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EIGENVALUE CHARACTERIZATION FOR (n, p) BOUNDARY-VALUE PROBLEMS

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Abstract

We consider the (n, p) boundary value problem

$$y^{(n)} + \lambda H(t, y) = \lambda K(t, y), \quad n \geq 2, \ t \in (0, 1),$$
$$y^{(p)}(1) = y^{(i)}(0) = 0, \quad 0 \leq i \leq n-2,$$

where $\lambda > 0$ and $0 \leq p \leq n-1$ is fixed. We characterize the values of λ such that the boundary value problem has a positive solution. For the special case $\lambda = 1$, we also offer sufficient conditions for the existence of positive solutions of the boundary value problem.

1. Introduction

In this paper we shall consider the nth order differential equation

$$y^{(n)} + \lambda H(t, y) = \lambda K(t, y), \ t \in (0, 1), \tag{1.1}$$

together with the (n, p) boundary conditions

$$\begin{aligned} &y^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ &y^{(p)}(1) = 0, \end{aligned} \tag{1.2}$$

where $n \geq 2$, $\lambda > 0$ and p is a fixed integer satisfying $0 \leq p \leq n-1$. Throughout it is assumed that there exist continuous functions $f : [0, \infty) \to (0, \infty)$ and $k, k_1, h, h_1 : (0, 1) \to \mathbb{R}$ such that

(H_1) f is nondecreasing;

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(H_2) for $u \in [0, \infty)$,

$$h(t) \le \frac{H(t,u)}{f(u)} \le h_1(t), \; k(t) \le \frac{K(t,u)}{f(u)} \le k_1(t);$$

(H_3) $h(t) - k_1(t)$ is nonnegative and is not identically zero on any subinterval of $(0, 1)$;

(H_4) $\int_0^1 (1-t)^{n-p-1}[h_1(t) - k(t)]\,dt < \infty$.

We shall characterize the values of λ for which the (n, p) boundary value problem (1.1), (1.2) has a positive solution. By a *positive solution* y of (1.1), (1.2), we mean $y \in C^{(n)}(0, 1)$ satisfying (1.1) on $(0, 1)$ and fulfilling (1.2), and y is nonnegative and is not identically zero on $[0, 1]$. If, for a particular λ the boundary value problem (1.1), (1.2) has a positive solution y, then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of (1.1), (1.2). We let

$$E = \{\lambda > 0 \mid (1.1), (1.2) \text{ has a positive solution}\}$$

be the set of eigenvalues of the boundary value problem (1.1), (1.2).

Next, for the special case $\lambda = 1$, we shall give an existence result for positive solutions of the boundary value problem (1.1), (1.2), assuming that f is either superlinear or sublinear. To be precise, introduce the notation

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

The function f is said to be *superlinear* if $f_0 = 0$, $f_\infty = \infty$, and f is *sublinear* provided $f_0 = \infty$, $f_\infty = 0$. The technique used here is a generalization and extension of that initiated by Fink, Gatica and Hernandez [19] and Erbe and Wang [17] for second-order boundary value problems.

The motivation for the present work stems from many recent investigations. In fact, when $n = 2$ the boundary value problem (1.1), (1.2) describes a vast spectrum of scientific phenomena such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, where only positive solutions are meaningful, for example, see [5, 9, 11, 12, 21, 24, 29]. Recently, several eigenvalue characterizations for particular cases of (1.1), (1.2) have been carried out. To cite a few examples, Fink, Gatica and Hernandez [19] have dealt with the boundary value problem

$$\begin{gathered} y'' + \lambda q(t) f(y) = 0, \quad t \in (0, 1), \\ y(0) = y(1) = 0. \end{gathered} \tag{1.3}$$

Their results are extended in [20] to systems of second-order boundary-value problems. In [8] and [18], a different boundary value problem is tackled

$$y'' + \frac{N-1}{t}y' + \lambda q(t)f(y) = 0, \quad t \in (0,1), \qquad (1.4)$$
$$y(0) = y(1) = 0.$$

Further, Chyan and Henderson [10] have studied a more general problem than (1.3), namely,

$$y^{(n)} + \lambda q(t)f(y) = 0, \quad t \in (0,1), \qquad (1.5)$$
$$y^{(i)}(0) = y^{(n-2)}(1) = 0, \quad 0 \le i \le n-2.$$

Our results not only generalize and extend the known eigenvalue theorems for (1.3)–(1.5), but also complement the work of Wong and Agarwal [33, 34], as well as including several other known criteria offered in [2].

For the special case $\lambda = 1$, particular and related cases of (1.1), (1.2) have been the subject matter of many recent publications on singular boundary value problems, for example, see the monograph of O'Regen [28] and also [3, 4, 13, 23, 25, 26, 31]. Further, for the case of second-order boundary value problems, (1.1), (1.2) arise in applications involving nonlinear elliptic problems in annular regions. For this we refer to [6, 7, 22, 30]. In all these applications, it is frequent that only solutions that are positive are useful. Recently, Eloe and Henderson [14, 15] have considered the nth-order differential equation

$$y^{(n)} + q(t)f(y) = 0, \quad t \in (0,1),$$

subject to the boundary conditions

$$y^{(i)}(0) = y^{(n-2)}(1) = 0, \quad 0 \le i \le n-2,$$
$$y^{(i)}(0) = y(1) = 0, \quad 0 \le i \le n-2.$$

Our result not only generalizes and extends their work, but also complements other related investigations in [16, 17, 32, 34].

