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AN INTERPOLATORY RATIONAL APPROXIMATION

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1. The classical Hermite–Fejér interpolation process is a positive linear mapping from $C[-1, 1]$ into the space of polynomials of degree $\leq 2n-1$. If $T_n(x)$ denotes the Tchebisheff polynomial of degree n and $x_k = x_{nk}$ $(k = 1, 2, \ldots, n)$ its roots, then for any given $f \in C[-1, 1]$ the Hermite–Fejér image H_nf of f is defined by

$$(H_nf)(x) = \sum_{k=1}^{n} h_{nk}(x)f(x_k) \tag{1.1}$$

where

$$h_{nk}(x) = \frac{T_n^2(x)}{n^2(x-x_k)^2}(1-xx_k) \tag{1.2}$$

for $k = 1, 2, \ldots, n$. It is known ([2], p. 69) that

$$\sum_{k=1}^{n} h_{nk}(x) \equiv 1 \tag{1.3}$$

and

$$h'_{nk}(x_j) = 0, \qquad k, j = 1, 2, \ldots, n.$$

As to the behavior of H_nf as an approximant, it has been shown ([1]) that if $\omega_f(\cdot)$ denotes the modulus of continuity of $f \in C[-1, 1]$, then the approximation error $\|f - H_nf\|$ is of the order $n^{-1}\sum_{k=1}^{n} \omega_f(k^{-1})$. This quantity is essentially larger than $\omega_f(n^{-1})$, the best order of approximation achievable by polynomials of degree n. The purpose of this paper is to present a sequence of *positive, linear, interpolatory* operators $\Lambda_n (n = 1, 2, \ldots)$, which map $C[-1, 1]$ into the set of *rational* functions of degree $\leq 4n-1$ and for which the error bound $\|f - \Lambda_nf\|$ is fo the order $\omega_f(n^{-1})$.

For functions in particular subclasses of $C[-1, 1]$ it is known ([3]) that there exist rational approximations of degree n which yield an error bound better than $\omega_f(n^{-1})$; the specific rational operators Λ_n to be introduced here are *interpolatory, positive* and *linear*. In addition, the denominator of each Λ_nf is independent of the function f and it remains between the fixed bounds $\frac{1}{3}$ and 1 for all n.

2. For $n = 1, 2, \ldots$, let $T_n(x)$ denote the classical Tchebisheff polynomial of degree n with roots $x_n < x_{n-1} < \cdots < x_1$ in $[-1, 1]$. It is easy to verify that the

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polynomials $p_{nk}(x)$ of degree $4n-1$ defined for $k=1,2,\ldots,n$ by

$$(2.1)\qquad p_{nk}(x)=\frac{T_n^4(x)}{2n^4(x-x_k)^4}\left\{(1-x^2)(1-x_k^2)+(1-xx_k)^2+\frac{4n^2-1}{3}(x-x_k)^2(1-xx_k)\right\}$$

satisfy the relations

$$p_{nk}(x_j)=\delta_{kj},\quad k,j=1,2,\ldots n$$

$$p'_{nk}(x_j)=p''_{nk}(x_j)=p'''_{nk}(x_j)=0,\quad k,j=1,2,\ldots,n$$

$$(2.2)\qquad \sum_{k=1}^{n} p_{nk}(x)\equiv 1.$$

If we set for $k=1,2,\ldots,n$

$$(2.3)\qquad \lambda_{nk}(x)=\frac{T_n^4(x)}{2n^4(x-x_k)^4}\{(1-x^2)(1-x_k^2)+(1-xx_k)^2,$$

then by (1.2) and (2.1)

$$\lambda_{nk}(x)=p_{nk}(x)-\frac{4n^2-1}{6n^2}T_n^2(x)h_{nk}(x),$$

so that

$$(2.4)\qquad \lambda_{nk}(x_j)=\delta_{kj},\quad k,j=1,2,\ldots,n.$$

Moreover, (1.3) and (2.2) imply that

$$(2.5)\qquad \sum_{k=1}^{n}\lambda_{nk}(x)=1-\frac{4n^2-1}{6n^2}T_n^2(x).$$

3. For any given $f\in C[-1,1]$ we define the rational function $\Lambda_n f$ by

$$(3.1)\qquad (\Lambda_n f)(x)=\sum_{k=1}^{n}\lambda_{nk}(x)f(x_k)/Q_n(x)$$

where

$$(3.2)\qquad Q_n(x)=1-\frac{4n^2-1}{6n^2}T_n^2(x).$$

It follows from (2.4) and (2.5) that $(\Lambda_n f)(x_j)=f(x_j)$ for $j=1,2,\ldots,n$, i.e. Λ_n is an interpolatory operator. From (2.3) and (3.1) it is clear that Λ_n is a positive and linear operator. Observe also that the denominator $Q_n(x)$ of every $\Lambda_n f$ is of degree $2n$ and that $\frac{1}{3}<Q_n(x)\le 1$ for $x\in[-1,1]$. We now state our result concerning the error of approximation.

