JOINT SPECTRA OF COMMUTING NORMAL OPERATORS ON BANACH SPACES

by MUNEO CHŌ

(Received 2 June, 1987; revised 26 October, 1987)

Dedicated to Professor Kisuke Tsuchida on his retirement on March 31 1989.

1. Introduction. The joint spectrum for a commuting n-tuple in functional analysis has its origin in functional calculus which appeared in J. L. Taylor's epoch-making paper [**19**] in 1970. Since then, many papers have been published on commuting n-tuples of operators on Hilbert spaces (for example, [**3**], [**4**], [**5**], [**8**], [**9**], [**10**], [**21**], [**22**]).

For those on Banach spaces, however, only a few results have come out. Recently, A. McIntosh, A. Pryde and W. Ricker in [**16**] characterized the joint spectrum for a strongly commuting n-tuple of operators on a Banach space. In this paper, we shall show among others that the joint spectrum for a strongly commuting n-tuple of operators on a Banach space is the joint approximate point spectrum for it.

Let X be a complex Banach space. We denote by X^* the dual space of X and by $B(X)$ the space of all bounded linear operators on X. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on X. And let $\sigma(\mathbf{T})$ be the Taylor joint spectrum of $\mathbf{T}$. We refer the reader to Taylor [**19**] for the definition of $\sigma(\mathbf{T})$.

A point $z = (z_1, \ldots, z_n)$ of C^n is in the approximate point spectrum $\sigma_\pi(\mathbf{T})$ of $\mathbf{T}$ if there exists a sequence $\{x_k\}$ of unit vectors in X such that

$$\|(T_i - z_i)x_k\| \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad i = 1, 2, \ldots, n.$$

A point $z = (z_1, \ldots, z_n)$ of C^n is said to be a joint eigenvalue of $\mathbf{T}$ if there exists a non-zero vector x such that

$$T_i x = z_i x \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Let us set

$$\pi = \{(x, f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1\}.$$

The spatial joint numerical range $V(\mathbf{T})$ and joint numerical range $V(B(X), \mathbf{T})$ of $\mathbf{T}$ are defined by

$$V(\mathbf{T}) = \{(f(T_1x), \ldots, f(T_nx)) : (x, f) \in \pi\}$$

and

$$V(B(X), \mathbf{T}) = \{(F(T_1), \ldots, F(T_n)) : F \text{ is a state on } B(X)\}.$$

respectively. The joint spectral radius and joint numerical radius of $\mathbf{T} = (T_1, \ldots, T_n)$ are defined by

$$r(\mathbf{T}) = \sup\{|z| : z \in \sigma(\mathbf{T})\}$$

and
$$\nu(\mathbf{T}) = \sup\{|z| : z \in V(\mathbf{T})\},$$
respectively.

For an operator $S \in B(X)$, the usual spectrum, approximate point spectrum, spatial numerical range and numerical range of S are denoted by $\sigma(S)$, $\sigma_\pi(S)$, $V(S)$ and $V(B(X), S)$, respectively.

If $V(S) \subset R$, then S is called hermitian. An operator $S \in B(X)$ is called normal if there are hermitian operators H and K such that $S = H + iK$ and $HK = KH$. We denote then the operator $H - iK$ by $\bar{S}$. Then the following are well-known:

(1) $\overline{\text{co}}\, V(S) = V(B(X), S)$, where $\overline{\text{co}}\, E$ is the closed convex hull of E.

(2) $\text{co}\, \sigma(S) \subset \overline{V(S)}$, where $\text{co}\, E$ and $\bar{E}$ are the convex hull and closure of E, respectively.

(3) If S is normal, then $\text{co}\, \sigma(S) = \overline{V(S)} = V(B(X), S)$. We refer the reader to Bonsall and Duncan [**1**] and [**2**]. We denote the boundary of E by δE.

An n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators is called strongly commuting if, for each $1 \le j \le n$, there exist operators U_j and V_j with real spectra, such that $T_j = U_j + iV_j$ and $(U_1, \ldots, U_n, V_1, \ldots, V_n)$ is a commuting $2n$-tuple.

Remark Since the Fuglde theorem holds for Banach space operators, $\mathbf{T} = (T_1, \ldots, T_n)$ is strongly commuting if $\mathbf{T}$ is a commuting n-tuple of normal operators.