The plan of this paper is as follows. In Section 2 we shall state a fixed-point theorem due to Krasnosel'skii [27], and present some properties of a certain Green's function which are needed later. In Section 3, by defining an appropriate Banach space and cone, we characterize the set E. Finally, the special case $\lambda = 1$ is treated in Section 4 and a fixed-point theorem from [27] is used to give an existence result for positive solutions of (1.1), (1.2).

2. Preliminaries

THEOREM 2.1 ([27]). *Let B be a Banach space, and let $C(\subset B)$ be a cone. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\bar{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that, either

(a) $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_2$, *or*
(b) $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_2$.

Then, S has a fixed point in $C \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

To obtain a solution for (1.1), (1.2), we require a mapping whose kernel $G(t, s)$ is the Green's function of the boundary-value problem

$$-y^{(n)} = 0,$$
$$y^{(p)}(1) = y^{(i)}(0) = 0, \quad 0 \le i \le n-2,$$

where $0 \le p \le n-1$ but fixed. From [1] we have

$$G(t,s) = \frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-p-1} - (t-s)^{n-1}, & 0 \le s \le t \le 1 \\ t^{n-1}(1-s)^{n-p-1}, & 0 \le t \le s \le 1 \end{cases} \tag{2.1}$$

and

$$\frac{\partial^i}{\partial t^i} G(t,s) \ge 0, \quad 0 \le i \le p, \quad (t,s) \in [0,1] \times [0,1].$$

LEMMA 2.1. *For $(t, s) \in [0, 1] \times [0, 1]$, we have*

$$G(t,s) \le \frac{1}{(n-1)!}(1-s)^{n-p-1}. \tag{2.2}$$

PROOF. This is immediate from (2.1).

LEMMA 2.2. *For $(t, s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [0, 1]$, we have*

$$G(t,s) \ge \left(\frac{1}{4}\right)^{n-1} \frac{1}{(n-1)!}(1-s)^{n-p-1}\phi(s), \tag{2.3}$$

where $0 \le \phi(s) \le 1$ is given by

$$\phi(s) = \begin{cases} 1 - (1-s)^p, & s \le t \\ 1, & t \le s. \end{cases} \tag{2.4}$$

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PROOF. For $0 \le s \le t$, from (2.1) we find

$$\begin{aligned}(n-1)!G(t,s) &\ge t^{n-1}(1-s)^{n-p-1} - (t-ts)^{n-1}\\ &= t^{n-1}(1-s)^{n-p-1}[1-(1-s)^p]\\ &\ge \left(\frac{1}{4}\right)^{n-1}(1-s)^{n-p-1}\phi(s).\end{aligned}$$

For $t \le s \le 1$, the inequality (2.3) is obvious.

We shall need the following notation later. Let

$$v(t) = h_1(t) - k(t) \quad \text{and} \quad u(t) = h(t) - k_1(t). \tag{2.5}$$

For a nonnegative y on $[0, 1]$, we denote

$$\alpha = \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-p-1}v(s)f(y(s))\,ds \tag{2.6}$$

and

$$\beta = \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-p-1}\phi(s)u(s)f(y(s))\,ds. \tag{2.7}$$

In view of (H_2) and (H_3), it is clear that $\alpha \ge \beta > 0$. Further, we define the constant

$$\gamma = \left(\frac{1}{4}\right)^{n-1}\frac{\beta}{\alpha} \tag{2.8}$$

and note that $0 < \gamma < 1$.

3. Eigenvalue characterization

Let the Banach space

$$B = \{y \mid y \in C[0, 1]\}$$

be equipped with norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$, and let

$$C = \left\{ y \in B \,\middle|\, y(t) \text{ is nonnegative on } [0, 1];\; \min_{t\in[\frac{1}{4},\frac{3}{4}]} y(t) \ge \gamma\|y\| \right\}.$$

We note that C is a cone in B. Further, let

$$C_M = \{y \in C \mid \|y\| \le M\}.$$

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We define the operator $S : C \to B$ by

$$Sy(t) = \int_0^1 G(t,s)\,[H(s,y) - K(s,y)]\,ds, \;\; t \in [0,1]. \tag{3.1}$$

To obtain a positive solution of (1.1), (1.2), we shall seek a fixed point of the operator λS in the cone C.

