic over C with the canonical basis of the generically stable type $tp(a_1a_2/acl(Cb_3))$. Moreover, we have $a_1a_2 \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{acl(Cb_3)} b_1b_2$, because $a_1 \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C b_1b_2b_3$ and $a_2 \in acl(Ca_1b_3)$. Thus, by stationarity of $tp(a_1a_2/acl(Cb_3))$, we have the following equalities:

$$Cb(a_1a_2/acl(Cb_3)) = Cb(a_1a_2/acl(Cb_1b_2b_3)) = Cb(a_1a_2/acl(Cb_1b_2)).$$

Since $a_2 \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C b_1b_2$, we have by stationarity $Cb(a_2/acl(Cb_1b_2)) = Cb(a_2/C) \subseteq C$. Then, Proposition 2.32, applied to $f_c = h_{b_1b_2}$ and $a = a_2$, implies that $Cb(a_1a_2/acl(Cb_1b_2))$ is interdefinable over C with $[h_{b_1b_2}]$. This concludes the proof. ⊣

DEFINITION 4.7. Let K be the set of germs of the form $f^{-1} \circ f'$ where $(f, f') \models F \otimes F|_C$.

Similarly, let L be the set of germs of the form $f' \circ f^{-1}$ where $(f, f') \models F \otimes F|_C$. Finally, let Γ be the C-type-definable set of germs of the form $k \circ k'$ where $k, k' \in K$.

REMARK 4.8. The set K is then defined by a complete C-definable type, also denoted as K. Indeed, the type K is the image of the tensor product $F \otimes F$ under the definable map which composes a germ with the inverse of another germ. This map is well-defined, by Corollary 4.3. Note that, thanks to the strong germs property (Corollary 2.36) and Corollary 4.3, the realizations of F, K and Γ act generically (recall that we introduced this notion in Definition 2.40) on the definable type $tp(a_2/C)$, and those of G and L on the type $tp(a_3/C)$.

The type-definable set Γ is the underlying set of the group we are going to construct.

REMARK 4.9. For ease of notation, let us write $p = tp(b_1/C)$, $q = tp(b_2/C)$.

1. Since $F \otimes F$ is a complete type, the image of $p \otimes p$ under the C-definable map $(b, b') \mapsto ([f_b], [f_{b'}])$ is equal to the type $F \otimes F$. More generally, any finite tensor product whose factors are F and G is the image under the appropriate function of the tensor product of corresponding factors p and q. This follows from the definitions.
2. Therefore, the type K is the image of the type $p \otimes p$ under the C-definable map $(b, b') \mapsto [f_b^{-1} \circ f_{b'}]$. Similarly, the type L is the image of $p \otimes p$ by the function $(b, b') \mapsto [f_{b'} \circ f_b^{-1}]$.

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3. By Remark 4.5, we know that $F \otimes F$ commutes with F, G and r, where r is the type $tp(a_1a_2a_3b_1b_2b_3/M)$. Then, by Corollary 2.42, the type K commutes with F, G and r. Again, we deduce that F, G, K, and r are in a commutative family, then that the family $\{F, G, K, L, r\}$ is commutative.
4. Since F commutes with itself, the inverse of a germ k realizing $K|_D$ is still a realization of $K|_D$, for all $D \supseteq C$. Similarly for L.

The following lemma shows that the collection of germs is, in some sense, homogeneous. It will be used for several key results.

LEMMA 4.10. *Let D be a small set containing C.*

1. *Let g, f_1, f_2 be such that gf_1f_2 realizes $G \otimes F \otimes F|_D$. Then, there exists b_2^2 realizing $tp(b_2/C)|_D$ such that $g \circ f_1 = [g_{b_2^2}] \circ f_2$, $b_2^2 \mathop{\smile\!\!\!\!\!\!|}_C Dgf_1$ and $b_2^2 \mathop{\smile\!\!\!\!\!\!|}_C Dgf_2$.*
2. *Let g, g', f_1 be such that $gg'f_1$ realizes $G \otimes G \otimes F|_D$. Then, there exists b_1^2 realizing $tp(b_1/C)|_D$ such that $g \circ f_1 = g' \circ [f_{b_1^2}]$, $b_1^2 \mathop{\smile\!\!\!\!\!\!|}_C Dgf_1$ and $b_1^2 \mathop{\smile\!\!\!\!\!\!|}_C Dg'f_1$.*

PROOF. Let us prove the first result, and then explain how to prove the second one.

CLAIM 4.11. *Assuming $b_2^2 \models tp(b_2/C)|_D$, to prove that $b_2^2 \mathop{\smile\!\!\!\!\!\!|}_C Dgf_1$ and $b_2^2 \mathop{\smile\!\!\!\!\!\!|}_C Dgf_2$, it suffices to show $b_2^2 \mathop{\smile\!\!\!\!\!\!|}_D gf_1$ and $b_2^2 \mathop{\smile\!\!\!\!\!\!|}_D gf_2$.*

PROOF. This is a direct application of transitivity (Proposition 2.21). ⊣

To simplify notations, let us assume that $tp(a_1a_2a_3b_1b_2b_3/D) = tp(a_1a_2a_3b_1b_2b_3/C)|_D$, i.e., $a_1a_2a_3b_1b_2b_3 \mathop{\smile\!\!\!\!\!\!|}_C D$. Recall that $a_1a_2a_3b_1b_2b_3$ is the configuration we built in Proposition 3.38, and D is not necessarily contained in M, so this assumption is not vacuous. We wish to be able to simply write, say $\alpha \equiv_D a_1$, instead of $\alpha \models tp(a_1/C)|_D$.

Let $q(x, y, z)$ denote the tensor product $tp(b_2/D)(x) \otimes tp(b_1/D)(y) \otimes tp(b_1/D)(z)$. By the first point of Remark 4.9, there exist b_2^1, b_1^1, b_1^2 such that $[g_{b_2^1}] = g$, $[f_{b_1^1}] = f_1$, $[f_{b_1^2}] = f_2$ and $b_2^1b_1^1b_1^2 \models q(x, y, z)$. We look for a suitable element b_2^2. Let $\beta_3 \in acl(Cb_2^1b_1^1)$ be such that $\beta_3b_2^1b_1^1 \equiv_D b_3b_2b_1$. Let α_2 be a realization of $tp(a_2/M)|_{Db_1^1b_1^2b_2^1\beta_3}$. Then, by stationarity, we have $\alpha_2b_1^1b_2^1\beta_3 \equiv_D a_2b_1b_2b_3$. Let $a_3^1 = f_{b_1^1}(\alpha_2)$ and $\alpha_1 = g_{b_2^1}(a_3^1)$. Thus, we have $\alpha_1\alpha_2a_3^1b_1^1b_2^1\beta_3 \equiv_D a_1a_2a_3b_1b_2b_3$.

By choice of α_2, we have $\alpha_2 \mathop{\smile\!\!\!\!\!\!|}_D b_1^1b_2^1b_1^2$. By construction and commutativity, we also have $b_1^2 \mathop{\smile\!\!\!\!\!\!|}_D b_1^1b_2^1$. Thus, we may apply Lemma 2.23, to deduce that $\alpha_2b_1^1b_2^1 \mathop{\smile\!\!\!\!\!\!|}_D b_1^2$. So, by Proposition 2.10(3)(b), we have

$$\alpha_1\alpha_2a_3^1b_1^1b_2^1\beta_3 \mathop{\smile\!\!\!\!\!\!|}_D b_1^2. \tag{2}$$

Then, by symmetry and stationarity, $\alpha_1\alpha_2\beta_3b_1^2 \equiv_D a_1a_2b_3b_1$. Let b_2^2, a_3^2 be such that $b_2^2a_3^2\alpha_1\alpha_2\beta_3b_1^2 \equiv_D b_2a_3a_1a_2b_3b_1$. We will show that b_2^2 has the required properties.

We end up with the following generically stable group configurations, which have the $(\beta_3, \alpha_2, \alpha_1)$-line in common, and whose type over D is $r|_D = tp(a_1, a_2, a_3, b_1, b_2, b_3/D)$.

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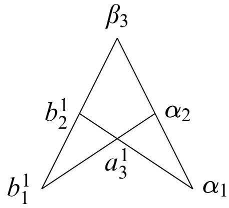

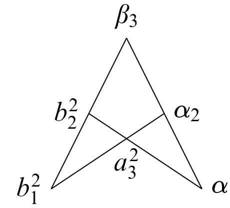

CLAIM 4.12. *We have* $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D gf_1$.

