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ON EXTREMAL POINTS OF THE UNIT BALL IN THE BANACH SPACE OF LIPSCHITZ CONTINUOUS FUNCTIONS

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Abstract

It is shown that for arbitrary $\varepsilon > 0$ there is a function $x(t,x)$ defined on the square $[0,1] \times [0,1]$ such that $x(t,s)$ represents an extremal point of the unit ball in the space of Lipschitz continuous functions, and the gradient of $x(t,s)$ is equal to 0 except on a set of measure at most ε.

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The investigations of Lipschitz observability and theorems concerning forms of optimal observations [1] stimulate investigations of extremal points of the unit ball in the space of Lipschitz functions defined on a metric space Ω with a metric ρ, and with the classical Lipschitz distance

$$d(f,g) = \sup_{t,t_1 \in \Omega} \frac{|[f(t) - g(t)] - [f(t_1) - g(t_1)]|}{\rho(t,t_1)}. \tag{1}$$

The semimetric d is invariant, and hence it induces a seminorm

$$\|f\| = d(f,0) = \sup_{t,t_1 \in \Omega} \frac{|f(t) - f(t_1)|}{\rho(t,t_1)}. \tag{2}$$

Observe that the semimetric d and the seminorm $\| \ \|$ do not distinguish the functions which differ by a constant. In fact, we consider the quotient space L/C

of all Lipschitz continuous functions L divided by the space of constant functions C. The seminorm $\| \|$ induces a norm on L/C, and L/C is a Banach space with this norm.

In further considerations it will be easier for us to consider an isomorphic and isometric image of L/C, namely the space L_a, where $a \in \Omega$, of all Lipschitz functions vanishing at a. Equation (2) above defines the isometric norm on L_a.

If Ω is an interval $[\alpha, \beta]$, then each function f belonging to L_a, $\alpha \leqslant a \leqslant \beta$, is differentiable almost everywhere. If f is an extremal point of the unit ball in L_a, then its derivative $f(t)$ has modulus equal to one almost everywhere; $|f(t)| = 1$ a.e. [1].

It is interesting that a similar theorem does not hold in the case when Ω is the square $[0,1] \times [0,1]$. Of course in this case also, every Lipschitz function is differentiable almost everywhere. However we have

THEOREM 1. *For each $\varepsilon > 0$, there is a function $f \in L_a$, being an extremal point of the unit ball of L_a, such that the support of the gradient has Lebesgue measure less than ε;*

$$|\{t: \nabla f|_t \neq 0\}| < \varepsilon, \tag{3}$$

where $|E|$ denotes the two dimensional Lebesgue measure of a set E.

The proof of Theorem 1 is based on the following notions and lemmas. Let Ω be a connected metric space. Let $\{\Omega_\alpha\}$ be a covering of Ω, i.e. $\bigcup_\alpha \Omega_\alpha \supset \Omega$. We say that the covering $\{\Omega_\alpha\}$ is *finitely connected* if each of the sets Ω_α is connected, and if, for arbitrary t, $\bar{t} \in \Omega$, there is a finite system of sets $\Omega_{\alpha_1}, \ldots, \Omega_{\alpha_n}$ for which there are $t_0 = t$, t_1, $t_2, \ldots, t_n = \bar{t}$ such that t_{i-1}, $t_i \in \Omega_{\alpha_i}$.

LEMMA 1. *Let $\{\Omega_\alpha\}$ be a finitely connected covering of Ω. Let f be an arbitrary Lipschitz function of seminorm 1 defined on Ω. Suppose that for all α the restriction f to Ω_α, $f|\Omega_\alpha$, is an extremal point of the unit ball of the space of Lipschitz functions defined on Ω_α. Then f is an extremal point of the space of Lipschitz functions defined on Ω.*

PROOF. Suppose that f is not an extremal point of the unit ball of the space of Lipschitz functions defined on Ω. Then by definition there are two Lipschitz functions f_1, f_2 with $\|f_1\| = \|f_2\| = 1$ such that $f_1 \neq f_2$ and $(f_1 + f_2)/2 = f$. This leads to a contradiction. Take any two points t and $\bar{t}$ such that

$$f_1(t) - f_2(t) \neq f_1(\bar{t}) - f_2(\bar{t}). \tag{4}$$

Such points must exist since $f_1 \neq f_2$. Since the covering Ω_α is finitely connected, there is a system of points $t_0 = t$, $t_1, \ldots, t_n = \bar{t}$ and a system of indices $\alpha_1, \ldots, \alpha_n$

such that t_{i-1}, $t_i \in \Omega_{\alpha_i}$, $i = 1, 2, \ldots, n$. Since the restrictions of f to the Ω_{α_i} are extremal points of the unit ball in the space of Lipschitz functions, and since $f_1 \mid \Omega_{\alpha_i}$ and $f_2 \mid \Omega_{\alpha_i}$ have for each α_i a pseudonorm not greater than one, the function $f_1(t) - f_2(t)$ is constant on each Ω_{α_i}. This implies that

$$f_1(t_0) - f_2(t_0) = f_1(t_1) - f_2(t_1) = \cdots = f_1(t_n) - f_2(t_n),$$

and this contradicts (4).