2. Joint spectra of strongly commuting n-tuples. McIntosh, Pryde and Ricker in [**16**] showed that following theorem:

Theorem. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a strongly commuting n-tuple of operators. Then $z = (z_1, \ldots, z_n)$ is in $\sigma(\mathbf{T})$ if and only if*
$$\sum_{j=1}^{n} (U_j - a_j)^2 + \sum_{j=1}^{n} (V_j - b_j)^2 \textit{ is not invertible}$$
where $T_j = U_j + iV_j$ and $z_j = a_j + ib_j$ $(j = 1, 2, \ldots, n)$.

We shall prove the following theorem.

Theorem 2.1. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a strongly commuting n-tuple of operators. Then $\sigma(\mathbf{T}) = \sigma_\pi(\mathbf{T})$.*

We shall use the following two theorems.

Theorem A (Choi and Davis [**7**], Slodkowski and Zelazko [**18**]. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators and f an m-tuple of polynomials in n-variables. Then*
$$\sigma_\pi(f(\mathbf{T})) = f(\sigma_\pi(\mathbf{T})).$$

Theorem B (Taylor [**20**]). *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators and f an m-tuple of polynomials in n-variables. Then*
$$\sigma(f(\mathbf{T})) = f(\sigma(\mathbf{T})).$$

Proof of Theorem 2.1. Let $T_j = U_j + iV_j$ and let $\mathbf{S} = (U_1, \ldots, U_n, V_1, \ldots, V_n)$. Then,

$$\sigma(\mathbf{T}) = g(\sigma(\mathbf{S}))$$

and

$$\sigma_\pi(\mathbf{T}) = g(\sigma_\pi(\mathbf{S}))$$

by the Spectral mapping theorem, where $g(z_1, z_2, \ldots, z_{2n}) = (z_1 + iz_{n+1}, \ldots, z_n + iz_{2n})$.

It suffices to check, therefore, that the result is true for a commuting n-tuple of operators with real spectra.

Let $\mathbf{H} = (H_1, \ldots, H_n)$ be a commuting n-tuple of operators with real spectra.

Let $a = (a_1, \ldots, a_n) \in \sigma(\mathbf{H})$ and let $f(z) := \sum_{i=1}^{n} (z_1 - a_1)^2$. Then $\sigma(f(\mathbf{H})) = f(\sigma(\mathbf{H}))$ so $0 \in \delta\sigma(f(\mathbf{H})) \subset \sigma_\pi(f(\mathbf{H})) = f(\sigma_\pi(\mathbf{H}))$. It follows that there exists $\tilde{a} \in \sigma_\pi(\mathbf{H}) \subset R^n$ such that $f(\tilde{a}) = 0$. Clearly $\tilde{a}$ must be equal to a, and so $a \in \sigma_\pi(\mathbf{H})$, concluding the proof.

If $T_1, T_2, \ldots, T_n$ are commuting operators, we denote by $A(T_1, \ldots, T_n)$ the least closed subalgebra of $B(X)$ generated by $I, T_1, \ldots, T_n$. And we denote by $\Phi_{A(T_1,\ldots,T_n)}$ the set of all non-zero multiplicative linear functionals on $A(T_1, \ldots, T_n)$.

THEOREM 2.2. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a strongly commuting n-tuple of operators such that* $T_j = U_j + iV_j$ $(j = 1, 2, \ldots, n)$. *Let A be* $A(U_1, \ldots, U_n, V_1, \ldots, V_n)$. *Then*

$$\sigma(\mathbf{T}) = \{(\varphi(H_1) + i\varphi(K_1), \ldots, \varphi(H_n) + i\varphi(K_n)) : \varphi \in \Phi_A\}.$$

Proof. Let $\mathbf{S} = (U_1, \ldots, U_n, V_1, \ldots, V_n)$. Let $a = (a_1, \ldots, a_{2n})$ be in $\sigma_A(\mathbf{S})$ if the equation

$$\sum_{i=1}^{n} (U_i - a_i)D_i + \sum_{i=n+1}^{2n} (V_{i-n} - a_i)D_i = I$$

fails to have a solution for $D_1, \ldots, D_{2n} \in A$. Since

$$\sigma(\mathbf{S}) \subset \sigma_A(\mathbf{S}),$$

an application of the Spectral Mapping Theorem gives

$$\sigma(\mathbf{T}) \subset \sigma_A(\mathbf{T}).$$

Moreover, McIntosh, Pryde and Ricker's result says that

$$\sigma_A(\mathbf{T}) \subset \sigma(\mathbf{T}).$$

Therefore, $\sigma(\mathbf{T}) = \sigma_A(\mathbf{T})$, as desired.