PROOF. We know that $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_1^2$, because $b_2 \mathop{\smile\hskip-0.9em\downarrow}_D b_1$. Let us show that $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_2^1 b_1^1$. By (2), we know that $\alpha_1\alpha_2 a_3^1 b_1^1 b_2^1 \beta_3 \mathop{\smile\hskip-0.9em\downarrow}_D b_1^2$. Then, as $tp(\alpha_1\alpha_2 a_3^1 b_1^1 b_2^1 \beta_3/D)$ is generically stable, we can apply symmetry, to get $b_1^2 \mathop{\smile\hskip-0.9em\downarrow}_D \alpha_1\alpha_2 a_3^1 b_1^1 b_2^1 \beta_3$. So, by Proposition 2.10(1) and (3)(b), we have $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_{D\beta_3} \alpha_1\alpha_2 a_3^1 b_1^1 b_2^1$. Since $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D \beta_3$, by transitivity, we have $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D \beta_3\alpha_1\alpha_2 a_3^1 b_1^1 b_2^1$. By monotonicity, we deduce $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_1^1 b_2^1$. Hence, we have indeed $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D gf_1$. ⊣

CLAIM 4.13. *We have* $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D gf_2$.

PROOF. By construction of the elements b_2^1, b_1^1, b_1^2, we know that $b_1^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_1^1 b_2^1$, so $b_1^2 \mathop{\smile\hskip-0.9em\downarrow}_D \beta_3 b_2^1$. Since $\beta_3 \mathop{\smile\hskip-0.9em\downarrow}_D b_2^1$, by Lemma 2.23, we have $\beta_3 b_1^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_2^1$, so $b_2^2 b_1^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_2^1$. Applying Lemma 2.23 again, we have $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D b_2^1 b_1^2$. Therefore, we have indeed $b_2^2 \mathop{\smile\hskip-0.9em\downarrow}_D gf_2$. ⊣

Besides, we know that $g_{b_2^1} \circ f_{b_1^1}(\alpha_2) = \alpha_1 = g_{b_2^2} \circ f_{b_1^2}(\alpha_2)$. It then remains to show that $\alpha_2 \mathop{\smile\hskip-0.9em\downarrow}_D b_2^1 b_1^1 b_2^2 b_1^2$, so that we can conclude equality of the germs of $g_{b_2^1} \circ f_{b_1^1}$ and $g_{b_2^2} \circ f_{b_1^2}$. It suffices to prove that $\alpha_2 \mathop{\smile\hskip-0.9em\downarrow}_D b_1^2 b_2^1 \beta_3$, which is true by choice of α_2.

In order to prove the second statement, one can use symmetry of the setting: all the tensors product involved are commutative, and the data $(a_1, a_2, a_3, b_1, b_2, b_3)$, F, G, D satisfies the same properties as the permuted data $(a_2, a_1, a_3, b_2, b_1, b_3)$, G^{-1}, F^{-1}, D. In other words, it suffices to apply the first point to the permuted data. ⊣

LEMMA 4.14. *Let D be a small set containing C. Let $f, f' \in F$ be such that $(f, f') \models F \otimes F|_D$. Then $f^{-1} \circ f' \models K|_{Df}$ and $f^{-1} \circ f' \models K|_{Df'}$. On the other hand, $f' \circ f^{-1} \models L|_{Df}$ and $f' \circ f^{-1} \models L|_{Df'}$.*

PROOF. Let us start by proving the statements about $f^{-1} \circ f'$. By the first point of Remark 4.9, we can apply Remark 2.31 to the case where $q = tp(b_1/M)^{\otimes 2}$ and $h : (b, b') \mapsto [f_b^{-1} \circ f_{b'}]$. We then find β_1, β_1' such that $([f_{\beta_1}], [f_{\beta_1'}]) = (f, f')$ and $(\beta_1, \beta_1') \models q|_D$. Let $g = [g_{\beta_2}] \in G$, where $\beta_2 \models tp(b_2/M)|_{D\beta_1\beta_1'}$. By stationarity, we know that $tp(b_1 b_2/M) = tp(b_1/M) \otimes tp(b_2/M)$. Since M is a model, we know by Fact 2.6 that $tp(b_1 b_2/M)|_E = (tp(b_1/M) \otimes tp(b_2/M))|_E$ for all $E \supseteq C$. Then, $\beta_2\beta_1$ and $\beta_2\beta_1'$ realize the type $tp(b_2 b_1/M)|_D$. So, let β_3, β_3' be such that $\beta_3\beta_2\beta_1$ and $\beta_3'\beta_2\beta_1'$ realize $tp(b_3 b_2 b_1/M)|_D$.

CLAIM 4.15. *We have* $\beta_1 \mathop{\smile\hskip-0.9em\downarrow}_D \beta_3\beta_3'$.

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PROOF. By construction, we have $\beta_2\beta_1\beta_1' \models (tp(b_2/M) \otimes q)|_D$, so $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_D \beta_2\beta_1'$. Since $\beta_3' \in acl(D\beta_2\beta_1')$, this implies by Proposition 2.10(3)(b) that $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_D \beta_2\beta_3'$. By choice of β_3', we have $\beta_2 \mathop{\smile\!\!\!\!\!\mid}_D \beta_3'$. So, by Lemma 2.23, we have $\beta_1\beta_2 \mathop{\smile\!\!\!\!\!\mid}_D \beta_3'$. As $\beta_3 \in acl(D\beta_1\beta_2)$, this implies $\beta_1\beta_3 \mathop{\smile\!\!\!\!\!\mid}_D \beta_3'$. We also know that $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_D \beta_3$. So, again by Lemma 2.23, we can conclude that $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_D \beta_3\beta_3'$, as desired. ⊣

CLAIM 4.16. *We have $f \mathop{\smile\!\!\!\!\!\mid}_C Df^{-1} \circ f'$.*

PROOF. By Claim 4.15, and transitivity, we have $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_C D\beta_3\beta_3'$. Also, by Lemma 4.6, we have $[g_{\beta_2}] \circ [f_{\beta_1}] \in acl(C\beta_3)$ and $[g_{\beta_2}] \circ [f_{\beta_1'}] \in acl(C\beta_3')$. Thus, we have $f^{-1} \circ f' = ([g_{\beta_2}] \circ [f_{\beta_1}])^{-1} \circ [g_{\beta_2}] \circ [f_{\beta_1'}] \in acl(C\beta_3\beta_3')$. The result then follows from Proposition 2.10(3)(b). ⊣

CLAIM 4.17. *We have $f^{-1} \circ f' \models K|_{Df}$ and $f^{-1} \circ f' \models K|_{Df'}$.*

PROOF. First note that, by symmetry of the hypotheses on f and f', which comes from commutativity (see, for instance, Remark 4.9(3) and (4), and Fact 2.39), it is enough to prove that $f^{-1} \circ f' \mathop{\smile\!\!\!\!\!\mid}_C Df$. The definition of K implies that $f^{-1} \circ f' \models K|_D$, since we assumed that $(f, f') \models F \otimes F|_D$. We want to show that $(f^{-1} \circ f', f) \models K \otimes F|_D$. By the third point of Remark 4.9, K and F commute, so it is equivalent to prove that $(f, f^{-1} \circ f') \models F \otimes K|_D$. By stationarity of F, as we know that $f^{-1} \circ f' \models K|_D$, it suffices to show that $f \mathop{\smile\!\!\!\!\!\mid}_C Df^{-1} \circ f'$, which is precisely the content of the previous claim. ⊣

Now, to prove the result for $f' \circ f^{-1}$, we use Lemma 4.10. By commutativity and stationarity, it suffices to show $f \mathop{\smile\!\!\!\!\!\mid}_C Df' \circ f^{-1}$ and $f' \mathop{\smile\!\!\!\!\!\mid}_C Df' \circ f^{-1}$. By transitivity, it suffices to show $f \mathop{\smile\!\!\!\!\!\mid}_D f' \circ f^{-1}$ and $f' \mathop{\smile\!\!\!\!\!\mid}_D f' \circ f^{-1}$. Let $g \models G|_{Dff'}$. Then, by Lemma 4.10, there exists $g' \models G|_D$ such that $g \circ f = g' \circ f'$, $g' \mathop{\smile\!\!\!\!\!\mid}_C Dgf$ and $g' \mathop{\smile\!\!\!\!\!\mid}_C Dgf'$. In particular, we have $g' \mathop{\smile\!\!\!\!\!\mid}_D gf$ and $g' \mathop{\smile\!\!\!\!\!\mid}_D gf'$. So, by Lemma 2.23 and symmetry, we have $f \mathop{\smile\!\!\!\!\!\mid}_D g \frown g'$ and $f' \mathop{\smile\!\!\!\!\!\mid}_D g \frown g'$. From the equality $g \circ f = g' \circ f'$, we deduce $g'^{-1} \circ g = f' \circ f^{-1}$. This implies $f \mathop{\smile\!\!\!\!\!\mid}_D f' \circ f^{-1}$ and $f' \mathop{\smile\!\!\!\!\!\mid}_D f' \circ f^{-1}$, as required. ⊣

COROLLARY 4.18.

