JOINT SPECTRA OF COMMUTING NORMAL OPERATORS ON BANACH SPACES
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Dedicated to Professor Kisuke Tsuchida on his retirement on March 31 1989.

1. Introduction. The joint spectrum for a commuting n-tuple in functional analysis has its origin in functional calculus which appeared in J. L. Taylor’s epoch-making paper [19] in 1970. Since then, many papers have been published on commuting n-tuples of operators on Hilbert spaces (for example, [3], [4], [5], [8], [9], [10], [21], [22]).

For those on Banach spaces, however, only a few results have come out. Recently, A. McIntosh, A. Pryde and W. Ricker in [16] characterized the joint spectrum for a strongly commuting n-tuple of operators on a Banach space. In this paper, we shall show among others that the joint spectrum for a strongly commuting n-tuple of operators on a Banach space is the joint approximate point spectrum for it.

Let \( X \) be a complex Banach space. We denote by \( X^* \) the dual space of \( X \) and by \( B(X) \) the space of all bounded linear operators on \( X \). Let \( T = (T_1, \ldots, T_n) \) be a commuting n-tuple of operators on \( X \). And let \( \sigma(T) \) be the Taylor joint spectrum of \( T \). We refer the reader to Taylor [19] for the definition of \( \sigma(T) \).

A point \( z = (z_1, \ldots, z_n) \) of \( C^n \) is in the approximate point spectrum \( \sigma_n(T) \) of \( T \) if there exists a sequence \( \{x_k\} \) of unit vectors in \( X \) such that
\[
\|(T_i - z_i)x_k\| \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

A point \( z = (z_1, \ldots, z_n) \) of \( C^n \) is said to be a joint eigenvalue of \( T \) if there exists a non-zero vector \( x \) such that
\[
T_ix = z_ix \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

Let us set
\[
\pi = \{(x, f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1\}.
\]
The spatial joint numerical range \( V(T) \) and joint numerical range \( V(B(X), T) \) of \( T \) are defined by
\[
V(T) = \{(f(T_1x), \ldots, f(T_nx)) : (x, f) \in \pi\}
\]
and
\[
V(B(X), T) = \{(F(T_1), \ldots, F(T_n)) : F \text{ is a state on } B(X)\}.
\]
respectively. The joint spectral radius and joint numerical radius of \( T = (T_1, \ldots, T_n) \) are defined by
\[
r(T) = \sup\{|z| : z \in \sigma(T)\}
\]
and
\[
\]
and
\[ v(T) = \sup \{|z|: z \in V(T)\}, \]
respectively.

For an operator \( S \in B(X) \), the usual spectrum, approximate point spectrum, spatial numerical range and numerical range of \( S \) are denoted by \( \sigma(S) \), \( \sigma_n(S) \), \( V(S) \) and \( V(B(X), S) \), respectively.

If \( V(S) \subset R \), then \( S \) is called hermitian. An operator \( S \in B(X) \) is called normal if there are hermitian operators \( H \) and \( K \) such that \( S = H + iK \) and \( HK = KH \). We denote then the operator \( H - iK \) by \( \hat{S} \). Then the following are well-known:

(1) \( \overline{co} V(S) = V(B(X), S) \), where \( \overline{co} E \) is the closed convex hull of \( E \).

(2) \( \overline{co} \sigma(S) = \overline{V(S)} \), where \( \overline{co} \) and \( \overline{E} \) are the convex hull and closure of \( E \), respectively.

(3) If \( S \) is normal, then \( \overline{co} \sigma(S) = \overline{V(S)} = V(B(X), S) \).

We refer the reader to Bonsall and Duncan [1] and [2]. We denote the boundary of \( E \) by \( \delta E \).

An \( n \)-tuple \( T = (T_1, \ldots, T_n) \) of operators is called strongly commuting if, for each \( 1 \leq j \leq n \), there exist operators \( U_j \) and \( V_j \) with real spectra, such that \( T_j = U_j + iV_j \) and \( (U_1, \ldots, U_n, V_1, \ldots, V_n) \) is a commuting \( 2n \)-tuple.

Remark Since the Fuglede theorem holds for Banach space operators, \( T = (T_1, \ldots, T_n) \) is strongly commuting if \( T \) is a commuting \( n \)-tuple of normal operators.