It is clear from (H_2) that

$$Uy(t) \le Sy(t) \le Vy(t), \quad t \in [0,1], \tag{3.2}$$

where

$$Uy(t) = \int_0^1 G(t,s)u(s)f(y(s))\,ds \tag{3.3}$$

and

$$Vy(t) = \int_0^1 G(t,s)v(s)f(y(s))\,ds. \tag{3.4}$$

We shall now show that the operator S is compact on the cone C. Let us consider the case when $u(t)$ is unbounded in a deleted right neighborhood of 0 and also in a deleted left neighborhood of 1. Clearly, $v(t)$ is also unbounded near 0 and 1. For $m \in \{1, 2, 3, \dots\}$, define $u_m, v_m : [0,1] \to \mathbb{R}$ by

$$u_m(t) = \begin{cases} u\left(\dfrac{1}{m+1}\right), & 0 \le t \le \dfrac{1}{m+1} \\ u(t), & \dfrac{1}{m+1} \le t \le \dfrac{m}{m+1} \\ u\left(\dfrac{m}{m+1}\right), & \dfrac{m}{m+1} \le t \le 1, \end{cases} \tag{3.5}$$

$$v_m(t) = \begin{cases} v\left(\dfrac{1}{m+1}\right), & 0 \le t \le \dfrac{1}{m+1} \\ v(t), & \dfrac{1}{m+1} \le t \le \dfrac{m}{m+1} \\ v\left(\dfrac{m}{m+1}\right), & \dfrac{m}{m+1} \le t \le 1, \end{cases} \tag{3.6}$$

and the operators $U_m, V_m : C \to B$ by

$$U_m y(t) = \int_0^1 G(t,s)u_m(s)f(y(s))\,ds, \tag{3.7}$$

$$V_m y(t) = \int_0^1 G(t,s)v_m(s)f(y(s))\,ds. \tag{3.8}$$

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It is standard that for each m, both U_m and V_m are compact operators on C. Let $M > 0$ and $y \in C_M$. Then, in view of Lemma 2.1, we find

$$
\begin{aligned}
&|V_m y(t) - Vy(t)| \\
&= \int_0^1 G(t,s)|v_m(s) - v(s)| f(y(s))\, ds \\
&= \int_0^{\frac{1}{m+1}} G(t,s)|v_m(s) - v(s)| f(y(s))\, ds + \int_{\frac{m}{m+1}}^1 G(t,s)|v_m(s) - v(s)| f(y(s))\, ds \\
&\le \frac{f(M)}{(n-1)!} \left[\int_0^{\frac{1}{m+1}} (1-s)^{n-p-1} \left| v\left(\frac{1}{m+1}\right) - v(s) \right| ds \right. \\
&\qquad \left. + \int_{\frac{m}{m+1}}^1 (1-s)^{n-p-1} \left| v\left(\frac{m}{m+1}\right) - v(s) \right| ds \right].
\end{aligned}
$$

The integrability of $(1-t)^{n-p-1}v(t)$ (condition (H_4)) implies that V_m converges uniformly to V on C_M. Hence, V is compact on C. Similarly, we can verify that U_m converges uniformly to U on C_M and therefore U is compact on C. It follows from (3.2) that the operator S is compact on C.

THEOREM 3.1. *There exists a $c > 0$ such that the interval $(0, c] \subseteq E$.*

PROOF. Let $M > 0$ be given. Define

$$
c = M \left[\frac{f(M)}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s)\, ds \right]^{-1}. \tag{3.9}
$$

Let $\lambda \in (0, c]$. We shall prove that $(\lambda S)(C_M) \subseteq C_M$. For this, let $y \in C_M$ and we shall first show that $\lambda S y \in C$. Clearly, from (3.2) and (H_3), we find

$$
(\lambda S y)(t) \ge \lambda \int_0^1 G(t,s) u(s) f(y(s))\, ds \ge 0, \quad t \in [0,1]. \tag{3.10}
$$

Further, it follows from (3.2) and Lemma 2.1 that

$$
\begin{aligned}
Sy(t) &\le \int_0^1 G(t,s) v(s) f(y(s))\, ds \\
&\le \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) f(y(s))\, ds = \alpha, \quad t \in [0,1].
\end{aligned}
$$

Thus

$$
\|Sy\| \le \alpha. \tag{3.11}
$$

Now, on using (3.2), Lemma 2.2 and (3.11), we find for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$ that

$$\begin{aligned}
(\lambda S y)(t) &\geq \lambda \int_0^1 G(t,s) u(s) f(y(s))\, ds \\
&\geq \lambda \left(\frac{1}{4}\right)^{n-1} \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-p-1} \phi(s) u(s) f(y(s))\, ds \\
&= \lambda \left(\frac{1}{4}\right)^{n-1} \beta \\
&\geq \lambda \left(\frac{1}{4}\right)^{n-1} \beta \cdot \frac{\|Sy\|}{\alpha} = \lambda \gamma \|Sy\| = \gamma \|\lambda S y\|.
\end{aligned}$$

Therefore

$$\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} (\lambda S y)(t) \geq \gamma \|\lambda S y\| \tag{3.12}$$

and (3.10) and (3.12) lead to $\lambda S y \in C$.

Next, we shall show that $\|\lambda S y\| \leq M$. For this, on using (3.2), Lemma 2.1 and (3.9) successively, we get

$$\begin{aligned}
(\lambda S y)(t) &\leq \lambda \int_0^1 G(t,s) v(s) f(y(s))\, ds \\
&\leq \frac{\lambda}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) f(M)\, ds \leq M, \quad t \in [0,1].
\end{aligned}$$

Consequently,

$$\|\lambda S y\| \leq M.$$

Hence $(\lambda S)(C_M) \subseteq C_M$. Also, standard arguments yield that λS is completely continuous. By the Schauder fixed point theorem, λS has a fixed point in C_M. Clearly this fixed point is a positive solution of (1.1), (1.2) and therefore λ is an eigenvalue of (1.1), (1.2). Since $\lambda \in (0, c]$ is arbitrary, it follows immediately that the interval $(0, c] \subseteq E$.