1. *The C-definable type K is generically stable.*
2. *The type-definable set Γ is the set of germs of the form $f^{-1} \circ f'$, where f, f' realize $F|_C$.*

PROOF. 1. Let f realize $F|_C$. Then, by Lemma 4.14, there exists a Cf-definable bijection $F|_{Cf} \simeq K|_{Cf}$, which maps any f' to $f^{-1} \circ f'$. Thus, by Proposition 2.25, the type $K|_{Cf}$ is generically stable. However, this type is definable over C, so $K|_C$ is generically stable, as stated.

2. Let γ be an element of Γ. By definition, there exist k_1, k_2 realizing $K|_C$ such that $\gamma = k_1 \circ k_2$. Let $f \models F|_{Ck_1k_2}$. Then, by Lemma 4.14, and completeness of the type $K|_{Cf} = K^{-1}|_{Cf}$ (see Remark 4.9(4)), there exist $f_1, f_2 \in F$ such that $k_1 = f_1^{-1} \circ f$ and $k_2 = f^{-1} \circ f_2$. Then, we compute that $\gamma = f_1^{-1} \circ f_2$, as desired. ⊣

The following lemma will be used to prove transitivity of the action of the group Γ, and regularity in the case of a regular group configuration.

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LEMMA 4.19. *Let* (α_2, α_2') *be a realization of* $tp(a_2/C) \otimes tp(a_2/C)$. *Then, there exists a germ* $k \in K$ *such that* $k \models K|_{C\alpha_2} \cup K|_{C\alpha_2'}$ *and* $k(\alpha_2) = \alpha_2'$.

For any (α, k) *realizing the tensor product* $tp(a_2/C) \otimes K|_C$, *the pair* $(k(\alpha), \alpha)$ *realizes the tensor product* $tp(a_2/C) \otimes tp(a_2/C)$.

Moreover, under the hypothesis (R) *of Theorem 3.37, i.e., if we started from a generically stable regular group configuration, there exist only finitely many germs* k *in* K *such that* $k \models K|_{C\alpha_2}$ *and* $k(\alpha_2) = \alpha_2'$.

PROOF. First, recall that if $k \in K$ realizes $K|_{C\alpha_2}$, then the pair (k, α_2) realizes the tensor product $K \otimes tp(a_2/M)|_C$. So, by commutativity (see the third point of Remark 4.9), the element α_2 realizes $tp(a_2/M)|_{Ck}$. Then, since K acts generically on $tp(a_2/C)$ (see Remark 4.8), the element $k(\alpha_2)$ realizes $tp(a_2/M)|_{Ck}$.

Let $p = tp(b_1/M)$ and $t = tp(a_2/M)$. By stationarity of t, the hypothesis is equivalent to $\alpha_2\alpha_2'$ realizing $t \otimes t|_C$.

Let us prove existence: Let $\alpha_3 \equiv_C a_3$ be such that $\alpha_3 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_2\alpha_2'$, and let $\beta_1\beta_1'$ realize $p \otimes p|_{C\alpha_3}$.

CLAIM 4.20. *The pair* $(f_{\beta_1}^{-1}(\alpha_3), f_{\beta_1'}^{-1}(\alpha_3))$ *realizes the type* $t \otimes t|_{C\alpha_3}$.

PROOF. By definition, we have $\beta_1 \models p|_{C\beta_1'\alpha_3}$. So $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_{C\alpha_3} \beta_1'$, so $f_{\beta_1}^{-1}(\alpha_3) \mathop{\smile\!\!\!\!\!\mid}_{C\alpha_3} f_{\beta_1'}^{-1}(\alpha_3)$. Also, by stationarity, since $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_3$ (and $b_1 \mathop{\smile\!\!\!\!\!\mid}_C a_3$), we have $\beta_1\alpha_3 \equiv_C b_1a_3$. Moreover, in the configuration $a_1a_2a_3b_1b_2b_3$, we have $f_{b_1}^{-1}(a_3) = a_2$, and $a_2 \mathop{\smile\!\!\!\!\!\mid}_C a_3$. So $f_{\beta_1}^{-1}(\alpha_3) \mathop{\smile\!\!\!\!\!\mid}_C \alpha_3$, and similarly $f_{\beta_1'}^{-1}(\alpha_3) \mathop{\smile\!\!\!\!\!\mid}_C \alpha_3$. So, by transitivity, $f_{\beta_1}^{-1}(\alpha_3) \mathop{\smile\!\!\!\!\!\mid}_C f_{\beta_1'}^{-1}(\alpha_3)\alpha_3$. In other words, $f_{\beta_1}^{-1}(\alpha_3)f_{\beta_1'}^{-1}(\alpha_3)$ realizes the type $t \otimes t|_{C\alpha_3}$, as required. ⊣

CLAIM 4.21. *We may assume that* $f_{\beta_1}^{-1}(\alpha_3) = \alpha_2$ *and* $f_{\beta_1'}^{-1}(\alpha_3) = \alpha_2'$, *without losing the properties* $\alpha_3 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_2\alpha_2'$, *and* $\beta_1\beta_1' \models p \otimes p|_{C\alpha_3}$.

PROOF. By choice of α_3, the tuple $\alpha_2\alpha_2'$ also realizes the type $t \otimes t|_{C\alpha_3}$. So, there exists an automorphism fixing $C\alpha_3$, and sending $f_{\beta_1}^{-1}(\alpha_3)$ to α_2, and $f_{\beta_1'}^{-1}(\alpha_3)$ to α_2'. ⊣

Then $(f_{\beta_1'}^{-1} \circ f_{\beta_1})(\alpha_2) = \alpha_2'$, considering the definable maps, and not their germs. We want to show that $k = [f_{\beta_1'}^{-1} \circ f_{\beta_1}]$ has the required properties.

CLAIM 4.22. *We have* $\beta_1\beta_1' \models p^{\otimes 2}|_{C\alpha_2} \cup p^{\otimes 2}|_{C\alpha_2'}$.

PROOF. By construction of β_1, β_1', we know that $\beta_1 \models p|_{C\alpha_3}$ and that $\beta_1' \models p|_{C\alpha_3}$. Then, by the third point of Proposition 4.2, we deduce that $\beta_1 \models p|_{C\alpha_2}$ and that $\beta_1' \models p|_{C\alpha_2'}$. Since p commutes with itself, it remains to show that $\beta_1' \models p|_{C\alpha_2\beta_1}$ and $\beta_1 \models p|_{C\alpha_2'\beta_1'}$. By symmetry of the construction, it suffices to prove the second point. Recall that, by definition of $\beta_1\beta_1'$ (right before Claim 4.20), we have $\beta_1 \models p|_{C\beta_1'\alpha_3}$. Since $\alpha_2' \in dcl(C\beta_1'\alpha_3)$, we have indeed that $\beta_1 \models p|_{C\alpha_2'\beta_1'}$. ⊣

We know by Corollary 2.42 that $p(x) \otimes p(y) \otimes t(z) = t(z) \otimes p(x) \otimes p(y)$. So, the claim implies that $\alpha_2 \models t|_{C\beta_1\beta_1'}$. Thus, k has the required properties.

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Let us now prove the second part of the statement. Note that it suffices to prove it for *some* realization of the complete type $tp(a_2/C) \otimes K|_C$. One may pick (α_2, k) as above, then one has indeed $(\alpha_2, \alpha_2') \models tp(a_2/C) \otimes tp(a_2/C)$, as required.