PROOF OF THEOREM 1. Take an arbitrary $\delta > 0$. Now take the Cantor set $K_\delta \subset [0, 1]$ of measure $1 - \delta$. This set we obtain by the classical Cantor construction changed in such a way that, at the nth step, we remove from each interval the central interval of length $(\delta/(1 + \delta))^n$. The complement of K_δ, $G^\delta = [0, 1] \setminus K_\delta$, is an open set of measure equal to δ, and it is dense in $[0, 1]$. Let

$$\Omega = \left(G^\delta \times [0,1]\right) \cup \left([0,1] \times \left[\tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta\right]\right).$$

It is easy to see that the set Ω is connected and dense in the whole square $[0, 1] \times [0, 1]$. Now we shall define a finitely connected covering in the following way. We enumerate all the components of G^δ. They are intervals (a_n, b_n). Now we define sets

$$\Omega_{1,n} = \left\{(x, y)\colon a_n < x < b_n, 0 \leqslant y \leqslant \tfrac{1}{2} - \delta + \tfrac{1}{2}(b_n - a_n) - \left|x - \frac{a_n + b_n}{2}\right|\right\},$$

$$\Omega_{2,n} = \left\{(x, y)\colon a_n < x < b_n, 1 \geqslant y \geqslant \tfrac{1}{2} + \delta - \tfrac{1}{2}(b_n - a_n) + \left|x - \frac{a_n + b_n}{2}\right|\right\}$$

and

$$\Omega_0 = \operatorname{cl}\left([0,1] \times \left[\tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta\right] \setminus \bigcup_{\substack{n=1 \\ j=1,2}} \Omega_{j,n}\right).$$

Since Ω_0 is closed, the sets

$$\Omega_0, \Omega_{1,n}, \Omega_{2,n}, \Omega_{1,2}, \Omega_{2,2}, \ldots$$

constitute a finitely connected covering. Now we define a function $f(x, y)$ on Ω in the following way:

$$f(x, y) = \begin{cases} \dfrac{a_n + b_n}{2} - \left|\dfrac{a_n + b_n - 2x}{2}\right| & \text{for } (x, y) \in \Omega_{j,n},\ j = 1, 2, \\ \delta - \left|y - \dfrac{1}{2}\right| & \text{for } (x, y) \in \Omega_0. \end{cases}$$

It is easy to verify that $f(x, y)$ is a continuous function on Ω and that its restriction to Ω_0 and $\Omega_{j,n}$ is, for each of these sets, an extremal point of the unit ball of the corresponding space of Lipschitz functions. Thus by Lemma 1 it is an extremal point of the unit ball of the Lipschitz functions defined on Ω. Observe that if $(x, y) \in [0,1] \times [0,1]$, and if (x, y) does not belong to Ω, then for any sequence $(x_n, y_n) \in \Omega$ such that (x_n, y_n) tends to (x, y), we have $\lim f(x_n, y_n) = 0$. Thus we can extend the function f from Ω to a continuous function on the square $[0,1] \times [0,1]$ by taking

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{for } (x, y) \in \Omega, \\ 0 & \text{for } (x, y) \notin \Omega. \end{cases}$$

Since Ω is dense in $[0,1] \times [0,1]$, the extension is unique, and thus $\tilde{f}(x, y)$ is an extremal point of the unit ball of the Lipschitz functions defined on $[0,1] \times [0,1]$.

Observe that the gradient of $\tilde{f}(x, y)$ vanishes outside Ω, and the set Ω has two dimensional Lebesgue measure less than 3δ. Taking $\delta \leqslant \varepsilon/3$, we obtain the theorem.

References

[1] S. Rolewicz, 'On optimal observability of Lipschitz systems', in *Selected topics in operations research and mathematical economics*, Lecture Notes in Economics and Mathematical Systems 226, Springer-Verlag, pp. 151–158.

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