2. Joint spectra of strongly commuting \( n \)-tuples. McIntosh, Pryde and Ricker in [16] showed that following theorem:

**Theorem.** Let \( T = (T_1, \ldots, T_n) \) be a strongly commuting \( n \)-tuple of operators. Then \( z = (z_1, \ldots, z_n) \) is in \( \sigma(T) \) if and only if

\[ \sum_{j=1}^{n} (U_j - a_j)^2 + \sum_{j=1}^{n} (V_j - b_j)^2 \text{ is not invertible} \]

where \( T_j = U_j + iV_j \) and \( z_j = a_j + ib_j \) (\( j = 1, 2, \ldots, n \)).

We shall prove the following theorem.

**Theorem 2.1.** Let \( T = (T_1, \ldots, T_n) \) be a strongly commuting \( n \)-tuple of operators. Then \( \sigma(T) = \sigma_n(T) \).

We shall use the following two theorems.

**Theorem A** (Choi and Davis [7], Slodkowski and Zelazko [18]. Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple of operators and \( f \) an \( m \)-tuple of polynomials in \( n \)-variables. Then

\[ \sigma_n(f(T)) = f(\sigma_n(T)). \]

**Theorem B** (Taylor [20]). Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple of operators and \( f \) an \( m \)-tuple of polynomials in \( n \)-variables. Then

\[ \sigma(f(T)) = f(\sigma(T)). \]
Proof of Theorem 2.1. Let \( T_j = U_j + iV_j \) and let \( S = (U_1, \ldots, U_n, V_1, \ldots, V_n) \). Then, 
\[ \sigma(T) = g(\sigma(S)) \]
and
\[ \sigma_n(T) = g(\sigma_n(S)) \]
by the Spectral mapping theorem, where 
\[ g(z_1, z_2, \ldots, z_{2n}) = (z_1 + iz_{n+1}, \ldots, z_n + iz_{2n}). \]
It suffices to check, therefore, that the result is true for a commuting \( n \)-tuple of operators with real spectra.

Let \( H = (H_1, \ldots, H_n) \) be a commuting \( n \)-tuple of operators with real spectra.

Let \( a = (a_1, \ldots, a_n) \in \sigma(H) \) and let \( f(z) = \sum_{i=1}^{n} (z_i - a_i)^2 \). Then \( \sigma(f(H)) = f(\sigma(H)) \) so 
\[ 0 \in \delta(\sigma(f(H))) \subset \sigma_n(f(H)) = f(\sigma_n(H)) \]. It follows that there exists \( \tilde{a} \in \sigma_n(H) \subset R^n \) such that 
\[ f(\tilde{a}) = 0. \]
Clearly \( \tilde{a} \) must be equal to \( a \), and so \( a \in \sigma_n(H) \), concluding the proof.

If \( T_1, T_2, \ldots, T_n \) are commuting operators, we denote by \( A(T_1, \ldots, T_n) \) the least closed subalgebra of \( B(X) \) generated by \( I, T_1, \ldots, T_n \). And we denote by \( \Phi_{A(T_1, \ldots, T_n)} \) the set of all non-zero multiplicative linear functionals on \( A(T_1, \ldots, T_n) \).

**Theorem 2.2.** Let \( T = (T_1, \ldots, T_n) \) be a strongly commuting \( n \)-tuple of operators such that \( T_j = U_j + iV_j \) \( (j = 1, 2, \ldots, n) \). Let \( A \) be \( A(U_1, \ldots, U_n, V_1, \ldots, V_n) \). Then 
\[ \sigma(T) = \{ (\varphi(H_1) + i\varphi(K_1), \ldots, \varphi(H_n) + i\varphi(K_n)) : \varphi \in \Phi_A \} \]

**Proof.** Let \( S = (U_1, \ldots, U_n, V_1, \ldots, V_n) \). Let \( a = (a_1, \ldots, a_{2n}) \) be in \( \sigma_A(S) \) if the equation 
\[ \sum_{i=1}^{n} (U_i - a_i)D_i + \sum_{i=n+1}^{2n} (V_{i-n} - a_i)D_i = I \]
fails to have a solution for \( D_1, \ldots, D_{2n} \in A \). Since 
\[ \sigma(S) \subset \sigma_A(S), \]
an application of the Spectral Mapping Theorem gives 
\[ \sigma(T) \subset \sigma_A(T). \]

Moreover, McIntosh, Pryde and Ricker's result says that 
\[ \sigma_A(T) \subset \sigma(T). \]

Therefore, \( \sigma(T) = \sigma_A(T) \), as desired.