The next theorem makes use of the monotonicity and compactness of the operator S on the cone C. We refer to [19, Theorem 3.2] for its proof.

THEOREM 3.2 ([19]). *Suppose that* $\lambda_0 \in E$. *Then, for each* $0 < \lambda < \lambda_0$, $\lambda \in E$.

The following corollary is immediate from Theorem 3.2.

COROLLARY 3.1. *E is an interval.*

We shall establish conditions under which E is a bounded or unbounded interval. For this, we need the following results.

THEOREM 3.3. *Let λ be an eigenvalue of* (1.1), (1.2) *and $y \in C$ be a corresponding eigenfunction. If $y^{(n-1)}(0) = q$ for some $q > 0$, then λ satisfies*

$$g(v)q\left[f\left(\frac{q}{(n-1)!}\right)\right]^{-1} \le \lambda \le g(u)q[f(0)]^{-1}, \tag{3.13}$$

where

$$g(z) = \left[\int_0^1 (1-s)^{n-p-1} z(s)\, ds\right]^{-1}. \tag{3.14}$$

PROOF. For $m \in \{1, 2, 3, \ldots\}$, we define $f_m = f * \psi_m$, where ψ_m is a standard mollifier [10, 19] such that f_m is Lipschitz and converges uniformly to f.

For a fixed m, let λ_m be an eigenvalue and y_m, with $y_m^{(n-1)}(0) = q$, be a corresponding eigenfunction of the boundary-value problem

$$y_m^{(n)} + \lambda_m H_m(t, y_m) = \lambda_m K_m(t, y_m),\ t \in [0, 1], \tag{3.15}$$

$$\begin{aligned} y_m^{(i)}(0) &= 0, \quad 0 \le i \le n-2, \\ y_m^{(p)}(1) &= 0, \end{aligned} \tag{3.16}$$

where H_m and K_m converge uniformly to H and K respectively, and

$$u_m(t) \le \frac{H_m(t, z) - K_m(t, z)}{f_m(z)} \le v_m(t) \tag{3.17}$$

(see (3.5) and (3.6) for the definitions of $u_m(t)$ and $v_m(t)$).

Clearly, y_m is the unique solution of the initial value problem (3.15),

$$\begin{aligned} y_m^{(i)}(0) &= 0, \quad 0 \le i \le n-2, \\ y_m^{(n-1)}(1) &= q. \end{aligned} \tag{3.18}$$

Since

$$y_m^{(n)}(t) = \lambda_m[K_m(t, y_m) - H_m(t, y_m)] \le -\lambda_m u_m(t) f_m(y_m(t)) \le 0,$$

we have $y_m^{(n-1)}$ is nonincreasing and hence

$$y_m^{(n-1)}(t) \le y_m^{(n-1)}(0) = q, \quad t \in [0, 1]. \tag{3.19}$$

Noting that

$$y_m^{(i)}(t) = \int_0^t y_m^{(i+1)}(s)\,ds, \quad 0 \le i \le n-2,\ t \in [0,1], \tag{3.20}$$

we obtain, on using (3.19),

$$y_m^{(n-2)}(t) = \int_0^t y_m^{(n-1)}(s)ds \le \int_0^t q\,ds = qt, \quad t \in [0,1].$$

Applying the above inequality and continuing integrating, we find

$$y_m(t) \le q\frac{t^{n-1}}{(n-1)!} \le \frac{q}{(n-1)!}, \quad t \in [0,1]. \tag{3.21}$$

Now, from (3.15), (3.17) and (3.21) we get for $t \in [0,1]$,

$$\lambda_m u_m(t) f_m(0) \le -y_m^{(n)}(t) \le \lambda_m v_m(t) f_m\left(\frac{q}{(n-1)!}\right). \tag{3.22}$$

An integration of (3.22) from 0 to t provides

$$\theta_1(t) \le y_m^{(n-1)}(t) \le \theta_2(t), \quad t \in [0,1], \tag{3.23}$$

where

$$\theta_1(t) = q - \lambda_m f_m\left(\frac{q}{(n-1)!}\right)\int_0^t v_m(s)\,ds$$

and

$$\theta_2(t) = q - \lambda_m f_m(0)\int_0^t u_m(s)\,ds.$$

Continuing the integration process, we get for $0 \le p \le n-1$,

$$\theta_3(t) \le y_m^{(p)}(t) \le \theta_4(t), \quad t \in [0,1], \tag{3.24}$$

where

$$\theta_3(t) = \frac{q}{(n-p-1)!}t^{n-p-1} - \lambda_m f_m\left(\frac{q}{(n-1)!}\right)\int_0^t \frac{(t-s)^{n-p-1}}{(n-p-1)!}v_m(s)\,ds$$

and

$$\theta_4(t) = \frac{q}{(n-p-1)!}t^{n-p-1} - \lambda_m f_m(0)\int_0^t \frac{(t-s)^{n-p-1}}{(n-p-1)!}u_m(s)\,ds.$$