Now, let us assume that the hypothesis (R) of Theorem 3.37 holds, and prove finiteness. Forget the previous definitions of α_3, β_1, and β_1'. Let k in K be such that $k \models K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha_2'$. Let us show that $k \in acl(C\alpha_2\alpha_2')$. As K is the image of $p \otimes p$ (see the second point of Remark 4.9), we can apply Remark 2.31. We then find β_1, β_1' realizations of $p = tp(b_1/C)$ such that $\beta_1\beta_1' \models p \otimes p|_{C\alpha_2}$ and $k = [f_{\beta_1'}^{-1} \circ f_{\beta_1}]$. Then, by commutativity, α_2 realizes $t|_{C\beta_1\beta_1'}$. Thus, we have

$$f_{\beta_1'}^{-1} \circ f_{\beta_1}(\alpha_2) = k(\alpha_2) = \alpha_2'.$$

Let $\alpha_3 = f_{\beta_1}(\alpha_2) = f_{\beta_1'}(\alpha_2')$. In order to symmetrize the information on α_2 and α_2', let us prove the following.

Claim 4.23. *We have* $\beta_1\beta_1' \models p \otimes p|_{C\alpha_2'}$.

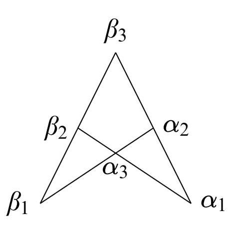
Proof. Since $\beta_1\beta_1' \models p \otimes p|_{C\alpha_2}$, we have, by commutativity, $\beta_1' \models p|_{C\beta_1\alpha_2}$, so $\beta_1' \models p|_{C\beta_1\alpha_2\alpha_3}$. Moreover, $\beta_1 \models p|_{\alpha_2}$, so, by Proposition 4.2(1), commutativity and stationarity, we have $\beta_1 \models p|_{\alpha_3}$. So $\beta_1'\beta_1 \models p \otimes p|_{C\alpha_3}$. So, by commutativity, $\beta_1\beta_1' \models p \otimes p|_{C\alpha_3}$. So $\beta_1 \models p|_{C\alpha_3\beta_1'}$, so $\beta_1 \models p|_{C\beta_1'\alpha_2'}$. Also, applying Proposition 4.2(3) (and commutativity and stationarity) to the hypothesis "$\beta_1' \models p|_{C\alpha_3}$" , we have $\beta_1' \models p|_{C\alpha_2'}$. Then, by definition of a tensor product, we have $\beta_1\beta_1' \models p \otimes p|_{C\alpha_2'}$, as claimed. ⊣

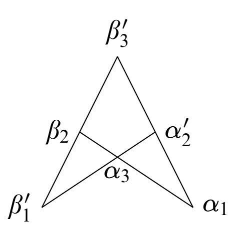
Let β_2 realize $tp(b_2/M)|_{C\alpha_3\beta_1\beta_1'\alpha_2\alpha_2'}$. By stationarity, we have $\beta_1\beta_2 \equiv_C \beta_1'\beta_2 \equiv_C b_1b_2$. Then, let $\beta_3\beta_3'$ be such that $\beta_1\beta_2\beta_3 \equiv_C \beta_1'\beta_2\beta_3' \equiv_C b_1b_2b_3$. Then, since $\beta_2 \mathop{\smile\!\!\!\!\!\mid}_C \beta_1\alpha_2$, and $\beta_1 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_2$, we have, by Lemma 2.23, $\beta_2\beta_1 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_2$, so $\alpha_2 \mathop{\smile\!\!\!\!\!\mid}_C \beta_2\beta_1$, so $\alpha_2 \mathop{\smile\!\!\!\!\!\mid}_C \beta_3\beta_2\beta_1$. Then, by stationarity, we have $\alpha_2\beta_1\beta_2\beta_3 \equiv_C a_2b_1b_2b_3$. So, we have

$$\alpha_2\alpha_3\beta_1\beta_2\beta_3 \equiv_C a_2a_3b_1b_2b_3,$$

because $\alpha_3 = f_{\beta_1}(\alpha_2)$. Let $\alpha_1 := g_{\beta_2}(\alpha_3)$. Then $\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \equiv_C a_1a_2a_3b_1b_2b_3$. By symmetric arguments, we also have $\alpha_1\alpha_2'\alpha_3\beta_1'\beta_2\beta_3' \equiv_C a_1a_2a_3b_1b_2b_3$.

Thus, we get the following configurations, which have the $(\beta_2, \alpha_3, \alpha_1)$-line in common:





By Lemma 4.6, the germ $k = [f_{\beta_1'}^{-1} \circ f_{\beta_1}] = [f_{\beta_1'}^{-1} \circ g_{\beta_2}^{-1} \circ g_{\beta_2} \circ f_{\beta_1}]$ is algebraic over $C\beta_3\beta_3'$. Besides, using the hypothesis (R) of Theorem 3.37, and Remark

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3.39(1), we know that $\beta_1\beta_1' \in acl(C\alpha_3\alpha_2\alpha_2')$, so $k \in acl(C\alpha_3\alpha_2\alpha_2')$. If we show that $\alpha_3 \mathop{\smile\!\!\!\!\!\mid}_{C\alpha_2\alpha_2'} \beta_3\beta_3'$, we can then apply Proposition 2.10(4), to deduce that $k \in acl(C\alpha_2\alpha_2')$. To that end, using Proposition 2.10(1) and (3), and recalling that $\alpha_2' \in acl(C\beta_3'\alpha_1) \subseteq acl(C\beta_3'\beta_3\alpha_2)$, it suffices to prove the

Claim 4.24. *We have* $\alpha_3 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_2\beta_3\beta_3'$.

Proof. By Claim 4.23, we have $\beta_1' \models p|_{C\beta_1\alpha_2}$. So $\beta_1' \mathop{\smile\!\!\!\!\!\mid}_C \beta_1\alpha_2$. Moreover, by choice of β_2, we have $\beta_2 \mathop{\smile\!\!\!\!\!\mid}_C \beta_1'\beta_1\alpha_2$. The type $tp(\beta_1\alpha_2/C)$ being generically stable, because $\beta_1\alpha_2 \equiv_C b_1a_2$, we may apply Lemma 2.23, which yields $\beta_2\beta_1\alpha_2 \mathop{\smile\!\!\!\!\!\mid}_C \beta_1'$. By symmetry, we deduce that $\beta_1' \mathop{\smile\!\!\!\!\!\mid}_C \beta_2\beta_1\alpha_2$, so $\beta_1' \mathop{\smile\!\!\!\!\!\mid}_{C\beta_2} \beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3$. Since $\beta_3' \in acl(C\beta_2\beta_1')$, this implies by Proposition 2.10(3)(b) that

$$\beta_3' \mathop{\smile\!\!\!\!\!\mid}_{C\beta_2} \beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3.$$

As $\beta_3' \mathop{\smile\!\!\!\!\!\mid}_C \beta_2$, we have, by transitivity for generically stable types, $\beta_3' \mathop{\smile\!\!\!\!\!\mid}_C \beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3$. So $\beta_3' \mathop{\smile\!\!\!\!\!\mid}_C \beta_3\alpha_2\alpha_3$. Since $\alpha_3 \mathop{\smile\!\!\!\!\!\mid}_C \beta_3\alpha_2$, and $tp(\beta_3\alpha_2/C)$ is generically stable, we can apply Lemma 2.23 again, to get $\beta_3'\beta_3\alpha_2 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_3$. Now, the type $tp(\beta_3'\beta_3\alpha_2/C)$ is extensible, for it is the tensor product $tp(\beta_3'/C) \otimes tp(\beta_3\alpha_2/C)$. Thus, we may apply Proposition 2.22, to deduce that $\alpha_3 \mathop{\smile\!\!\!\!\!\mid}_C \alpha_2\beta_3\beta_3'$, as desired. ⊣

Thus, we have proved that $k \in acl(C\alpha_2\alpha_2')$. This holds for all realizations of the partial type defined by "$k \models K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha_2'$" so, by compactness, there are only finitely many $k \in K$ satisfying $k \models K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha_2'$. ⊣

We can now prove that K behaves like the generic of a group:

Corollary 4.25. *Let* (f_1, f_2, f_3, f_4) *be a family of elements of* F *which realizes the tensor product* $F^{\otimes 4}|_C$. *Let* $E \supseteq C$. *Assume that* f_1f_2 *realizes* $F \otimes F|_{Ef_3f_4}$. *Then, there exist* $f, f' \in F$ *such that*:

1. *We have* $(f_1^{-1} \circ f_2) \circ (f_3^{-1} \circ f_4) = f^{-1} \circ f'$.
2. *The pair* (f, f') *realizes* $F \otimes F|_{Cf_1^{-1}\circ f_2}$ *and* $F \otimes F|_{Ef_3f_4}$.
3. *We have* $(f_1^{-1} \circ f_2) \circ (f_3^{-1} \circ f_4) \models K|_{Cf_1^{-1}\circ f_2}$ *and* $(f_1^{-1} \circ f_2) \circ (f_3^{-1} \circ f_4) \models K|_{Ef_3f_4}$.

