*Second Meeting, 8th December 1899.*

R. F. MUIRHEAD, Esq., M.A., B.Sc., President, in the Chair.

## On the Evaluation of a certain Determinant.

By LAWRENCE CRAWFORD, M.A., D.Sc.

1. In a paper by Mr Arthur Berry, M.A., in the *Proceedings of the Cambridge Philosophical Society*, Volume X. Pt. I., "On the Evaluation of a certain Determinant which occurs in the mathematical theory of statistics and in that of elliptic geometry of any number of dimensions," a remark is made that in the case of $n = 3$ this determinant was readily evaluated by me by means of the formulæ of spherical trigonometry. I have thought that it might be of interest to show this evaluation, but I shall merely state the determinant at once of order 3, and leave the reader to refer to the paper quoted for the general determinant.

2. The question then is, to evaluate the determinant

$$\frac{\partial(\xi_{12},\ \xi_{13},\ \xi_{23})}{\partial(r_{12},\ r_{13},\ r_{23})},$$

where $\xi_{12} = R_{12}/\sqrt{R_{11}R_{22}}$, &c., where $R_{pq}$ is the minor (with proper sign) of the element in the $p$th row and $q$th column of R, where

$$R = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}$$

3. We have $R_{11} = 1 - r_{23}^2$, $\quad R_{22} = 1 - r_{13}^2$, $\quad R_{33} = 1 - r_{12}^2$;

$R_{12} = r_{13}r_{23} - r_{12}$, $\quad R_{13} = r_{12}r_{23} - r_{13}$, $\quad R_{23} = r_{12}r_{13} - r_{23}$.

Put then $r_{23} = \cos a$, $r_{13} = \cos b$, $r_{12} = \cos c$, where $a$, $b$, $c$ are the sides of a spherical triangle, then $R_{11} = \sin^2 a$, $R_{22} = \sin^2 b$, $R_{33} = \sin^2 c$,

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and $R_{12} = \cos b \cos a - \cos c = -\sin a \sin b \cos C$, $R_{13} = -\sin a \sin c \cos B$,
and $R_{23} = -\sin b \sin c \cos A$,

$$\therefore \quad \xi_{12} = -\cos C, \quad \xi_{13} = -\cos B, \quad \xi_{23} = -\cos A.$$

$$\therefore \text{ we wish J, where } J = -\frac{\partial(\cos C, \cos B, \cos A)}{\partial(\cos c, \cos b, \cos a)}$$

$$= -\frac{\partial(\cos A, \cos B, \cos C)}{\partial(\cos a, \cos b, \cos c)};$$

$$\therefore \quad J \times \frac{\partial(\cos a, \cos b, \cos c)}{\partial(a, b, c)} = -\frac{\partial(\cos A, \cos B, \cos C)}{\partial(a, b, c)}.$$

In this $a$, $b$, $c$ are independent, A, B, C functions of $a$, $b$, $c$.

$$\cos A = (\cos a - \cos b \cos c)/\sin b \sin c,$$

$$\therefore \quad \frac{\partial(\cos A)}{\partial a} = -\sin a/\sin b \sin c,$$

and
$$\frac{\partial(\cos A)}{\partial b} = \{\sin^2 b \cos c - (\cos a - \cos b \cos c)\cos b\}/\sin^2 b \sin c,$$

$$= (\cos c - \cos a \cos b)/\sin^2 b \sin c,$$

$$= \sin a \cos C/\sin b \sin c,$$

and
$$\frac{\partial(\cos A)}{\partial c} = \sin a \cos B/\sin b \sin c.$$

$$\therefore J \times \frac{\partial(\cos a, \cos b, \cos c)}{\partial(a, b, c)} = \frac{-1}{\sin^2 a \sin^2 b \sin^2 c} \begin{vmatrix} -\sin a & \sin a \cos C & \sin a \cos B \\ \sin b \cos C & -\sin b & \sin b \cos A \\ \sin c \cos B & \sin c \cos A & -\sin c \end{vmatrix}$$

$$= \frac{-1}{\sin a \sin b \sin c} \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$= -(-1 + 2\cos A \cos B \cos C + \cos^2 A + \cos^2 B + \cos^2 C)/\sin a \sin b \sin c$$

$$= (1 + 2\cos a' \cos b' \cos c' - \cos^2 a' - \cos^2 b' - \cos^2 c')/\sin a \sin b \sin c$$

(using the polar triangle)

$$= \sin^2 A' \sin^2 b' \sin^2 c'/\sin a \sin b \sin c$$

$$= \sin^2 a \sin^2 B \sin^2 C / \sin a \sin b \sin c$$

$$= \sin^2 A \sin B \sin C / \sin a.$$

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$$\text{Now } \frac{\partial(\cos a, \cos b, \cos c)}{\partial(a, b, c)} = \begin{vmatrix} -\sin a & 0 & 0 \\ 0 & -\sin b & 0 \\ 0 & 0 & -\sin c \end{vmatrix} = -\sin a \sin b \sin c;$$

$$\therefore \quad J = -\sin^2 A \sin B \sin C / \sin^2 a \sin b \sin c$$

$$= -(\sin A/\sin a)^4.$$

$$\text{Also } R = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$$

$$= \sin^2 A \sin^2 b \sin^2 c;$$

$$\therefore \quad J = -(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)^2 / \sin^4 a \sin^4 b \sin^4 c;$$

$$\therefore \quad J = -R^2/R_{11}^2 R_{22}^2 R_{33}^2.$$