In order to have $y_m^{(p)}(1) = 0$ (see (3.16)), from (3.24) it is necessary that $\theta_3(1) \le 0$ and $\theta_4(1) \ge 0$, or equivalently,

$$\lambda_m \ge g(v_m)q\left[f_m\left(\frac{q}{(n-1)!}\right)\right]^{-1} \tag{3.25}$$

and

$$\lambda_m \le g(u_m)q[f_m(0)]^{-1}. \tag{3.26}$$

Coupling (3.25) and (3.26), we get

$$g(v_m)q\left[f_m\left(\frac{q}{(n-1)!}\right)\right]^{-1} \le \lambda_m \le g(u_m)q[f_m(0)]^{-1}. \tag{3.27}$$

It follows from (3.23) that $\{y_m^{(n-1)}\}_{m=1}^{\infty}$ is a uniformly bounded sequence on [0, 1]. Using the initial conditions (3.18) and repeated integrations, we find that $\{y_m^{(i)}\}_{m=1}^{\infty}$, $0 \le i \le n-1$ is a uniformly bounded sequence. Thus there exists a subsequence, which can be relabelled as $\{y_m\}_{m=1}^{\infty}$, that converges uniformly (in fact, in $C^{(n-1)}$-norm) to some y on [0, 1]. We note that each $y_m(t)$ can be expressed as

$$y_m(t) = \lambda_m \int_0^1 G(t,s)[H_m(s,y_m) - K_m(s,y_m)]\,ds, \quad t \in [0,1]. \tag{3.28}$$

Since $\{\lambda_m\}_{m=1}^{\infty}$ is a bounded sequence (from (3.27)), there is a subsequence, which can be relabelled as $\{\lambda_m\}_{m=1}^{\infty}$, that converges to some λ. Letting $m \to \infty$ in (3.28) yields

$$y(t) = \lambda \int_0^1 G(t,s)[H(s,y) - K(s,y)]\,ds, \ t \in [0,1].$$

This means that y is an eigenfunction of (1.1), (1.2) corresponding to the eigenvalue λ. Further, $y^{(n-1)}(0) = q$, and (3.13) follows from (3.27) immediately.

THEOREM 3.4. *Let λ be an eigenvalue of* (1.1), (1.2) *and $y \in C$ be a corresponding eigenfunction. Further, let $\eta = \|y\|$ and $\rho = \max_{t\in[0,1]}|y^{(n-2)}(t)|$. Then*

$$\lambda \ge \frac{\eta}{f(\eta)}(n-1)!\left[\int_0^1 (1-s)^{n-p-1}v(s)\,ds\right]^{-1} \tag{3.29}$$

and

$$\lambda \le \frac{\eta}{f(\gamma\eta)}\left[\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2},s\right)u(s)\,ds\right]^{-1}. \tag{3.30}$$

Also, there exists a $c > 0$ such that

$$\lambda \leq \frac{\rho}{f(c\rho)} \frac{1}{(n-2)!} \left[\int_{\frac{1}{4}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) u(s)\, ds \right]^{-1}. \tag{3.31}$$

PROOF. First we shall prove (3.29). For this, let $t_0 \in [0, 1]$ be such that

$$\eta = \|y\| = y(t_0).$$

Then, applying (3.2) and Lemma 2.1 we find

$$\begin{aligned} \eta = y(t_0) = (\lambda S y)(t_0) &\leq \lambda \int_0^1 G(t_0, s) v(s) f(y(s))\, ds \\ &\leq \frac{\lambda}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) f(y(s))\, ds \\ &\leq \frac{\lambda}{(n-1)!} f(\eta) \int_0^1 (1-s)^{n-p-1} v(s)\, ds \end{aligned}$$

from which (3.29) is immediate.

Next, using (3.2) and the fact that $\min_{t \in [\frac{1}{4}, \frac{3}{4}]} y(t) \geq \gamma \eta$, we get

$$\begin{aligned} \eta \geq y\left(\frac{1}{2}\right) &\geq \lambda \int_0^1 G\left(\frac{1}{2}, s\right) v(s) f(y(s))\, ds \\ &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) v(s) f(y(s))\, ds \\ &\geq \lambda f(\gamma\eta) \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) v(s)\, ds \end{aligned}$$

which gives (3.30).

Finally, to prove (3.31) we note from the relation

$$y^{(i)}(t) = \int_0^t y^{(i+1)}(s)\, ds, \quad 0 \leq i \leq n-3,\ t \in [0, 1] \tag{3.32}$$

and the nonnegativity of y that $y^{(n-2)}$ is nonnegative on $[0, 1]$. It is also observed that $y^{(n)}$ is nonpositive and hence $y^{(n-2)}$ is concave on $[0, 1]$. Thus, there exists a unique $t \in [0, 1]$ such that $\rho = \max_{t \in [0,1]} y^{(n-2)}(t) = y^{(n-2)}(t_1)$. We shall consider two cases.

