THE COMMUTATION RELATION $i[Y, Z] = 2Y$
AND THE ABSOLUTELY CONTINUOUS SPECTRUM OF $Y$

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Abstract

A relation between positive commutators and absolutely continuous spectrum is obtained. If $i[Y, Z] = 2Y$ holds on a core for $Z$ and if $Y$ is positive then we have a system of imprimitivity for the group $\mathbb{R}^+_*$ on $\mathbb{R}^+_*$, from which it follows that $Y$ has no singular continuous spectrum.

Assume that $Y$ and $Z$ are self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and that

$$i[Y, Z]f = 2Yf$$

(1)

for all $f$ belonging to a dense subset $D$ of $\mathcal{H}$. We obtain conditions under which the relation (1) implies that the singular continuous spectrum of $Y$ is empty.

The argument is simple. We first show that if $Y$ is positive and if

$$e^{-iZs}Ye^{iZs}u = e^{2s}Yu$$

(2)

for all $u \in D(Y)$ and all $s \in \mathbb{R}$, then the singular continuous spectrum of $Y$ is empty. We then obtain conditions on the subset $D$ that ensure that whenever (1) holds then (2) holds. We also obtain a converse to this, namely, if $Y$ is a positive self-adjoint operator with absolutely continuous spectrum on $[0, \infty)$ and uniform spectral multiplicity then there exists a self-adjoint operator $Z$ such that (1) holds.

**Theorem 1.** Let $Y$ be a positive self-adjoint operator and $U_s$ a unitary representation of the real line, such that for all $u \in D(Y)$ and all $s \in \mathbb{R}$

$$U_s^{-1}YU_su = e^{-2s}Yu,$$

then if the spectrum of $Y$ is continuous it is absolutely continuous.
PROOF. For any complex number \( \omega \),

\[
U_s^{-1}(Y - \omega I)U_s u = e^{-2s}(Y - e^{2s}\omega I)u
\]

for all \( u \in D(Y) \) and all real \( s \). Therefore, if the imaginary part of \( \omega \) is non-zero, \( (Y - \omega I) \) is invertible, and

\[
U_s^{-1}(Y - \omega I)^{-1}U_s = e^{2s}(Y - e^{2s}\omega I)^{-1}.
\]

This last equation holds as an operator identity in \( B(\mathcal{H}) \) for all \( s \in \mathbb{R} \).

Assume that the continuous spectrum of \( Y \) is non-empty and contains the interval \( \Delta \). The spectral projection \( E_\Delta(Y) \) is given by Stone’s formula

\[
E_\Delta(Y) = \lim_{\epsilon \to 0^+} (2\pi i)^{-1} \int_{\Delta} \left[ (Y - \omega I)^{-1} - (Y - e^{2\pi i \epsilon})^{-1} \right] d\mu
\]

where we have written \( \omega = \mu + i\epsilon \) and \( \omega = \mu - i\epsilon \).

Therefore

\[
U_s^{-1}E_\Delta(Y)U_s = \lim_{\epsilon \to 0^+} e^{2s}(2\pi i)^{-1} \int_{\Delta} \left[ (Y - e^{2\pi i \epsilon})^{-1} - (Y - e^{2\pi i \epsilon})^{-1} \right] d\mu
\]

\[
= \lim_{\epsilon_0 \to 0^+} (2\pi i)^{-1} \int_{\epsilon_0}^{2\pi} \left[ (Y - \eta - i\epsilon_0)^{-1} - (Y - \eta + i\epsilon_0)^{-1} \right] d\eta
\]

\[
= E_{e^{2\pi}Y}(Y)
\]

where we have put \( \eta + i\epsilon_0 = e^{2\pi} \).

Let \( \beta \) be any Borel subset in the continuous spectrum of \( Y \), then by the usual construction of Borel subsets from intervals we obtain

\[
U_s^{-1}E_\beta(Y)U_s = E_{e^{2\pi}Y}(Y).
\]

Let \( \mathbb{R}_+^* = (0, \infty) \) denote the multiplicative group of positive real numbers. We obtain a representation \( V_a \) of \( \mathbb{R}_+^* \) from the representation \( U_s \) of \( \mathbb{R} \) by putting \( a = e^{2s} \) for all \( s \in \mathbb{R} \), and observing that

\[
V_a = U_{\frac{1}{\ln a}} \text{ for all } a \in \mathbb{R}_+^*.
\]

By hypothesis \( Y \) is positive definite and so its spectrum is contained in \([0, \infty)\). By spectral multiplicity theory, the set of all spectral projections \( \{ E_\beta(Y); \beta \text{ a Borel subset of } [0, \infty) \} \) has a separating vector \( \Phi \). In fact, \( \Phi \) is a cyclic vector for the commutant of this family of projections.