Note that neither the hypotheses nor the conclusion are symmetric: f_3, f_4 do not necessarily realize $F|_E$, whereas f_1, f_2, f, and f' do.

Proof. Let g_2 realize $G|_{Ef_1f_2f_3f_4}$. Then, by Lemma 4.10, there exists $g_3 \in G$ such that $g_2 \circ f_3 = g_3 \circ f_2$ and $g_3 \mathop{\smile\!\!\!\!\!\mid}_C g_2f_3$. We know that $g_2 \models G|_{Cf_1f_2f_3f_4}$, and $f_2f_3f_4 \models F^{\otimes 3}|_{Cf_1}$. By choice of g_2, this implies $g_2f_2f_3f_4 \models G \otimes F^{\otimes 3}|_{Cf_1}$. Then, by commutativity, we have $f_2f_3 \models F \otimes F|_{Cg_2f_1f_4}$.

Then, by Lemma 4.14 applied to $D = Cg_2f_1f_4$, we have $f_3 \circ f_2^{-1} \models L|_{Cg_2f_1f_3f_4}$. So $g_2 \circ f_3 \circ f_2^{-1} \mathop{\smile\!\!\!\!\!\mid}_{Cg_2} f_1f_3f_4$. In other words, $g_3 \mathop{\smile\!\!\!\!\!\mid}_{Cg_2} f_1f_3f_4$. Since $g_3 \mathop{\smile\!\!\!\!\!\mid}_C g_2$, we then have $g_3 \mathop{\smile\!\!\!\!\!\mid}_C g_2f_1f_3f_4$. Thus, by stationarity, $g_3 \models G|_{Cg_2f_1f_3f_4}$. Moreover, we have chosen g_2 so that $g_2 \models G|_{Ef_1f_2f_3f_4}$, hence

$$g_3g_2f_4 \models G \otimes G \otimes F|_{Cf_1f_3}.$$

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Then, we can again apply Lemma 4.10, for the germs g_2, g_3 and f_4. We thus obtain a germ $f_5 \in F$ such that $g_3 \circ f_5 = g_2 \circ f_4$ and $f_5 \downarrow_C g_2 f_4$. We will show that $f = f_1$ and $f' = f_5$ have the required properties.

Compute: $f_1^{-1} \circ f_2 \circ f_3^{-1} \circ f_4 = f_1^{-1} \circ g_3^{-1} \circ g_2 \circ f_4 = f_1^{-1} \circ f_5$. In other words, $f_5 = f_2 \circ f_3^{-1} \circ f_4$. In particular, $f_5 \downarrow_C f_1$, so, by stationarity, $f_1 f_5 \models F \otimes F|_C$.

CLAIM 4.26. *The pair* (f_1, f_5) *realizes* $F \otimes F|_{Cf_1^{-1}\circ f_2}$.

PROOF. By Lemma 4.14, we know that $f_1^{-1} \circ f_2 \models K|_{Ef_1}$, so $(f_1^{-1} \circ f_2, f_1) \models K \otimes F|_E$. So, by commutativity, f_1 realizes $F|_{Ef_1^{-1}\circ f_2}$, so a fortiori f_1 realizes $F|_{Cf_1^{-1}\circ f_2}$. It then remains to show that $f_5 \models F|_{Cf_1 \cup (f_1^{-1}\circ f_2)}$, i.e., $f_5 \models F|_{Cf_1 f_2}$. By stationarity, it is enough to prove that $f_5 \downarrow_C f_1 f_2$. Using the hypotheses on (f_1, f_2, f_3, f_4), we have, by commutativity, $f_2 f_3 \models F \otimes F|_{Cf_1 f_4}$. So, by Lemma 4.14 applied to (f_3, f_2), with $D = Cf_1 f_4$, we have $f_2 \circ f_3^{-1} \models L|_{Cf_1 f_2 f_4}$. So $f_5 = f_2 \circ f_3^{-1} \circ f_4 \downarrow_{Cf_4} f_1 f_2$. By construction, we have $f_5 \downarrow_C g_2 f_4$. Thus, by transitivity, $f_5 \downarrow_C f_1 f_2 f_4$. In particular, $f_5 \downarrow_C f_1 f_2$, as desired. ⊣

CLAIM 4.27. *The pair* (f_1, f_5) *realizes* $F \otimes F|_{Ef_3 f_4}$.

PROOF. By commutativity, it suffices to show that $f_5 f_1$ realizes $F \otimes F|_{Ef_3 f_4}$. We know by hypothesis that f_1 realizes $F|_{Ef_3 f_4}$. By stationarity of $tp(f_5/C)$, it remains to show that $f_5 \downarrow_C Ef_1 f_3 f_4$. On the one hand, by hypothesis (and symmetry), we have $f_2 \downarrow_C Ef_1 f_3 f_4$. So $f_2 \circ f_3^{-1} \circ f_4 \downarrow_{Cf_3 f_4} Ef_1$, i.e.,

$$f_5 \underset{Cf_3 f_4}{\downarrow} Ef_1. \tag{3}$$

On the other hand, Lemma 4.14 applied with $D = Cf_1 f_4$ implies that $f_2 \circ f_3^{-1}$ realizes $L|_{Cf_1 f_3 f_4}$, so $f_2 \circ f_3^{-1} \downarrow_C f_1 f_3 f_4$, so $f_5 \downarrow_{Cf_4} f_1 f_3$. We also know that $f_5 \downarrow_C g_2 f_4$. So, by transitivity, $f_5 \downarrow_C f_1 f_3 f_4$. So, by transitivity in (3), we have $f_5 \downarrow_C Ef_1 f_3 f_4$, as desired. ⊣

Finally, the third point follows from the first two points and the definition of K. ⊣

COROLLARY 4.28. *Let* k_1, k_2 *be realizations of* $K|_C$, *and* $D \supseteq C$, *such that* $k_1 \models K|_{Dk_2}$. *Then* $k_1 \circ k_2 \models K|_{Dk_2}$ *and* $k_2 \circ k_1 \models K|_{Dk_2}$.

PROOF. Let us show that $k_2 \circ k_1 \models K|_{Dk_2}$, the other result being more straightforward. By definition of K, there are f_3, f_4 such that $(f_3, f_4) \models F \otimes F|_C$ and $f_3^{-1} \circ f_4 = k_2$. Let f_1, f_2 in F be such that $(f_1, f_2) \models F \otimes F|_{Df_3 f_4}$. Then, if $k' = f_1^{-1} \circ f_2$, we know that $k' \models K|_{Df_3 f_4}$, so in particular $k' \models K|_{Dk_2}$, so $k' \equiv_{Dk_2} k_1$. So, it suffices to prove that $k_2 \circ k' \models K|_{Dk_2}$.

Since F commutes with itself, we can apply Corollary 4.25 to the family (f_2, f_1, f_4, f_3). It yields that $f_2^{-1} \circ f_1 \circ f_4^{-1} \circ f_3$ realizes $K|_{Dk_2}$. Then, by Remark 4.9(4), the inverse $f_3^{-1} \circ f_4 \circ f_1^{-1} \circ f_2$ still realizes $K|_{Dk_2}$. In other words, $k_2 \circ k' \models K|_{Dk_2}$, as stated.

To show that $k_1 \circ k_2 \models K|_{Dk_2}$, we also apply Corollary 4.25, without permuting functions, nor considering inverses. ⊣

Recall that, by Definition 4.7, the set Γ is the set of composites $k_1 \circ k_2$, where k_1, k_2 realize $K|_C$.

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PROPOSITION 4.29. *The C-type-definable-set* Γ *is closed under composition of germs.*

PROOF. Let k_1, k_2, k_3, k_4 realize $K|_C$. Let $D = Ck_1k_2k_3k_4$. Let $k \models K|_D$. Then, using Corollary 4.28 four times, we deduce that $k_1 \circ k_2 \circ k_3 \circ k_4 \circ k \models K|_D$.

Finally, we notice that $k_1 \circ k_2 \circ k_3 \circ k_4 = (k_1 \circ k_2 \circ k_3 \circ k_4 \circ k) \circ (k^{-1})$. Since k^{-1} and $(k_1 \circ k_2 \circ k_3 \circ k_4 \circ k)$ are in K, the germ $k_1 \circ k_2 \circ k_3 \circ k_4$ is indeed in $K \circ K = \Gamma$. ⊣

COROLLARY 4.30. *Composition of germs induces a definable group structure on the type-definable set* Γ.

