THE D-PROPERTY AND THE SORGENFREY LINE

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Abstract

We show that for the Sorgenfrey line \( S \), the minimal dense linearly ordered extension of \( S \) is a D-space, but not a monotone D-space; the minimal closed linearly ordered extension of \( S \) is not a monotone D-space; the monotone D-property is inversely preserved by finite-to-one closed mappings, but cannot be inversely preserved by perfect mappings.


Keywords and phrases: Sorgenfrey line, real line, linearly ordered topological space, monotone D-space.

1. Introduction

The notion of D-spaces was introduced by van Douwen and interesting results for the D-property and the Sorgenfrey line were demonstrated in [7].

A neighborhood assignment for a space \( X \) is a function \( \varphi \) from \( X \) to the topology of \( X \) such that \( x \in \varphi(x) \) for all \( x \in X \). A space \( X \) is a D-space if, for each neighborhood assignment \( \varphi \) for the space \( X \), there exists a closed discrete subset \( F \) of \( X \) satisfying \( X = \bigcup \{ \varphi(x) \mid x \in F \} \).

A space \( X \) is a monotone D-space ([5]) if, for each neighborhood assignment \( \varphi \) for \( X \), we can pick a closed discrete subset \( F(\varphi) \) of \( X \) with \( X = \bigcup \{ \varphi(x) \mid x \in F(\varphi) \} \) such that if \( \psi \) is a neighborhood assignment for \( X \) and \( \varphi(x) \subseteq \psi(x) \) for all \( x \), then \( F(\psi) \subseteq F(\varphi) \). Monotone D-spaces are D-spaces, but the converse is not true (see [5]).

The Sorgenfrey line \( S \) (that is, the set of all real numbers topologized by letting all half-open intervals \( [a, b) \) be a base) is one of the most important elementary examples in general topology. In [7], it is shown that the Sorgenfrey line \( S \) is a D-space. However, the Sorgenfrey line \( S \) is not a monotone D-space [5].

The main result of this note is as follows:

(1) the minimal dense linearly ordered extension of the Sorgenfrey line is a D-space, but not monotonically D;

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(2) the minimal closed linearly ordered extension of Sorgenfrey line is not a monotone $D$-space;
(3) the monotone $D$-property is inversely preserved by finite-to-one closed mappings, but cannot be inversely preserved by perfect mappings.

Throughout the note, spaces are topological spaces and are Hausdorff. Mappings are continuous. We reserve the symbols $\mathbb{R}$ and $\mathbb{Z}$ for the sets of all real numbers and all integers, respectively. For a neighborhood assignment $\varphi$ for the space $X$ and $F \subset X$, we denote $\bigcup \{\varphi(x) \mid x \in F\}$ by $\varphi(F)$. Undefined terminology and symbols will be found in [2].

2. Main results

Let $\ell(S) = \mathbb{R} \times \{0, -1\}$ be with the linearly ordered topology generated by the lexicographical order $\preceq$ on $\ell(S)$.

Note that the Sorgenfrey line $S$ is homeomorphic to the dense subspace $\mathbb{R} \times \{0\}$ of the space $\ell(S)$. By [4, Theorem 2.1], the space $\ell(S)$ is the minimal dense linearly ordered extension of $S$.

**Theorem 1.** The minimal dense linearly ordered extension $\ell(S)$ of the Sorgenfrey line $S$ is a $D$-space.

**Proof.** Note that the subset $\mathbb{R} \times \{0\}$ of $\ell(S)$ with the restricted order $\preceq |_{\mathbb{R} \times \{0\}}$ is a linearly ordered set. By the linearly ordered topological space $\mathbb{R} \times \{0\}$ we mean the subset $\mathbb{R} \times \{0\}$ of $\ell(S)$ with the open interval topology generated by the linear order $\preceq |_{\mathbb{R} \times \{0\}}$. Obviously the linearly ordered topological space $\mathbb{R} \times \{0\}$ is homeomorphic to the real line $\mathbb{R}$ (the set $\mathbb{R}$ with the Euclidean topology).