The measure \( \nu(\Delta) = \langle \Phi, E_\Delta(Y)\Phi \rangle \), defined on the Borel subsets of \( \mathbb{R}_+^* \), is equivalent to the Haar measure of \( \mathbb{R}_+^* \). To see this, first observe that because \( \Phi \) is separating if \( \Delta_0 \) is a Borel subset of \( \mathbb{R}_+^* \) such that \( \nu(\Delta_0) = 0 \) then \( E_{\Delta_0}(Y) = 0 \), and therefore

\[
\langle \Phi, V_a^{-1}E_{\Delta_0}(Y)V_a\Phi \rangle = 0
\]

for all \( a \in \mathbb{R}_+^* \).
for all \( a \in \mathbb{R}^*_+ \). On the other hand when equation (4) is written in terms of the representation \( V_a \) of the multiplicative group \( \mathbb{R}^*_+ \) we obtain \( V_a^{-1}E_{a\Delta_0}(Y)V_a = E_{a\Delta_0}(Y) \). Therefore

\[
\nu(a\Delta_0) = \langle \Phi, E_{a\Delta_0}(Y)\Phi \rangle = 0
\]

for all \( a \in \mathbb{R}^*_+ \). This means that \( \nu \) is a Borel measure on \( \mathbb{R}^*_+ \) that is quasi-invariant with respect to the action of \( \mathbb{R}^*_+ \) on itself, and therefore \( \nu \) is equivalent to Haar measure of \( \mathbb{R}^*_+ \) on \( \mathbb{R}^*_+ \).

The absolute continuity of the spectrum of \( Y \) follows because the Haar measure of \( \mathbb{R}^*_+ \) on \( \mathbb{R}^*_+ \) is absolutely continuous with respect to Lebesgue measure. Let the Borel subset \( S \) of \( \mathbb{R} \) have Lebesgue measure zero, that is, \( |S| = 0 \). If \( S \) is a subset of \( \mathbb{R}^*_+ \), \( \nu(S) = 0 \) and therefore \( E_S(Y)\phi = 0 \) and \( E_S(Y) = 0 \) because \( \Phi \) is separating. If \( S \) is not a subset of \( \mathbb{R}^*_+ \) then \( S = S_1 \cup S_2 \) where \( S_2 \) is a subset of \( \mathbb{R}^*_+ \) and \( S_1 \) lies in the complement of \( \mathbb{R}^*_+ \). Now \( E_S(Y) = E_{S_1}(Y) + E_{S_2}(Y) \) where \( E_{S_2}(Y) = 0 \) by the argument given above and \( E_S(Y) = 0 \) by the positivity of \( Y \) and the continuity of spectrum of \( Y \).

This theorem shows that the spectral measure class of the positive operator \( Y \) is equivalent to the Haar measure of the multiplicative group of the positive reals, \( \mathbb{R}^*_+ \), on itself. The equation (1) defines a system of imprimitivity of the group \( \mathbb{R}^*_+ \). The proof is modelled on Mackey’s approach to the representations of the canonical commutation relations [4].

**Definition.** Let \( Y \) be a positive self-adjoint operator in a Hilbert space \( \mathcal{H} \). A subset \( D \) of \( \mathcal{H} \) is said to be a domain of integration for the self-adjoint operator \( Z \) with respect to the relation

\[
i[Y, Z] = 2Y
\]

if

\[
(YZ - ZY)f = -2iYf
\]

for all \( f \in D \) implies that

\[
e^{iZs}Ye^{-iZs}u = e^{-2s}Yu
\]

for all \( u \in D(Y) \) and all \( s \in \mathbb{R} \).