PROOF. Corollary 4.3 implies that the composition of germs is associative, for it is induced by composition of functions. More precisely, let $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. By Corollary 4.18, let $\beta_1, \beta_1', \beta_2, \beta_2', \beta_3, \beta_3'$ be realizations of $tp(b_1/C)$ such that, for $i = 1, 2, 3$, we have $\gamma_i = [f_{\beta_i}^{-1} \circ f_{\beta_i'}]$. Then, by definition, we have $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = ([f_{\beta_1}^{-1} \circ f_{\beta_1'}] \cdot [f_{\beta_2}^{-1} \circ f_{\beta_2'}]) \cdot [f_{\beta_3}^{-1} \circ f_{\beta_3'}]$. This is equal to the germ of the definable map $(f_{\beta_1}^{-1} \circ f_{\beta_1'} \circ f_{\beta_2}^{-1} \circ f_{\beta_2'}) \circ f_{\beta_3}^{-1} \circ f_{\beta_3'}$. By associativity of composition for functions, that definable map is equal to $f_{\beta_1}^{-1} \circ f_{\beta_1'} \circ (f_{\beta_2}^{-1} \circ f_{\beta_2'} \circ f_{\beta_3}^{-1} \circ f_{\beta_3'})$. Thus, by computing the germs of these maps, we get $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$, as desired.

Moreover, by the fourth point of Remark 4.9, K is closed under taking inverses, and so is Γ. Besides, we have proved that Γ is closed under composition.

Finally, the germ of the identity is indeed in Γ, for, if $k \in K$, then $id = k \circ k^{-1} \in \Gamma$. ⊣

4.2. Properties of the group. *In this subsection, we retain the context of the end of the previous subsection. In particular, the conclusion of Proposition 3.38 holds.*

PROPOSITION 4.31. *The type-definable group* Γ *is type-connected, with generic K.*

PROOF. First, recall that K is a C-definable type. We will prove the following:

$$Stab_\Gamma(K) = \Gamma. \tag{4}$$

Let $g \in \Gamma(M)$. Let k be a realization of $K|_M$. By definition of Γ, and since M is sufficiently saturated, there exist $k_1, k_2 \in K(M)$ such that $g = k_1 \circ k_2$. Then, applying Corollary 4.28 twice, we show that $g \circ k = k_1 \circ k_2 \circ k$ still realizes $K|_M$. Thus (4) holds. We then apply Lemma 3.7, to conclude that K is the unique generic of Γ. ⊣

DEFINITION 4.32. Let Y be the set of pairs (γ, α) where $\gamma \in \Gamma$ and $\alpha \models tp(a_2/C)$. Let E be the binary relation on Y defined by $(\gamma, \alpha)\ E\ (\gamma', \alpha')$ if and only if for some, equivalently, for any, element $\sigma \models K|_{C\gamma\gamma'\alpha\alpha'}$, we have $(\sigma \cdot \gamma)(\alpha) = (\sigma \cdot \gamma')(\alpha')$.

PROPOSITION 4.33. *The set* Y *is type-definable over* C*, and* E *is relatively* C*-definable. The relation* E *is an equivalence relation.*

PROOF. Type-definability over C of Y is immediate. Since K is a complete definable type, the formula $\phi(\gamma_1, \alpha_1, \gamma_2, \alpha_2)$ defines E inside $Y \times Y$, where $\phi(\gamma_1, \alpha_1, \gamma_2, \alpha_2) = d_K z\, [(z \cdot \gamma_1)(\alpha_1) = (z \cdot \gamma_2)(\alpha_2)]$.

Let us now prove that E is an equivalence relation. Reflexivity and symmetry are clear. Let us prove transitivity. So, assume that $(\gamma, \alpha)E(\gamma', \alpha')$ and $(\gamma', \alpha')E(\gamma'', \alpha'')$.

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Let σ be a realization of $K|_{C\gamma\gamma'\gamma''\alpha\alpha'\alpha''}$. Then, we have $(\sigma \cdot \gamma)(\alpha) = (\sigma \cdot \gamma')(\alpha') = (\sigma \cdot \gamma'')(\alpha'')$, which proves transitivity. ⊣

DEFINITION 4.34. Let X denote the type-definable set Y/E. Let P_2 denote the type-definable set of realizations of $tp(a_2/C)$.

LEMMA 4.35. *Let $(k, a) \models K|_C \otimes tp(a_2/C)$. Then we have $(k, a)E(1, k(a))$.*

PROOF. Let $\sigma \models K|_{Cak}$. Then, by definition of the product in the group Γ, we have $\sigma(k(a)) = (\sigma \cdot k)(a)$, which proves the result. ⊣

PROPOSITION 4.36. *For each $\sigma \in \Gamma$, the map $(\gamma, a) \mapsto (\sigma \cdot \gamma, a)$ factorizes through the equivalence relation E, and this induces a definable action of Γ on X.*

PROOF. Let $\sigma \in \Gamma$. Pick (γ, a) and (γ', a') that are in the same E-class. Let us show that $(\sigma \cdot \gamma, a)$ and $(\sigma \cdot \gamma', a')$ are in the same E-class. By assumption, there exists $\tau \models K|_{C\gamma\gamma' aa'}$ such that $(\tau \cdot \gamma) \cdot a = (\tau \cdot \gamma') \cdot a'$. In fact, by completeness of the type $K|_{C\gamma\gamma' aa'}$, the equality holds for all such τ. Let τ_1 realize $K|_{C\sigma\gamma\gamma' aa'}$. Then, by genericity of K and Lemma 3.7, the element $\tau_2 = \tau_1 \cdot \sigma$ also realizes $K|_{C\sigma\gamma\gamma' aa'}$. Thus, we have $(\tau_2 \cdot \gamma) \cdot a = (\tau_2 \cdot \gamma') \cdot a'$, which implies $(\tau_1 \cdot \sigma \cdot \gamma) \cdot a = (\tau_1 \cdot \sigma \cdot \gamma') \cdot a'$, which proves that the map does factor through E.

The fact that this induces a definable action follows from the universal property of the quotient map $\pi : Y \to Y/E$, and the fact that Γ acts on itself by left translation. More explicitly, let $\sigma, \tau \in \Gamma$, and $c = (\gamma, a) \in Y$. By construction, we have $\tau \cdot (\sigma \cdot \pi(c)) = \tau \cdot (\pi(\sigma \cdot \gamma, a)) = \pi(\tau \cdot (\sigma \cdot \gamma), a) = \pi((\tau \cdot \sigma) \cdot \gamma, a) = (\tau \cdot \sigma) \cdot \pi(\gamma, a)$, as desired. ⊣

PROPOSITION 4.37. *The type-definable set P_2 embeds definably into X, via the injective map $a \mapsto [1, a]_E$. Moreover, the action of Γ on X extends the generic action of K on P_2.*

PROOF. Let us prove injectivity. Let $a, a' \in P_2$ be such that $[1, a]_E = [1, a']_E$. Then, there exists σ realizing $K|_{Caa'}$ such that $\sigma(a) = \sigma(a')$. Now, recall that σ is the germ of a definable injection. Thus, we have $a = a'$, which proves injectivity. The fact that the action of Γ on X extends the generic action of K on P_2 follows from Lemma 4.35. ⊣

PROPOSITION 4.38.

1. *The action of Γ on X is transitive.*
2. *The (image of the) type $tp(a_2/C)$ is generic in the space X, which is type-connected.*
3. *The action of Γ on X is faithful.*
4. *Under the hypothesis (R) of Theorem 3.37, the action is almost free: the stabilizers are finite.*

PROOF. 1. We start with the following.

CLAIM 4.39. *Let a, a' realize $tp(a_2/C)$. Then, there exists $\sigma \in \Gamma$ such that $(\sigma, a)\, E\, (1, a')$.*

PROOF. Given such a, a', let a'' realize $tp(a_2/C)|_{Caa'}$. Then, by Lemma 4.19, there exist τ_1, τ_2 such that $\tau_1 \models K|_{Ca} \cup K|_{Ca''}$, $\tau_2 \models K|_{Ca'} \cup K|_{Ca''}$, $\tau_1(a) = a''$ and

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$\tau_2(a'') = a'$. Let σ be $\tau_2 \cdot \tau_1 \in \Gamma$. Since we already know that Γ acts on X, it suffices to note that, by Lemma 4.35, the element τ_1 sends the class $[1, a]_E$ to $[1, a'']_E$, which is then sent by τ_2 to $[1, a']_E$, as desired. ⊣

Then, let $[\gamma, a]_E$ be an arbitrary element of X. By the claim, let $\sigma \in \Gamma$ be such that $(\sigma, a)E(1, a_2)$. Then, $\gamma \cdot \sigma^{-1}[1, a_2]_E = \gamma \cdot \sigma^{-1}([\sigma, a]_E) = [\gamma, a]_E$. So, we have proved that any element is in the orbit of $[1, a_2]_E$, which shows transitivity.