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Case 1 $y^{(n-2)}(1) = 0$

Here, $y^{(n-2)}(0) = y^{(n-2)}(1) = 0$. Thus, it follows from the concavity of $y^{(n-2)}$ that

$$y^{(n-2)}(t) \geq \begin{cases} \dfrac{\rho}{t_1} t, & t \in [0, t_1] \\ \dfrac{\rho}{1 - t_1}(1 - t), & t \in [t_1, 1] \end{cases}$$
$$\geq \rho t(1 - t), \quad t \in [0, 1]. \tag{3.33}$$

Using (3.32) and (3.33), we get

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)\, ds \geq \int_0^t \rho s(1 - s)\, ds = \rho \left(\frac{t^2}{2} - \frac{t^3}{3}\right), \quad t \in [0, 1].$$

Continuing the integration process, we obtain

$$y(t) \geq \rho \psi(t), \quad t \in [0, 1], \tag{3.34}$$

where

$$\psi(t) = \frac{t^{n-1}}{(n-1)!} - 2\frac{t^n}{n!}.$$

We note that

$$\psi'(t) = \frac{t^{n-2}}{(n-2)!}\left(1 - \frac{2t}{n-1}\right)$$

is nonnegative for $t \in I \equiv \left[0, \frac{n-1}{2}\right]$. Hence in particular $\psi(t)$ is nondecreasing for $t \in \left[\frac{1}{4}, \frac{1}{2}\right] \subseteq I$. It follows from (3.34) that

$$y(t) \geq c\rho, \quad t \in \left[\frac{1}{4}, \frac{1}{2}\right], \tag{3.35}$$

where

$$c = \psi\left(\frac{1}{4}\right) = \frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!} > 0. \tag{3.36}$$

Now, relation (3.32) provides

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)\, ds \leq \int_0^t \rho\, ds = \rho t, \quad t \in [0, 1].$$

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Using the above inequality and (3.32) again leads to

$$y(t) \leq \rho \frac{t^{n-2}}{(n-2)!} \leq \frac{\rho}{(n-2)!}, \quad t \in [0, 1]. \tag{3.37}$$

In view of (3.37), (3.2) and (3.35), we find

$$\begin{aligned} \frac{\rho}{(n-2)!} \geq y\left(\frac{1}{2}\right) &\geq \lambda \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq \lambda \int_{\frac{1}{4}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq \lambda f(c\rho) \int_{\frac{1}{4}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) u(s)\, ds \end{aligned}$$

from which (3.31) follows immediately.

Case 2 $\quad y^{(n-2)}(1) > 0$

In this case, $y^{(n-2)}(0) = 0$, $y^{(n-2)}(1) \neq 0$. Hence, by the concavity of $y^{(n-2)}$, we have

$$y^{(n-2)}(t) \geq y^{(n-2)}(1)t \geq y^{(n-2)}(1)t(1-t), \quad t \in [0, 1]. \tag{3.38}$$

Using a similar technique to that of Case 1, it follows from (3.38) and successive integrations that

$$y(t) \geq y^{(n-2)}(1)\psi(t), \quad t \in [0, 1]. \tag{3.39}$$

This leads to (3.35), where

$$c = \frac{y^{(n-2)}(1)}{\rho} \left[\frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!} \right] > 0. \tag{3.40}$$

The rest of the proof is similar to that of Case 1.

This completes the proof of the theorem.

THEOREM 3.5. *Let*

$$F_B = \left\{ f \,\middle|\, \frac{u}{f(u)} \text{ is bounded for } u \in [0, \infty) \right\},$$

$$F_0 = \left\{ f \,\middle|\, \lim_{u \to \infty} \frac{u}{f(u)} = 0 \right\}, \qquad F_\infty = \left\{ f \,\middle|\, \lim_{u \to \infty} \frac{u}{f(u)} = \infty \right\}.$$

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(a) *If* $f \in F_B$, *then* $E = (0, c)$ *or* $(0, c]$ *for some* $c \in (0, \infty)$.
(b) *If* $f \in F_0$, *then* $E = (0, c]$ *for some* $c \in (0, \infty)$.
(c) *If* $f \in F_\infty$, *then* $E = (0, \infty)$.

PROOF. (a) This is immediate from (3.30) as well as from (3.31).

(b) Since $F_0 \subseteq F_B$, it follows from case (a) that $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$. In particular,

$$c = \sup E. \tag{3.41}$$

Let $\{\lambda_m\}_{m=1}^\infty$ be a monotonically increasing sequence in E which converges to c, and let $\{y_m\}_{m=1}^\infty$ in C be a corresponding sequence of eigenfunctions. Further, let $\eta_m = \|y_m\|$. Then, (3.30) implies that no subsequence of $\{\eta_m\}_{m=1}^\infty$ can diverge to infinity. Thus, there exists $M > 0$ such that $\eta_m \le M$ for all m. So y_m is uniformly bounded. Hence, there is a subsequence of $\{y_m\}_{m=1}^\infty$, relabelled as the original sequence, which converges uniformly to some $y \in C$. Noting that $\lambda_m S y_m = y_m$, we have

$$cSy_m = \frac{c}{\lambda_m} y_m. \tag{3.42}$$

Since $\{cSy_m\}_{m=1}^\infty$ is relatively compact, y_m converges to y and λ_m converges to c, letting $m \to \infty$ in (3.42) gives $cSy = y$, that is, $c \in E$. This completes the proof for Case (b).