THEOREM 2.3. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a strongly commuting n-tuple of operators. If* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *is in* $\sigma(\bar{\mathbf{T}}\mathbf{T})$, *then there exists* $z = (z_1, \ldots, z_n)$ *in* $\sigma(\mathbf{T})$ *such that* $|z_i|^2 = \alpha_i$ $(i = 1, 2, \ldots, n)$ *where* $\bar{\mathbf{T}}\mathbf{T} = (\bar{T}_1 T_1, \ldots, \bar{T}_n T_n)$.

Proof. Let $f : R^{2n} \to R^n$ be the polynomial given by $f(x) = f(x_1, \ldots, x_{2n}) = (x_1^2 + x_{n+1}^2, \ldots, x_n^2 + x_{n+2n}^2)$, $x \in R^{2n}$. Then the the Spectral Mapping Theorem for the

Taylor joint spectrum, it follows that

$$f(\sigma(\mathbf{S})) = \sigma(\bar{\mathbf{T}}\mathbf{T}),$$

where $T_j = U_j + iV_j$ $(j = 1, \ldots, n)$ and $\mathbf{S} = (U_1, \ldots, V_1, \ldots, V_n)$. Hence there exists $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ in $\sigma_\pi(\mathbf{S})$ such that $a_j^2 + b_j^2 = \alpha_j$ $(j = 1, 2, \ldots, n)$.

Let $z_j = a_j + ib_j$ $(j = 1, 2, \ldots, n)$. Then the scalar $z = (z_1, \ldots, z_n)$ is a desired element.

So the proof is complete.

3. Joint numerical ranges of commuting normal operators. V. Wrobel proved the following theorem.

THEOREM 3.1 (Wrobel [**23**]). *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a commuting n-tuple of operators. Then*

$$\text{co}\,\sigma(\mathbf{T}) \subset \overline{V(\mathbf{T})}.$$

THEOREM 3.2. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of normal operators. Then

$$\text{co}\,\sigma(\mathbf{T}) = \overline{V(\mathbf{T})} = V(B(X), \mathbf{T}).$$

Proof. We assume that $\alpha = (\alpha_1, \ldots, \alpha_n)$ is in $V(B(X), \mathbf{T}) - \text{co}\,\sigma(\mathbf{T})$. By the separation theorem for a convex set, there exists a linear functional ϕ on C^n and $r \in R$ such that $\text{Re}\,\phi(\lambda) < r < \text{Re}\,\phi(\alpha)$ $(\lambda \in \text{co}\,\sigma(\mathbf{T}))$. We let $\phi(z) = \sum_{j=1}^{n} a_{1j} z_j$ $(z = (z_1, \ldots, z_n) \in C^n)$, and choose a non-singular $n \times n$ matrix M with $(a_{11}, \ldots, a_{1n})$ in its first row. Then

$$\text{Re}\, z_1 < r < \text{Re}\,\beta_1 \qquad (z = (z_1, \ldots, z_n) \in \sigma(M\mathbf{T})),$$

where $(\beta_1, \ldots, \beta_n) = M\alpha$. It follows that

$$\text{co}\,\sigma\Big(\sum_{j=1}^{n} a_{1j} T_j\Big) \subsetneqq V\Big(B(X), \sum_{j=1}^{n} a_{1j} T_j\Big).$$

Since $\sum_{j=1}^{n} a_{1j} T_j$ is a normal operator, this is a contradiction. So the proof is complete.

COROLLARY 3.3. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a commuting n-tuple of normal operators. Then* $r(\mathbf{T}) = v(\mathbf{T})$.

REMARK There is a normal operator N such that $r(N) = 1$ and $\|N\| = 2$ (see Theorem 25.6 in [**2**]).