Let $\varphi'$ be a neighborhood assignment for $\ell(S)$. We now define a neighborhood assignment $\varphi$ for the linearly ordered topological space $\mathbb{R} \times \{0\}$ as follows. For any $x \in \mathbb{R}$, take an $s_x \in \mathbb{R}$ such that $x < s_x$ and the open interval $((x, 0), (s_x, 0)) \subset \varphi'((x, 0))$. We can also take an $a_x \in \mathbb{R}$ such that $a_x < x$ and $[(a_x, 0), (x, 0)) \subset \varphi'((x, -1))$. Define $\varphi((x, 0)) = (a_x, s_x) \times \{0\}$.

Since metrizability implies the $D$-property, the real line $\mathbb{R}$ is a $D$-space. So for $\varphi$ there exists a closed discrete subset $F$ of the real line $\mathbb{R}$ such that $\varphi(F \times \{0\}) = \mathbb{R} \times \{0\}$.

Put $F' = F \times \{0, -1\}$. Then $F'$ is closed in $\ell(S)$.

In fact, for any $x' = \langle x, i \rangle \in \ell(S) \setminus F'$, since $x \not\in F$ and $F$ is closed in the real line $\mathbb{R}$ there exist real numbers $a_x$ and $s_x$ with $a_x < x < s_x$ such that $(a_x, s_x) \cap F = \emptyset$. Then the open neighborhood $I_{x'} = ((a_x, 0), (s_x, -1))$ of $x'$ satisfies $I_{x'} \cap F' = \emptyset$.

To show that $F'$ is discrete, let $x' = \langle x, i \rangle \in F'$. Then $x \in F$ and thus there exists an open interval $(c_x, d_x)$ containing $x$ such that $(c_x, d_x) \cap F = \{x\}$ since $F$ is discrete in the real line $\mathbb{R}$. If $i = 0$, put $U_{x'} = ((x, -1), (d_x, -1))$. If $i = -1$, put $U_{x'} = ((c_x, 0), (x, 0))$. Then the open neighborhood $U_{x'}$ of $x'$ satisfies $U_{x'} \cap F' = \{x'\}$.

Finally, we will show that $\{\varphi(x') \mid x' \in F'\}$ covers $\ell(S)$. For any $y' = \langle y, i \rangle \in \ell(S) \setminus F'$, since $\varphi(F \times \{0\}) = \mathbb{R} \times \{0\}$ there exists an $x \in F$ such that

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The following are true:
\[ (y, 0) \in \varphi((x, 0)). \]
Since \( y \neq x \), by the definition of \( \varphi \) we have the following: if \( x < y \), then \( y' \in \varphi'((x, 0)) \); if \( y < x \), then \( y' \in \varphi'((x, -1)) \). So \( \varphi'(F') = \ell(S) \) and thus \( \ell(S) \) is a \( D \)-space.

The minimal closed linearly ordered extension \( S^* \) of \( S \) is defined as follows. Put
\[ S^* = \mathbb{R} \times \{ k \in \mathbb{Z} | k \leq 0 \}. \]
Let the linear order \( \preceq \) be the lexicographic order on \( S^* \). Equip \( S^* \) with the linearly ordered topology generated by the order \( \preceq \) on \( S^* \) (that is, the topology on \( S^* \) is generated by \( \{(a, \rightarrow) | a \in S^*\} \cup \{(\leftarrow, a) | a \in S^*\} \) as a subbase), where \( (a, \rightarrow) = \{x \in S^* | a < x\} \) and \( (\leftarrow, a) = \{x \in S^* | x < a\} \).

The Sorgenfrey line \( S \) is homeomorphic to the closed subspace \( \mathbb{R} \times \{0\} \) of the linearly ordered topological space \( S^* \). The space \( S^* \) is called a closed linearly ordered extension of \( S \) (see [3]). By [6, Theorem 9], the space \( S^* \) is the minimal closed linearly ordered extension of \( S \).

**Theorem 2.** The following are true:
1. the space \( S^* \) is not a monotone \( D \)-space;
2. the space \( \ell(S) \) is not a monotone \( D \)-space.

**Proof.** (1) Assume that \( S^* \) is a monotone \( D \)-space. Since the monotone \( D \)-property is hereditary with respect to closed subspaces (see [5, Theorem 1.7]) and \( S \) is homeomorphic to the closed subspace \( \mathbb{R} \times \{0\} \) of \( S^* \), \( S \) is a monotone \( D \)-space. By [5, Theorem 2.4] \( S \) is not a monotone \( D \)-space, which is a contradiction.