(c) This follows from Corollary 3.1 and (3.29).

EXAMPLE 3.1. Consider the boundary-value problem

$$y^{(4)} + \lambda \frac{1}{(5 + 2t^3 - t^4)^r}(12y + 5)^r = 0, \quad t \in (0, 1),$$
$$y(0) = y'(0) = y''(0) = y^{(p)}(1) = 0,$$

where $0 \le p \le 3$ but fixed, $\lambda > 0$ and $r \ge 0$.

Taking $f(y) = (12y + 5)^r$, we find

$$\frac{H(t, y)}{f(y)} = \frac{1}{(5 + 2t^3 - t^4)^r} \quad \text{and} \quad \frac{K(t, y)}{f(y)} = 0.$$

Hence, we may take

$$h_1(t) = \frac{2}{(5 + 2t^3 - t^4)^r}, \quad h(t) = \frac{1}{2(5 + 2t^3 - t^4)^r}$$

and $k(t) = k_1(t) = 0$. All the hypotheses (H_1)–(H_4) are satisfied.

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Case 1 $0 \le r < 1$

Since $f \in F_\infty$, by Theorem 3.5(c) the set $E = (0, \infty)$. For example when $p = \lambda = 2$, the boundary-value problem has a positive solution given by $y(t) = t^3(2-t)/12$.

Case 2 $r = 1$

Since $f \in F_B$, by Theorem 3.5(a) the set E is an open or a half-closed interval. Further, we note from Case 1 and Theorem 3.2 that when $p = 2$, E contains the interval $(0, 2]$.

Case 3 $r > 1$

Since $f \in F_0$, by Theorem 3.5(b) the set E is a half-closed interval. Again, it is noted that when $p = 2$, $(0, 2] \subseteq E$.

EXAMPLE 3.2. Consider the boundary-value problem

$$y'' + \lambda \frac{\sin \pi t}{(8 + 5\sin \pi t)^r}(5y+8)^r = 0, \quad t \in (0, 1),$$
$$y(0) = y^{(p)}(1) = 0,$$

where $p = 0$ or 1 (but fixed), $\lambda > 0$ and $r \ge 0$.

Choosing $f(y) = (5y+8)^r$, we may take

$$h_1(t) = \frac{3\sin \pi t}{(8 + 5\sin \pi t)^r}, \quad h(t) = \frac{\sin \pi t}{4(8 + 5\sin \pi t)^r},$$

and $k(t) = k_1(t) = 0$. All the hypotheses (H_1)–(H_4) are satisfied and we note that when $p = 0$ and $\lambda = \pi^2$, the boundary-value problem has a positive solution given by $y(t) = \sin \pi t$. With obvious modification, the three cases considered in Example 3.1 also apply here.

4. Special case: $\lambda = 1$

THEOREM 4.1. *Suppose that f is either superlinear or sublinear. Then the boundary-value problem* (1.1), (1.2) *has a positive solution.*

PROOF. To obtain a positive solution of (1.1) (1.2), we shall seek a fixed point of the operator S (defined in (3.1)) in the cone C. We have seen that S is compact on the cone C. Further, we observe from the proof of Theorem 3.1 that S maps C into itself. Also, the standard arguments yield that S is completely continuous.

Case 1 Suppose that f is superlinear. Since $f_0 = 0$, we may choose $\epsilon, \delta > 0$ such that

$$f(u) \le \epsilon u, \quad 0 < u \le \delta \tag{4.1}$$

and

$$\frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s)\, ds \le 1. \tag{4.2}$$

Let $y \in C$ be such that $\|y\| = \delta$. Then, applying (3.2), (4.1), Lemma 2.1 and (4.2) successively, we find for $t \in [0, 1]$,

$$\begin{aligned} Sy(t) &\le \int_0^1 G(t,s) v(s) f(y(s))\, ds \\ &\le \epsilon \int_0^1 G(t,s) v(s) y(s)\, ds \\ &\le \frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) y(s)\, ds \\ &\le \frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) \|y\|\, ds \le \|y\|. \end{aligned}$$

Hence

$$\|Sy\| \le \|y\|. \tag{4.3}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < \delta\}$, then (4.3) holds for $y \in C \cap \partial\Omega_1$.

Next, since $f_\infty = \infty$, we may choose $M, N > 0$ such that

$$f(u) \ge Mu, \quad u \ge N \tag{4.4}$$

and

$$M\gamma \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s)\, ds \ge 1. \tag{4.5}$$

Let $y \in C$ be such that $\|y\| = N_1 \equiv \max\left\{2\delta, \frac{N}{\gamma}\right\}$. Thus for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$,

$$y(t) \ge \gamma \|y\| \ge \gamma \cdot \frac{N}{\gamma} = N,$$

which in view of (4.4) leads to

$$f(y(t)) \geq My(t), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{4.6}$$

Using (3.2), (4.6) and (4.5), we find

$$\begin{aligned} Sy\left(\frac{1}{2}\right) &\geq \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq M \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) y(s)\, ds \\ &\geq M \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) \gamma \|y\|\, ds \geq \|y\|. \end{aligned}$$

Therefore

$$\|Sy\| \geq \|y\|. \tag{4.7}$$

If we set $\Omega_2 = \{y \in B \mid \|y\| < N_1\}$, then (4.7) holds for $y \in C \cap \partial\Omega_2$.