2. Let us show that the class of $(1, a_2)$ is generic in X.

Claim 4.40. *The stabilizer of the type $q = tp([1, a_2]_E/C)$, in the sense of the action of Γ on X, contains $K|_C$.*

Proof. Let $k \models K|_C$, let $\alpha_2 \models tp(a_2/C)|_{Ck}$, and $c = [1, \alpha_2]_E$, so that c realizes $q|_{Ck}$. Then, by Lemma 4.19, we have $k(\alpha_2) \models tp(a_2/C)|_{C\alpha_2}$, in particular $k(\alpha_2) \models tp(a_2/C)$.

Let us show that $k(\alpha_2) \mathop{\smile\!\!\!\!\!\!\mid}_C k$, which, by stationarity, will imply that $k(\alpha_2) \models tp(a_2/C)|_{Ck}$. By the second point of Remark 4.9, there exist β_1, β_1' realizing $tp(b_1/C)$ such that $k = [f_{\beta_1}^{-1} \circ f_{\beta_1'}]$, with $(\beta_1, \beta_1', \alpha_2) \models tp(b_1/C)^{\otimes 2} \otimes tp(a_2/C)$. In other words, the triple $(\beta_1, \beta_1', \alpha_2)$ is independent over C. Then, by the first point of Proposition 4.2 applied to $A = C\beta_1$, $D = C$, and β_1', we have $f_{\beta_1'}(\alpha_2) \mathop{\smile\!\!\!\!\!\!\mid}_C \beta_1\beta_1'$. Let α_3 denote $f_{\beta_1'}(\alpha_2)$. Then, we know that $\alpha_3 \equiv_C a_3$. We can thus apply the third point of Proposition 4.2, for $A = C\beta_1'$, $D = C$, and β_1, to deduce that $f_{\beta_1}^{-1}(\alpha_3) \mathop{\smile\!\!\!\!\!\!\mid}_C \beta_1'\beta_1$. In other words, we have $f_{\beta_1}^{-1}(\alpha_3) = f_{\beta_1}^{-1}(f_{\beta_1}(\alpha_2)) = k(\alpha_2) \mathop{\smile\!\!\!\!\!\!\mid}_C \beta_1'\beta_1$. In particular, we have $k(\alpha_2) \mathop{\smile\!\!\!\!\!\!\mid}_C k$, as required.

To conclude, recall that, by Lemma 4.35, we have $(1, k(a_2))E(k, a_2)$. Thus, the element $k(c)$ is equal to $[1, k(a_2)]_E$, which realizes $q|_{Ck}$, hence we have $k \in Stab(q)$, as required. ⊣

Moreover, the stabilizer of q is a C-type-definable subgroup of Γ. Since K generates Γ, the stabilizer of q is Γ itself, which proves genericity of q and type-connectedness of X.

3. Let $g \in \Gamma$ be an element that acts trivially on X. Let us show that $g = 1$. We know that there exist $k_1, k_2 \in K$ such that $g = k_1^{-1} \circ k_2$. Then, by the hypothesis on g, we deduce that, for all a realizing $tp(a_2/C)|_{Ck_1k_2}$, we have $k_1(a) = k_2(a)$. Thus, by definition of a germ, we get $k_1 = k_2$, which implies $g = 1$.

4. Now, let us work under the hypothesis (R) of Theorem 3.37. By transitivity of the action, it suffices to show that the stabilizer of the E-class of $(1, a_2)$ is finite. Let $\sigma \in \Gamma$ be such that $\sigma \cdot (1, a_2)\, E\, (1, a_2)$. By definition, there exists τ realizing $K|_{C\sigma a_2}$ such that $\tau \cdot \sigma(a_2) = \tau(a_2)$.

Claim 4.41. *The element $\tau(a_2)$ realizes $tp(a_2/C)|_{Ca_2}$.*

Proof. By choice of τ, we have $(\tau, a_2) \models K|_C \otimes tp(a_2/C)$. The result then follows from the second point of Lemma 4.19. ⊣

Moreover, since $\tau \models K|_{C\sigma a_2}$, and $\sigma \in Stab^r(K) = \Gamma$ (by Lemma 3.7(2)), we know that $\tau \cdot \sigma$ realizes $K|_{C\sigma a_2}$. Thus, by the second point of Lemma 4.19, we deduce that $\tau \cdot \sigma \in acl(Ca_2\tau(a_2))$. Then, $\sigma \in acl(Ca_2\tau)$. However, the element τ satisfies $\tau \mathop{\smile\!\!\!\!\!\!\mid}_C a_2\sigma$, so we have $\tau \mathop{\smile\!\!\!\!\!\!\mid}_{Ca_2} \sigma$. Then, by Proposition 2.10(4), we have $\sigma \in acl(Ca_2)$. Finally, by compactness, the stabilizer of the E-class of $(1, a_2)$ is finite, as desired. ⊣

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4.3. End of the proof. Here, we return to the context of Theorem 3.37.

Proof of Theorem 3.37. By Proposition 3.38, there are elements $b_1', b_2'', b_3''' \in M$ and a configuration $(a_1, a_2, a_3, b_1, b_2, b_3)$ equivalent over M to $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$, such that:

1. The tuple $b_1'b_2''b_3'''$ realizes $(p_{0_{y_1}} \otimes p_{0_{y_2}} \otimes p_{0_{y_3}})|_{C_0}$.
2. The type $tp(a_1, a_2, a_3, b_1, b_2, b_3/M)$ is generically stable over $C := acl(C_0b_1'b_2''b_3''')$.
3. We have $a_1 \in dcl(Ca_3b_2)$, $a_2 \in dcl(Ca_3b_1)$, and $a_3 \in dcl(Ca_1b_2) \cap dcl(Ca_2b_1)$.

Then, by the results proved above in Section 4, especially those between Propositions 4.31 and 4.38, there is a type-connected C-type-definable group Γ with a (unique) generically stable generic K, and a C-type-definable set X equipped with a transitive and faithful C-definable action of Γ, such that the C-type-definable set of realizations of $tp(a_2/C)$ embeds C-definably into X.

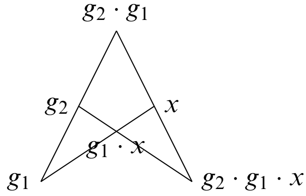
We shall now construct, in the general case, a definable group configuration equivalent over M to the initial one. *At the end of the proof, we will explain how to adjust the construction in the case where (R) holds. We will be replacing the space X with Γ itself, altering the configuration built in the general case, and using finiteness of the stabilizers to show that the altered configuration is equivalent to the original one.*

To build a definable group configuration equivalent over M to $(a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0)$, it suffices to build one equivalent over M to $(a_1, a_2, a_3, b_1, b_2, b_3)$. Let $b_1^1, b_2^1, b_3^1 \in M$ be such that $b_1^1b_2^1b_3^1 \equiv_C b_1b_2b_3$.

Claim 4.42. *We have $b_1^1b_2^1b_3^1a_3 \equiv_C b_1b_2b_3a_3$.*

Proof. Since $a_3 \mathop{\smile\!\!\!\!\!\vert}_C Mb_1b_2b_3$, and $tp(a_3/C)$ is generically stable, stationarity implies that $tp(a_3/Mb_1b_2b_3)$ admits a C-invariant extension. The result then follows from the hypothesis $b_1^1, b_2^1, b_3^1 \in M$ and $b_1^1b_2^1b_3^1 \equiv_C b_1b_2b_3$. ⊣

Consider the following quadrangle:



with the following definitions:

- $g_1 = [f_{b_1^1}^{-1}] \circ [f_{b_1}]$.
- $g_2 = [f_{b_1^1}^{-1}] \circ [g_{b_2^1}^{-1}] \circ [g_{b_2}] \circ [f_{b_1^1}]$.
- $x = a_2$.

There are several facts to check, in order to make sure this is well-defined and equivalent to the original quadrangle. Note that, by Proposition 3.32, proving the equivalence with the original quadrangle yields that the quadrangle is a generically stable group configuration over M. To simplify notations, let $g_3 = g_2 \cdot g_1$, $y_1 = g_2 \cdot g_1 \cdot x$, $y_2 = x$, and $y_3 = g_1 \cdot x$.