The terminology reflects the fact that equation (8) can be obtained from equation (9) by differentiating with respect to \( s \) at \( s = 0 \). An immediate consequence of this definition and Theorem 1 is the following result:

**Theorem 2.** Let \( D \) be a domain of integration for \( Z \) and the relation (7) and suppose that \( Y \) is positive definite, then whenever \( i[Y, Z]f = 2Yf \) for all \( f \in D \) the singular continuous spectrum of \( Y \) is empty.
The problem of finding a domain of integration for the operator \( Z \) and relation (7) is related to the problem of lifting a representation of a Lie algebra as skew-adjoint operators on a Hilbert space to a unitary representation of the corresponding Lie group. Nelson's theorem [5] gives necessary and sufficient conditions for the solution of the general problem, and can be used for our problem. Nevertheless, we present a criterion for \( D \) modelled on a result of Kato [2] for the problem of obtaining the Weyl commutation relations from those of Heisenberg (see also Cartier [1]).

**Theorem 3.** Let \( D \) be a subset of \( D(YZ) \cap D(ZY) \) on which equation (8) holds with \( Y \) positive. \( D \) is a domain of integration for \( Z \) and relation (7) if \( D \) is a core for \( Z \).

**Proof.** Since \( D \) is a core for \( Z \) there is an \( \alpha \neq 0 \) such that \( (Z - i\alpha)D \) is dense in \( \mathcal{H} \). If \( \epsilon > 0 \), \( (Y + \epsilon I) \) is strictly positive and symmetric and hence \((Y + \epsilon)^{-1}(Z - (\alpha + 2))^{-1}u = \epsilon(Y + \epsilon)^{-1}(Z - i(\alpha + 2))^{-1}(u - 2i\epsilon f) = (Y + \epsilon)^{-1}(Z - (\alpha + 2))^{-1}u + \epsilon(Y + \epsilon)^{-1}[(Z - i\alpha)^{-1} - (Z - i(\alpha + 2))^{-1}](Y + \epsilon)^{-1}u \). But \( u \in (Y + \epsilon)(Z - i\alpha)D \) and thus we have the operator equation

\[
(Z - i\alpha)^{-1}(Y + \epsilon)^{-1} - (Y + \epsilon)^{-1}(Z - i(\alpha + 2))^{-1} = \epsilon(Y + \epsilon)^{-1}((Z - i\alpha)^{-1} - (Z - i(\alpha + 2))^{-1})(Y + \epsilon)^{-1}. \tag{10}
\]

We now prove by induction that

\[
(Z - i\alpha)^{-n}(Y + \epsilon)^{-1} - (Y + \epsilon)^{-1}(Z - i(\alpha + 2))^{-n} = \epsilon(Y + \epsilon)^{-1}(Z - i\alpha)^{-n} - (Z - i(\alpha + 2))^{-n}(Y + \epsilon)^{-1}. \tag{11}
\]

for all positive integers \( n \). It is true for \( n = 1 \); assume it is true for \( n \) and write \( P_0 = (Z - i\alpha)^{-1}, P_2 = (Z - i(\alpha + 2))^{-1}, \) and \( Q = (Y + \epsilon)^{-1} \). Then

\[
P_0^{n+1}Q - QP_2^{n+1} = P_0^n(P_0Q - QP_2) + (P_0^nQ - QP_2^n)P_2
= \epsilon\{P_0^nQ(P_0Q - QP_2) + (QP_0^n - QP_2^n)QP_2\}
= \epsilon\{QP_0^{n+1}Q - QP_2^{n+1}Q\},
\]
on substituting for \( P_0^nQ \) and \( QP_2 \) in the penultimate line. The argument now goes exactly as in [2]. Use the Neumann series for \((Z - i\beta)^{-1}\) and the fact that \((Z - \omega)^{-1}\) is analytic for \( \text{Im} \omega \neq 0 \) to extend the validity of (11) from \( \omega = i\alpha \) to \( \omega = i\beta \) for all real \( \beta, \beta \neq 0, \beta \neq -2 \).
Multiply equation (11) by \((-i\alpha)^n\) and set \(\alpha = n/s\) with \(s \neq 0\). \((Z - i\alpha)^n\) becomes \((1 + in^{-1}sZ)^{-n}\) and \((Z - i(\alpha + 2))^{-n}\) becomes \((1 + n^{-1}s(2 + iZ))^{-n}\).