**Theorem 2.3.** Let \( T = (T_1, \ldots, T_n) \) be a strongly commuting \( n \)-tuple of operators. If 
\( \alpha = (\alpha_1, \ldots, \alpha_n) \) is in \( \sigma(\hat{T}T) \), then there exists \( z = (z_1, \ldots, z_n) \) in \( \sigma(T) \) such that \( |z_i|^2 = \alpha_i \) \((i = 1, 2, \ldots, n) \) where \( \hat{T}T = (\hat{T}_1T_1, \ldots, \hat{T}_nT_n) \).

**Proof.** Let \( f : R^{2n} \to R^n \) be the polynomial given by \( f(x) = f(x_1, \ldots, x_{2n}) = (x_1^2 + x_{n+1}^2, \ldots, x_n^2 + x_{2n+2}^2) \), \( x \in R^{2n} \). Then the the Spectral Mapping Theorem for the
Taylor joint spectrum, it follows that
\[ f(\sigma(S)) = \sigma(\bar{T}T), \]
where \( T_j = U_j + iV_j \) (\( j = 1, \ldots, n \)) and \( S = (U_1, \ldots, V_1, \ldots, V_n) \). Hence there exists \( (a_1, \ldots, a_n, b_1, \ldots, b_n) \) in \( \alpha_n(S) \) such that \( a_j^2 + b_j^2 = \alpha_j \) (\( j = 1, 2, \ldots, n \)).

Let \( z_j = a_j + ib_j \) (\( j = 1, 2, \ldots, n \)). Then the scalar \( z = (z_1, \ldots, z_n) \) is a desired element.

So the proof is complete.

3. Joint numerical ranges of commuting normal operators.

V. Wrobel proved the following theorem.

**Theorem 3.1** (Wrobel [23]). Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple of operators. Then
\[ \text{co} \sigma(T) \subset V(T). \]

**Theorem 3.2.** Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple of normal operators. Then
\[ \text{co} \sigma(T) = V(T) = V(B(X), T). \]

**Proof.** We assume that \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is in \( V(B(X), T) - \text{co} \sigma(T) \). By the separation theorem for a convex set, there exists a linear functional \( \phi \) on \( C^n \) and \( r \in R \) such that \( \text{Re} \phi(\lambda) < r < \text{Re} \phi(\alpha) \) (\( \lambda \in \text{co} \sigma(T) \)). We let \( \phi(z) = \sum_{j=1}^n a_jz_j \) (\( z = (z_1, \ldots, z_n) \in C^n \)), and choose a non-singular \( n \times n \) matrix \( M \) with \( (a_1, \ldots, a_n) \) in its first row. Then
\[ \text{Re} z_1 < r < \text{Re} \beta_1 \]
where \( (\beta_1, \ldots, \beta_n) = M\alpha \). It follows that
\[ \text{co} \sigma\left(\sum_{j=1}^n a_jT_j\right) \subset V(B(X), \sum_{j=1}^n a_jT_j). \]

Since \( \sum_{j=1}^n a_jT_j \) is a normal operator, this is a contradiction. So the proof is complete.

**Corollary 3.3.** Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple of normal operators. Then \( r(T) = v(T) \).

**Remark.** There is a normal operator \( N \) such that \( r(N) = 1 \) and \( \|N\| = 2 \) (see Theorem 25.6 in [2]).

**Problem.** The joint operator norm \( \|T\| \) of \( T = (T_1, \ldots, T_n) \) is defined by
\[ \|T\| = \sup \left\{ \left( \sum_{j=1}^n \|T_jx\|^2 \right)^{1/2} : x \in X \text{ and } \|x\| = 1 \right\}. \]
If \( H = (H_1, \ldots, H_n) \) is a commuting \( n \)-tuple of hermitian operators, is it then true that \( r(H) = \|H\| \)?

**Definition 1.** Banach space \( X \) is called *smooth* if the set \( \{ f : \|f\| = f(x) = 1 \} \) is a singleton for each \( x \in X \) with \( \|x\| = 1 \).