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Claim 4.43. *For $i = 1, 2, 3$, we have $acl(Mg_i) = acl(Mb_i)$.*

Proof. For $i = 1$, this is a consequence of the definition of definable group configurations, and of Proposition 2.32, applied to $a = a_2$ and $c = g_i$. In fact, the element g_1 is interalgebraic over M with the germ $[f_{b_1}]$, which is, by Proposition 2.32, interdefinable over $C \subseteq M$ with the canonical basis $Cb(a_2a_3/acl(Cb_1))$, which is, by point 4 of Definition 3.11, interalgebraic over C with b_1.

For $i = 2$, we apply Proposition 2.32 to $a = a_3$ and $c = b_2$. For $i = 3$, we use Lemma 4.6 and Proposition 2.32. ⊣

Claim 4.44. *The elements g_1, g_2, g_3 realize $K|_M$, and we have $g_1 \downarrow_M g_2$.*

Proof. Since K is generically stable, it suffices by stationarity (Proposition 2.15(6)) to check that these elements realize $K|_C$, and that each is independent from M over C. In fact, since K is the unique generic of Γ, it suffices to check it for g_1 and g_2, and to prove the independence $g_1 \downarrow_M g_2$. First note that, by definition of K, we have $tp(g_1/C) = K|_C$.

For $g_1 = [f^{-1}_{b^1_1}] \circ [f_{b_1}]$, we know that $b_1 \downarrow_C M$ and $b^1_1 \in M$, so that $g_1 \downarrow_{C[f_{b^1_1}]} M$. Then, applying Lemma 4.14, we have $g_1 \downarrow_C [f_{b^1_1}]$. Since $tp(g_1/C) = K|_C$ is generically stable, we can apply transitivity, to get $g_1 \downarrow_C M$, as desired.

For $g_2 = [f^{-1}_{b^1_1}] \circ [g^{-1}_{b^1_2}] \circ [g_{b_2}] \circ [f_{b^1_1}]$, the ideas are similar: one deduces from $b_2 \downarrow_C M$ the following:

$$g_2 \underset{C[f_{b^1_1}][g_{b^1_2}]}{\downarrow} M. \tag{5}$$

Also, we have by construction $[f_{b^1_1}] \frown [g_{b^1_2}] \models (F \otimes G)|_C$ and $b_2 \downarrow_C M$, so $b_2 \downarrow_C [f_{b^1_1}] \frown [g_{b^1_2}]$, thus $[g_{b_2}] \downarrow_C [f_{b^1_1}] \frown [g_{b^1_2}]$. Therefore, by stationarity of G, we have $[g_{b_2}] \models G|_{C[f_{b^1_1}] \frown [g_{b^1_2}]}$. In other words, the triple $[g_{b_2}] \frown [f_{b^1_1}] \frown [g_{b^1_2}]$ realizes the tensor product $(G \otimes F \otimes G)|_C$. Hence, by Lemma 4.10 and commutativity of the tensor product above (see Remark 4.9(3)), there exists an $f \in F$ such that $[g_{b_2}] \circ [f_{b^1_1}] = [g_{b^1_2}] \circ f$ and $f \downarrow_C [f_{b^1_1}][g_{b^1_2}]$, which implies that $g_2 = [f^{-1}_{b^1_1}] \circ f \models K|_{C[g_{b^1_2}]}$. Then, by Lemma 4.14, we also have $[f^{-1}_{b^1_1}] \circ f \downarrow_C [f_{b^1_1}][g_{b^1_2}]$, i.e., $g_2 \downarrow_C [f_{b^1_1}][g_{b^1_2}]$. We can then apply transitivity to (5), just as before.

Finally, to show $g_1 \downarrow_M g_2$, note that $g_1 = [f^{-1}_{b^1_1}] \circ [f_{b_1}] \in dcl(Mb_1)$, because $b^1_1 \in M$, and that $g_2 = [f^{-1}_{b^1_1}] \circ [g^{-1}_{b^1_2}] \circ [g_{b_2}] \circ [f_{b^1_1}] \in dcl(Mb_2)$. Then, recall that $b_1 \downarrow_M b_2$, and apply Proposition 2.10(3)(b). ⊣

Claim 4.45. *The elements $y_1 = g_3(y_2)$ and $y_3 = g_1(y_2)$ are well-defined, and satisfy the following: $y_1 = f^{-1}_{b^1_1} \circ g^{-1}_{b^1_2}(a_1)$ and $y_3 = f^{-1}_{b^1_1}(a_3)$.*

Proof. Let us first show that the elements are well-defined. Since $tp(a_2/C)$ is generically stable, it suffices to check that $y_2 \downarrow_C g_3$ and $y_2 \downarrow_C g_1$. These verifications rely on the facts that $y_2 = a_2 \downarrow_C Mb_1b_2$ and $b^1_1 b^1_2 \in M$, and are left to the reader.

For the equalities, we have by definition that $b_1b_2 \mathop{\smile\!\!\!\!\!\!\mid}_C a_2$ and $g_{b_2} \circ f_{b_1}(y_2) = a_1 \mathop{\smile\!\!\!\!\!\!\mid}_C b_1^1 b_2^1$, so that $y_1 = g_3(y_2) = [f^{-1}_{b_1^1}] \circ [g^{-1}_{b_2^1}] \circ [g_{b_2}] \circ [f_{b_1}](a_2) = [f^{-1}_{b_1^1}] \circ [g^{-1}_{b_2^1}](a_1) = f^{-1}_{b_1^1} \circ g^{-1}_{b_2^1}(a_1)$.

Similarly, we have $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_C b_1 b_1^1$, and $f_{b_1}(a_2) = a_3 \mathop{\smile\!\!\!\!\!\!\mid}_C b_1^1$, which implies the following: $y_3 = g_1(y_2) = [f^{-1}_{b_1^1}] \circ [f_{b_1}](a_2) = [f^{-1}_{b_1^1}](a_3) = f^{-1}_{b_1^1}(a_3)$. ⊣

Claim 4.46. *For $i = 1, 2, 3$, we have $acl(My_i) = acl(Ma_i)$.*

Proof. This follows from the equalities in Claim 4.45, the fact that b_1^1, b_2^1 are in M, and the equality $y_2 = a_2$. ⊣

For the general case, all that remains to show is genericity over M of the elements $g_1 \in \Gamma$, $g_2 \in \Gamma$, and $x = a_2 \in X$. For g_1 and g_2, this follows from Proposition 4.31, and Claim 4.44. For $x = a_2$, this follows from Proposition 4.38(2), the fact $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_C M$, and stationarity of the generically stable generic of the space X.

So, all that remains is the case where (R) holds. Recall that, in that case, by Proposition 4.38(4), the stabilizers for the action of Γ on X are finite. We wish to build a configuration made only of elements of Γ, and make sure that it is equivalent over M to the one we just defined. *This is where finiteness of the stabilizers comes into play.*

Let $a \in X(M)$, and $Z \leq \Gamma(M)$ be the finite stabilizer of a for the action of Γ. Since the action is transitive, we have a Γ-equivariant (relatively) M-definable bijection $\rho : \Gamma/Z \simeq X$. In particular, there is a Γ-equivariant (relatively) M-definable finite-to-one surjection $\pi : \Gamma \to X$.

Claim 4.47. *We have $\pi_*(K|_M) = tp(a_2/M)$. In particular, we have $\pi_*(K|_C) = tp(a_2/C)$.*

Proof. Since π is Γ-equivariant, this follows from Lemma 3.5, and uniqueness of the generic of X (see Proposition 3.10(1)). ⊣

So, let $g \models K|_C$ be such that $\pi(g) = a_2 = y_2$. In the previous configuration, replace y_1 with $g_3 \cdot g$, y_2 with g, and y_3 with $g_1 \cdot g$. Since π is equivariant, it is straightforward to compute that $\pi(g_3 \cdot g) = g_3(a_2) = y_1$, $\pi(g) = y_2$, and $\pi(g_1 \cdot g) = g_1(a_2) = y_3$. Since π is M-definable and has finite fibers, this shows that the quadrangle $(g_3 \cdot g, g, g_1 \cdot g, g_1, g_2, g_3)$ is equivalent over M to $(y_1, y_2, y_3, g_1, g_2, g_3)$. Hence, letting Γ act on itself by left translations, we just constructed a generic configuration for this action, since g realizes $K|_C$. This configuration is equivalent over M to the original one. ⊣

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