PROBLEM. The joint operator norm $\|\mathbf{T}\|$ of $\mathbf{T} = (T_1, \ldots, T_n)$ is defined by

$$\|\mathbf{T}\| = \sup\left\{\Big(\sum_{i=1}^{n} \|T_i x\|^2\Big)^{1/2} : x \in X \text{ and } \|x\| = 1\right\}.$$

If $\mathbf{H}=(H_1, \ldots, H_n)$ is a commuting n-tuple of hermitian operators, is it then true that $r(\mathbf{H})=\|\mathbf{H}\|$?

Definition 1. Banach space X is called *smooth* if the set $\{f : \|f\| = f(x) = 1\}$ is a singleton for each $x \in X$ with $\|x\| = 1$.

Definition 2. The Banach space X is called *strictly c-convex* if $y = 0$ whenever $\|x\| = 1$ and $\|x + \lambda y\| \leq 1$ for all complex numbers λ with $\|\lambda\| \leq 1$.

Let X be either smooth and reflexive, or strictly c-convex. Then K. Mattila proved the following:

Theorem 3.4 (Mattila [**14**] and [**15**]). *Let N be a normal operator. If λ is an extreme point of $\overline{V(N)}$ such that $\lambda \in V(N)$, then λ is a eigen value of N.*

We extend this theorem to a commuting n-tuple of operators. In the following, we assume that the space X is either smooth and reflexive, or strictly c-convex.

Lemma 3.5. *Let $\mathbf{H}=(H_1, \ldots, H_n)$ be a commuting n-tuple of hermitian operators. If $a=(a_1, \ldots, a_n)$ is an extreme point of $\overline{V(\mathbf{H})}$ and $a \in V(\mathbf{H})$, then a is a joint eigenvalue of* $\mathbf{H}$.

Proof. The proof is by induction on n. By the theorem above, the statement is true for $n=1$.

Given a positive integer n, suppose that the statement is true for $n-1$. We may assume that $a=(0, \ldots, 0)$. Since 0 is an extreme point of $\overline{V(\mathbf{H})}$, we can choose a linear map F on R^n, which has an orthogonal matrix, such that

$$F(\overline{V(\mathbf{H})}) \subset \{z=(z_1, \ldots, z_n) \in R^n : z_n \geq 0\}.$$

Let $\mathbf{K}=(K_1, \ldots, K_n)=F(H_1, \ldots, H_n)$. Then $\mathbf{K}$ is a commuting n-tuple of hermitian operators which has the property $V(K_n) \subset R^+$. Since 0 is an extreme point of $\overline{V(K_n)}$, it follows that 0 is the eigenvalue of K_m. Let Y be the kernel space of K_n. Then Y is a closed subspace of X and has the same property as X. Since all K_i commute with K_n, the restrictions K_i' of K_i to Y are commuting hermitian operators on Y $(i=1, 2, \ldots, n)$. Thus

$$(0, \ldots, 0) \in V(K_1', \ldots, K_n').$$

Clearly, $\overline{V(K_1', \ldots, K_{n-1}')} \times \{0\} \subset \overline{V(\mathbf{K})}$.

Therefore $(0, \ldots, 0)$ is an extreme point of $\overline{V(K_1', \ldots, K_{n-1}')}$. Hence, by the hypothesis, there exists a non-zero vector x in Y such that

$$K_i'x = 0 \qquad (i=1, 2, \ldots, n),$$

that is, 0 is a joint eigenvalue of $\mathbf{K}$. So 0 is a joint eigenvalue of $\mathbf{H}$. Thus the proof is complete.

Theorem 3.6. *Let $\mathbf{T}=(T_1, \ldots, T_n)$ be a commuting n-tuple of normal operators. If $z=(z_1, \ldots, z_n)$ is an extreme point of $\overline{V(\mathbf{T})}$ and $z \in V(\mathbf{T})$, then z is a joint eigenvalue of* $\mathbf{T}$.