Both these expressions have strong limits as \(n\) tends to infinity:

\[
(1 + in^{-1}sZ)^{-n} \to e^{isZ} \quad \text{and} \quad (1 + n^{-1}s(2 + iZ))^{-n} \to e^{-2s}e^{-iZs}.
\]

Therefore

\[
e^{-iZs}(Y + \varepsilon)^{-1} - (Y + \varepsilon)^{-1}e^{iZs}e^{-2s} = \varepsilon(Y + \varepsilon)^{-1}(e^{-iZs} - e^{iZs}e^{-2s})(Y + \varepsilon)^{-1},
\]

and, for all \(g \in D(Y)\),

\[
(Y + \varepsilon)e^{-iZs}g - e^{-iZs}e^{-2s}(Y + \varepsilon)g = \varepsilon(e^{-iZs} - e^{-iZs}e^{-2s})g,
\]

or

\[
e^{iZs}Ye^{-iZs}g = e^{-2s}Yg.
\]

Putting these results together we have the useful corollary of Theorem 3.

**Corollary.** Let \(Y\) and \(Z\) be self-adjoint operators on a separable Hilbert space \(\mathcal{H}\) and suppose that \(Y\) is positive. Let \(D\) be a subset of \(D(YZ) \cap D(ZY)\) such that for all \(f \in D\)

\[i[Y, Z]f = 2Yf\]

and suppose that \(D\) is a core for \(Z\). Then the singular continuous spectrum of \(Y\) is empty.

We will now use this corollary in a number of examples.

**Examples.** 1.

\[\mathcal{H} = L^2([a, b]), \quad 0 < a < b < \infty,\]

\[Y = -\frac{d^2}{dx^2} \quad \text{on} \quad D(Y), \quad Z = \frac{1}{2i}\left(x\frac{d}{dx} + \frac{d}{dx}x\right) \quad \text{on} \quad D(Z),\]

where

\[D(Y) = \{f \in \mathcal{H} | f \in AC^2[a, b], f(a) = 0 = f(b)\},\]

\[D(Z) = \{f \in \mathcal{H} | f \in AC[a, b], xf \in AC[a, b] \text{ and } af(a) = \sqrt{b}f(b)\},\]

\[AC[a, b] = \{f \in \mathcal{H} | f(x) \text{ is absolutely continuous on } [a, b] \text{ and } f'(x) \in \mathcal{H}\},\]

\[AC^2[a, b] = \{f \in \mathcal{H} | f \text{ is differentiable}, \quad f' \text{ is absolutely continuous and } f'' \in \mathcal{H}\}.\]
With these domains, $Y$ and $Z$ are self-adjoint and $Y$ is positive. We take $D \subset D(YZ) \cap D(XY)$ to be $C^\infty_0[a, b]$, the set of $C^\infty$ functions with compact support in $[a, b]$ whose support stays away from the end points. Then for all $f \in D$, 

$$i[Y, Z]f = 2Yf.$$ 

We know that the spectrum of $Y$ is not absolutely continuous, but this does not contradict Theorem 3 as $D$ is not a core for $Z$. For any real number $\alpha \neq 0$, $(Z - i\alpha)D$ is not dense in $L^2[a, b]$, because the function $u(x) = Ax^{\alpha - 1/2}$ is orthogonal to $(Z - i\alpha)D$. In fact this function is orthogonal to $(Z - i\alpha)(D(YZ) \cap D(ZY))$.

2.

$$\mathcal{H} = L^2([a, b]), \quad 0 < a < b < \infty,$$

$Y$ is the multiplicative operator, $(Yf)(x) = x^2f(x)$, with $D(Y) = \mathcal{H}$. $Z = -(1/2i)(xd/dx + (d/dx)x)$ on $D(Z)$ as in example (1).

Both $Y$ and $Z$ are self-adjoint, $Y$ is positive, and if we take $D \subset D(YX) \cap D(ZY)$ to be $C^\infty_0[a, b]$ as in example (1), then for all $f \in D$, 

$$i[Y, Z]f = 2Yf.$$ 

The argument of example (1) yields the result that $D$ is not a core for $Z$, even though we know that the spectrum of $Y$ is absolutely continuous. This shows that the conditions of Theorem 4 are not necessary. What goes wrong in this example is that it is not true that $e^{-izs}Ye^{izs}f = e^{-2zs}Yf$ for all $f \in D(Y)$. This example should be compared with the usual particle in a box counterexample to the uniqueness of the representation for the Heisenberg commutation relations.