THEOREM 1. *Let $f\in C[-1,1]$ with modulus of continuity $\omega_f(\cdot)$. Then for $n=1,2,\ldots$ we have (in the max norm)*

$$(3.3)\qquad \|f-\Lambda_n f\|\le(1+\sqrt{3})\omega_f(n^{-1}).$$

Proof. Let $x\in[-1,1]$. Then by (3.1)

$$|f(x)-(\Lambda_nf)(x)|\le\left\{\sum_{k=1}^{n}\lambda_{nk}(x)|f(x)-f(x_k)|\right\}\Big/ Q_n(x).$$

Since for all x, x_k one has

$$|f(x)-f(x_k)|\le\{1+n|x-x_k|\}\cdot\omega_f(n^{-1}),$$

we get from the above

$$|f(x)-(\Lambda_nf)(x)|\le\left(1+\frac{n}{Q_n(x)}\sum_{k=1}^{n}\lambda_{nk}(x)\,|x-x_k|\right)w_f(n^{-1}). \tag{3.4}$$

Now, using the inequality of Schwartz and (2.5), we obtain

$$\sum_{k=1}^{n}\lambda_{nk}(x)|x-x_k|\le(Q_n(x))^{1/2}\left(\sum_{k=1}^{n}\lambda_{nk}(x)(x-x_k)^2\right)^{1/2}. \tag{3.5}$$

In order to estimate the last sum on the right hand side, we first rewrite (2.3) as

$$\lambda_{nk}(x)=\frac{T_n^4(x)}{n^4(x-x_k)^4}\{1-xx_k-x^2(1-x_k^2)/2-x_k^2(1-x^2)/2\}.$$

It is then clear that

$$\lambda_{nk}(x)(x-x_k)^2\le\frac{T_n^2(x)}{n^2}\cdot\frac{T_n^2(x)}{n^2(x-x_k)^2}(1-xx_k),$$

which, on account of (1.2) and (1.3), yields that

$$\sum_{k=1}^{n}\lambda_{nk}(x)(x-x_k)^2\le\frac{T_n^2(x)}{n^2}\le\frac{1}{n^2}. \tag{3.6}$$

Combining (3.4)–(3.6) we obtain the inequality

$$|f(x)-(\Lambda_nf)(x)|\le\{1+(Q_n(x))^{-1/2}\}\cdot\omega_f(n^{-1}). \tag{3.7}$$

Since (3.2) implies that $Q_n(x)>\frac{1}{3}$ for all $x\in[-1,1]$, the required result follows from (3.7).

4. For functions that satisfy a Lipschitz condition, we can state a somewhat stronger result.

THEOREM 2. *Let $f\in\mathrm{Lip}_M\,1$ in $[-1,1]$. Then for $x\in[-1,1]$ and $n=1,2,\ldots,$ we have*

$$|f(x)-(\Lambda_nf)(x)|\le M\left(\frac{\sqrt{6(1-x^2)}}{n}+\frac{3}{2n^2}\right).$$

Proof. We again rewrite (2.3). We have for $k=1,2,\ldots,n$

$$\lambda_{nk}(x)=(1-x^2)(1-x_k^2)\frac{T_n^4(x)}{n^4(x-x_k)^4}+\frac{T_n^4(x)}{2n^4(x-x_k)^2}=u_k(x)+v_k(x). \tag{4.1}$$

It is easy to see that for all $x, x_k \in [-1, 1]$, the inequalities $|x-x_k| \leq 1-xx_k$ and $1-x_k^2 \leq 2(1-xx_k)$ hold. Hence by (1.2) and (1.3),

$$\sum_{k=1}^{n} v_k(x)|x-x_k| \leq T_n^2(x)/2n^2 \leq \frac{1}{2n^2} \tag{4.2}$$

and

$$\sum_{k=1}^{n} u_k(x)(x-x_k)^2 \leq 2(1-x^2)\frac{T_n^2(x)}{n^2} \leq \frac{2(1-x^2)}{n^2}. \tag{4.3}$$

From (4.3), on using the Schwartz inequality and the fact that $u_k(x) \leq \lambda_{nk}(x)$, we get

$$\sum_{k=1}^{n} u_k(x)|x-x_k| \leq (Q_n(x))^{1/2} \cdot \frac{\sqrt{2(1-x^2)}}{n}. \tag{4.4}$$

The result now follows from (4.2), (4.4) and the inequality $Q_n(x) > \frac{1}{3}$, since $|f(x)-f(x_k)| \leq M \cdot |x-x_k|$ for all x, x_k.

REMARK. The estimates given for $\|f-\Lambda_n f\|$ by Theorem 1 and for $|f(x)-(\Lambda_n f)(x)|$ by Theorem 2 are poor if $n=1$. In fact we have $\|f-\Lambda_1 f\| \leq \omega_f(1)$ and $\|f-\Lambda_1 f\| \leq M$ respectively.

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