Proof. Let $T_j = H_j + ik_j$ and $z_j = a_j + ib_j$ $(j = 1, 2, \ldots, n)$. Then $(H_1, \ldots, H_n, K_1, \ldots, K_n)$ is a commuting $2n$-tuple of hermitian operators. $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is an extreme point of $\overline{V(H_1, \ldots, H_n, K_1, \ldots, K_n)}$ and an element of $V(H_1, \ldots, H_n, K_1, \ldots, K_n)$. So, by the lemma above, this point is a joint eigenvalue of $H_1, \ldots, H_n, K_1, \ldots, K_n$.

So the proof is complete.

REMARK From the proof of Theorem 3.2, the following holds:

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of operators (not necessarily commuting). Then $\overline{\text{co}}\, V(\mathbf{T}) = V(B(X), \mathbf{T})$.

Acknowledgement. I would like to express my thanks to Prof. K. Mattila for her useful suggestion given when we met at Univ. College Cork on May 20, 1987. I also express my thanks to the referee.

REFERENCES

1. Bonsall, F. F. and Duncan, J., *Numerical ranges of operators on normed spaces and elements of normed algebras,* (Cambridge Univ. Press, 1971).

2. Bonsall, F. F. and Duncan, J., *Numerical ranges II,* (Cambridge Univ. Press, 1973).

3. Ceausescu, Z. and Vasilescu, F.-H., Tensor product and the joint spectrum in Hilbert spaces, *Proc. A.M.S.* **72** (1978), 505–508.

4. Ceausescu, Z. and Vasilescu, F.-H., Tensor product and Taylor's joint spectrum, *Studia Math.* **62** (1978), 305–311.

5. Chō, M. and Takaguchi, M., Some classes of commuting n-tuple of operators, *Studia Math.* **80** (1984), 245–259.

6. Chō, M., Joint spectra of operators on Banach spaces, *Glasgow Math. J.* **28** (1986), 69–72.

7. Choi, M.-D. and Davis, C., The spectral mapping theorem for joint approximate point spectrum, *Bull. A.M.S.* **80** (1974), 317–321.

8. Curto, R., On the connectedness of invertible n-tuples, *Indiana Univ. Math. J.* **29** (1980), 393–406.

9. Curto, R., Fredholm and invertible n-tuples of operators, The deformation problem, *Trans. A.M.S.* **266** (1981), 129–159.

10. Curto, R., Spectral inclusion for doubly commuting subnormal n-tuples, *Proc. A.M.S.* **83** (1981), 730–734.

11. Dekker, N., *Joint numerical ranges and joint spectrum of Hilbert space operators,* (Ph.D. thesis, Amsterdam, 1969).

12. Harte, R., The spectral mapping theorem in several variables, *Bull. A.M.S.* **78** (1972), 871–875.

13. Mattila, K., On proper boundary points of the spectrum and complemented eigenspaces, *Math. Scand.* **43** (1978), 363–368.

14. Mattila, K., Normal operators and proper boundary points of the spectra of operators on Banach space, *Ann. Acad. Sci. Fnn. A I, Math. Dissertations* **19** (1978).

15. Mattila, K., Complex strict and uniform convexity and hyponormal operators, *Math. Proc. Camb. Phil. Soc.* **96** (1984), 483–493.

16. McIntosh, A., Pryde, A. and Ricker, W., Comparison of joint spectra for certain classes of commuting operators, *to appear.*

17. McIntosh, A. and Pryde, A., A functional calculus for several commuting operators, *submitted.*

18. Slodkowski, Z. and Zelazko, W., On joint spectra of commuting families of operators, *Studia Math.* **50** (1974), 127–148.

19. Taylor, J. L., A joint spectrum for several commuting operators, *J. Functional Anal.* **6** (1970), 172–191.

20. Taylor, J. L., The analytic functional calculus for several commuting operators, *Acta Math.* **125** (1970), 1–38.

21. Vasilescu, F.-H., On pairs of commuting operators, *Studia Math.* **62** (1978), 203–207.

22. Vasilescu, F.-H., A characterization of the joint spectrum in Hilbert spaces, *Rev. Roum. Math. Pures Appl.* **22** (1977), 1003–1009.

23. Wrobel, V., Joint spectra and joint numerical ranges for pairwise commuting operators in Banach spaces, *Glasgow Math. J.* **30** (1988), 145–153.

Department of Mathematics
Joetsu University of Education
Joetsu, Niigata 943
Japan