3.

$$\mathcal{H} = L^2(0, \infty),$$

$Y$ is the operator of multiplication, $(Yf)(\lambda) = \lambda f(\lambda)$ and

$$D(Y) = \left\{ f \in \mathcal{H} | \int_0^\infty \lambda^2 |f(\lambda)|^2 \, d\lambda < \infty \right\},$$

$$Z = -\frac{1}{i} \left( \lambda \frac{d}{d\lambda} + \frac{d}{d\lambda} \lambda \right) \quad \text{with domain}$$

$$D(Z) = \left\{ f \in L^2(0, \infty) \mid f \in AC[0, \infty), \lambda f \in AC[0, \infty) \right\},$$

and

$$\lim_{a \to 0^+} \sqrt{a} f(a) = \lim_{b \to -\infty} \sqrt{b} f(b).$$

The last condition in the description of the domain of $Z$ should be taken to mean that both limits exist and are equal.
With these domains, $Y$ and $Z$ are self-adjoint and $Y$ is positive. Furthermore we know that the spectrum of $\lambda$ is absolutely continuous. This does follow from Theorem 4 because if $D$ is taken to be $C_0^\infty[0, \infty]$ with the support of the functions staying away from zero and infinity, then $D$ is a core for $Z$; in fact $(Z - i\alpha)D$ is dense in $L^2([0, \infty))$ for any real $\alpha \neq 0$. This is so because if $(Z - i\alpha)D$ were not dense there must be an element $\omega \neq 0$ that is perpendicular to $(Z - i\alpha)D$, but the only possible $\omega$ are of the form $Ax^{\alpha - 1/2}$ which are not in $L^2([0, \infty))$.

4. In non-relativistic quantum theory, the commutation relation (7) arises with $Y = H_0$, the kinetic energy or free Hamiltonian operator, and $Z = A$, the generator of the one parameter group of dilations. In the usual Schrödinger representation for a single particle, $H_0 = p^2$, $A = \frac{1}{i}(x \cdot p + p \cdot x)$ with $p$ representing the canonical momentum operator and $x$ the canonical position operator. Further, $H_0$ and $A$ are self-adjoint operators on their natural domains. It is well known that the spectrum of $H_0$ is $[0, \infty)$ and is purely absolutely continuous. The connection with this paper can be made directly but it is more interesting to notice that in the usual spectral representation of $H_0$, [3], we have a unitary map $U$ from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^+, d\lambda; \mathcal{H}'')$, where $\mathcal{H}'' = L^2(S^2, d\Omega)$, and $S^2$ is the unit sphere in $\mathbb{R}^3$, and $d\Omega$ its usual surface measure, that sends $H_0$ to multiplication by $\lambda$ and $A$ to the operator $Z = -(1/i)(\lambda d/d\lambda + (d/d\lambda)\lambda)$ that is discussed in example (3). Explicitly if $\hat{f}$ denotes the Fourier transform of an element of $f$ of $L^2(\mathbb{R}^3)$ then $(Uf)(\lambda; \omega) = (\sqrt{2})^{-1}\lambda^{1/4}\hat{f}(\lambda^{1/2}\omega)$.

As a result of these last two examples we are led to the following proposition.

**Proposition.** Let $\mathcal{H}$ be a separable Hilbert space. If $Y$ is a positive self-adjoint unbounded operator with absolutely continuous spectrum on $[0, \infty)$ and uniform spectral multiplicity then there exists a self-adjoint operator $Z$ such that

$$i[Y, Z]f = 2Yf$$

for all $f$ belonging to a domain of integration $Z$.

**Proof.** By hypothesis, $Y$ has a spectral representation as multiplication by $\lambda$ a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+, d\lambda; \mathcal{H}'')$ for some constant fibre $\mathcal{H}'$. But by Example 3 the operator $Z_0 = -(1/i)(\lambda d/d\lambda + (d/d\lambda)\lambda)$, with domain $D(Z_0)$ given in that example, is self-adjoint and for all $f \in C_0^\infty(\mathbb{R}^+; \mathcal{H}'')$

$$i[\lambda, Z_0]f = 2\lambda f.$$

Now the pre-image of $Z_0$ under the unitary map $U$ of Example 4 gives a self-adjoint operator $Z$ on $D(Z) \subset \mathcal{H}$ such that $i[Y, Z]f = 2Yf$ on a domain of integration for $Z$. 

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This proposition gives a partial converse to Theorem 2 and appears to be useful in non-relativistic scattering theory. We hope to discuss this connection in a subsequent paper.

References


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