**Definition 2.** The Banach space \( X \) is called *strictly c-convex* if \( y = 0 \) whenever \( \|x\| = 1 \) and \( \|x + \lambda y\| \leq 1 \) for all complex numbers \( \lambda \) with \( \|\lambda\| \leq 1 \).

Let \( X \) be either smooth and reflexive, or strictly c-convex. Then K. Mattila proved the following:

**Theorem 3.4 (Mattila [14] and [15]).** Let \( N \) be a normal operator. If \( \lambda \) is an extreme point of \( V(N) \) such that \( \lambda \in V(N) \), then \( \lambda \) is an eigenvalue of \( N \).

We extend this theorem to a commuting \( n \)-tuple of operators. In the following, we assume that the space \( X \) is either smooth and reflexive, or strictly c-convex.

**Lemma 3.5.** Let \( H = (H_1, \ldots, H_n) \) be a commuting \( n \)-tuple of hermitian operators. If \( a = (a_1, \ldots, a_n) \) is an extreme point of \( V(H) \) and \( a \in V(H) \), then \( a \) is a joint eigenvalue of \( H \).

**Proof.** The proof is by induction on \( n \). By the theorem above, the statement is true for \( n = 1 \).

Given a positive integer \( n \), suppose that the statement is true for \( n - 1 \). We may assume that \( a = (0, \ldots, 0) \). Since \( 0 \) is an extreme point of \( V(H) \), we can choose a linear map \( F \) on \( R^n \), which has an orthogonal matrix, such that

\[
F(V(H)) = \{ z = (z_1, \ldots, z_n) \in R^n : z_n \geq 0 \}.
\]

Let \( K = (K_1, \ldots, K_n) = F(H_1, \ldots, H_n) \). Then \( K \) is a commuting \( n \)-tuple of hermitian operators which has the property \( V(K_n) \subset R^{+} \). Since \( 0 \) is an extreme point of \( V(K_n) \), it follows that \( 0 \) is the eigenvalue of \( K_n \). Let \( Y \) be the kernel space of \( K_n \). Then \( Y \) is a closed subspace of \( X \) and has the same property as \( X \). Since all \( K_i \) commute with \( K_n \), the restrictions \( K_i \) of \( K_i \) to \( Y \) are commuting hermitian operators on \( Y \) (i = 1, 2, \ldots, n). Thus

\[
(0, \ldots, 0) \in V(K_1, \ldots, K_n).
\]

Clearly, \( V(K_1, \ldots, K_{n-1}, 0) \subset V(K) \).

Therefore \( (0, \ldots, 0) \) is an extreme point of \( V(K_1, \ldots, K_{n-1}) \). Hence, by the hypothesis, there exists a non-zero vector \( x \) in \( Y \) such that

\[
K_i x = 0 \quad (i = 1, 2, \ldots, n),
\]

that is, \( 0 \) is a joint eigenvalue of \( K \). So \( 0 \) is a joint eigenvalue of \( H \). Thus the proof is complete.

**Theorem 3.6.** Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple of normal operators. If \( z = (z_1, \ldots, z_n) \) is an extreme point of \( V(T) \) and \( z \in V(T) \), then \( z \) is a joint eigenvalue of \( T \).
Proof. Let \( T_j = H_j + ik_j \) and \( z_j = a_j + ib_j \) \((j = 1, 2, \ldots, n)\). Then \((H_1, \ldots, H_n, K_1, \ldots, K_n)\) is a commuting \(2n\)-tuple of hermitian operators. \((a_1, \ldots, a_n, b_1, \ldots, b_n)\) is an extreme point of \(V(H_1, \ldots, H_n, K_1, \ldots, K_n)\) and an element of \(V(H_1, \ldots, H_n, K_1, \ldots, K_n)\). So, by the lemma above, this point is a joint eigenvalue of \(H_1, \ldots, H_n, K_1, \ldots, K_n\).

So the proof is complete.

Remark. From the proof of Theorem 3.2, the following holds:

Let \( T = (T_1, \ldots, T_n) \) be an \(n\)-tuple of operators (not necessarily commuting). Then \(\overline{\cap} V(T) = V(B(X), T)\).

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