In view of (4.3) and (4.7), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$, such that $\delta \leq \|y\| \leq N_1$. This y is a positive solution of (1.1), (1.2).

Case 2 Suppose that f is sublinear. Since $f_0 = \infty$, there exist $L, \xi > 0$ such that

$$f(u) \geq Lu, \quad 0 < u \leq \xi \tag{4.8}$$

and

$$L\gamma \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s)\, ds \geq 1. \tag{4.9}$$

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Let $y \in C$ be such that $\|y\| = \xi$. On using (3.2), (4.8) and (4.9) successively, we get

$$\begin{aligned} Sy\left(\frac{1}{2}\right) &\geq \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq L \int_0^1 G\left(\frac{1}{2}, s\right) u(s) y(s)\, ds \\ &\geq L \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) y(s)\, ds \\ &\geq L \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) \gamma \|y\|\, ds \geq \|y\| \end{aligned}$$

from which (4.7) follows immediately. If we set $\Omega_1 = \{y \in B \mid \|y\| < \xi\}$, then (4.7) holds for $y \in C \cap \partial\Omega_1$.

Next, in view of $f_\infty = 0$, we may choose $J, \theta > 0$ such that

$$f(u) \leq \theta u, \quad u \geq J \tag{4.10}$$

and

$$\frac{\theta}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s)\, ds \leq 1. \tag{4.11}$$

Let $J_1 = \max\{2\xi, J\}$. Since f is nondecreasing, $f(u) \leq f(J_1)$ for $0 < u \leq J_1$. In view of (4.10), this implies that

$$f(u) \leq \theta J_1, \quad 0 < u \leq J_1. \tag{4.12}$$

Let $y \in C$ be such that $\|y\| = J_1$. Then it follows from (4.12) that

$$f(y(t)) \leq \theta J_1, \quad t \in [0, 1]. \tag{4.13}$$

On using (3.2), (4.13), Lemma 2.1 and (4.11) successively, we get for $t \in [0, 1]$ that

$$\begin{aligned} Sy(t) &\leq \int_0^1 G(t, s) v(s) f(y(s))\, ds \\ &\leq \theta J_1 \int_0^1 G(t, s) v(s)\, ds \\ &\leq \frac{\theta J_1}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s)\, ds \\ &\leq J_1 = \|y\| \end{aligned}$$

from which (4.3) follows immediately. If we set $\Omega_2 = \{y \in B \mid \|y\| < J_1\}$, then (4.3) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (4.7) and (4.3), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$, such that $\xi \le \|y\| \le J_1$. This y is a positive solution of (1.1), (1.2).

The proof of the theorem is complete.

The following two examples illustrate Theorem 4.1.

EXAMPLE 4.1. Consider the boundary-value problem

$$y^{(3)} + \frac{\pi^3 \sin \pi t}{(5 - 4\cos \pi t)^r}(4y + 1)^r = 0, \qquad t \in (0, 1),$$
$$y(0) = y'(0) = y^{(p)}(1) = 0,$$

where $0 \le p \le 2$ but fixed and $0 \le r < 1$.

Taking $f(y) = (4y + 1)^r$ (which is sublinear), we find that

$$\frac{H(t, y)}{f(y)} = \frac{\pi^3 \sin \pi t}{(5 - 4\cos \pi t)^r} \quad \text{and} \quad \frac{K(t, y)}{f(y)} = 0.$$

Hence we may choose

$$h_1(t) = \frac{\pi^4 \sin \pi t}{(5 - 4\cos \pi t)^r}, \qquad h(t) = \frac{\pi^2 \sin \pi t}{(5 - 4\cos \pi t)^r}$$

and $k(t) = k_1(t) = 0$. All the conditions of Theorem 4.1 are fulfilled and therefore the boundary-value problem has a positive solution. We note that when $p = 1$, one such solution is given by $y(t) = 1 - \cos \pi t$.

EXAMPLE 4.2. Consider the boundary-value problem

$$y'' + \frac{1}{(3 + 2t - t^2)^r}(2y + 3)^r = 0, \quad t \in (0, 1),$$
$$y(0) = y^{(p)}(1) = 0,$$

where $p = 0$ or 1 (but fixed) and $0 \le r < 1$.

Choosing $f(y) = (2y + 3)^r$ (which is sublinear), we may take

$$h_1(t) = \frac{5}{(3 + 2t - t^2)^r}, \qquad h_1(t) = \frac{1}{3(3 + 2t - t^2)^r}$$

and $k(t) = k_1(t) = 0$. Again, all the conditions of Theorem 4.1 are satisfied and so the boundary-value problem has a positive solution. Indeed, when $p = 1$, one such solution is given by $y(t) = t(2 - t)/2$.

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