(2) Assume that the space \( \ell(S) \) is a monotone \( D \)-space. Define a mapping \( f : \ell(S) \to \mathbb{R} \), where \( \mathbb{R} \) is the real line, as follows. For each \( (x, i) \in \ell(S), f(x') = x \).

Then \( f \) is continuous and closed surjective mapping. In fact, for an open interval \( (a, b) \) of the real line \( \mathbb{R} \), \( f^{-1}((a, b)) \) is obviously open in \( \ell(S) \), so \( f \) is continuous.

Let \( F' \) be a closed subset of \( \ell(S) \) and \( x \notin f(F') \). Then \( f^{-1}(x) = \{(x, 0), (x, -1)\} \) and \( f^{-1}(x) \cap F' = \emptyset \). Thus there exist open intervals \( U = ((a_x, 0), (x, 0)) \) and \( V = ((x, -1), (b_x, 0)) \) of \( \ell(S) \) with \( (x, -1) \in U \), \( U \cap F' = \emptyset \) and \( (x, 0) \in V \), \( V \cap F' = \emptyset \), where \( a_x, b_x \in \mathbb{R} \). Thus \( x \in (a_x, b_x) \) and \( (a_x, b_x) \cap f(F') = \emptyset \). Hence \( f(F') \) is closed.

Since the image of a monotone \( D \)-space under a continuous closed mapping is monotonically \( D \) ([5, Theorem 1.7]), the real line \( \mathbb{R} \) is a monotone \( D \)-space. Thus the closed subspace \( [0, 1] \) of \( \mathbb{R} \) is monotonically \( D \), which contradicts the fact that closed unit interval \( [0, 1] \) is not monotonically \( D \) (see [5, Theorem 2.3]).

It is shown that the closed image of a \( D \)-space is a \( D \)-space, and the perfect inverse image of a \( D \)-space is a \( D \)-space (see [1]), For the monotone \( D \)-property, although it is also preserved by closed mappings (see [5]), it cannot be inversely preserved by perfect mappings.

**Example 3.** There exists a perfect mapping \( f \) from \( X \) onto \( Y \) with \( Y \) a monotone \( D \)-space, but where \( X \) not a monotone \( D \)-space.
Let \( S_0 \) be a countable subspace of the Sorgenfrey line \( S \). Put \( X = S_0 \times [0, 1] \) and \( Y = S_0 \), where \([0, 1]\) is the usual unit closed interval. Define \( f : X \to Y \) such that, for each \( x = (s, t) \in X \), \( f(x) = s \). Clearly \( f \) is perfect. By \([5, \text{Theorem 2.4}]\), the countable subspace \( Y \) of the Sorgenfrey line \( S \) is a monotone \( D \)-space. Take an \( s \in S_0 \). Since the closed subspace \( \{s\} \times [0, 1] \) of \( X \) is homeomorphic to \([0, 1]\) and \([0, 1]\) is not a monotone \( D \)-space (see \([5, \text{Theorem 2.3}]\)), \( X \) is not monotonically \( D \).

Recall that a mapping \( f : X \to Y \) is called finite-to-one if, for each \( y \in Y \), \( f^{-1}(y) \) is finite.

**Theorem 4.** Let a closed mapping \( f : X \to Y \) be finite-to-one and surjective. If \( Y \) is a monotone \( D \)-space, then so is \( X \).

**Proof.** Let \( \varphi \) be a neighborhood assignment for \( X \). For each \( y \in Y \), put \( U_y = \bigcup \{ \varphi(x) \mid x \in f^{-1}(y) \} \) and \( \varphi'(y) = Y \setminus f(X \setminus U_y) \). Then \( \varphi' \) is a neighborhood assignment for \( Y \). Since \( Y \) is a monotone \( D \)-space, there exists a closed discrete subset \( D_{\varphi'} \) of \( Y \) such that \( Y = \bigcup \{ \varphi'(t) \mid t \in D_{\varphi'} \} \). Then \( D_{\varphi} = \bigcup \{ f^{-1}(t) \mid t \in D_{\varphi'} \} \) is a closed discrete subset of \( X \) and \( X = \bigcup \{ \varphi(x) \mid x \in D_{\varphi} \} \). Hence \( X \) is a monotone \( D \)-space.

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**References**


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