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SEMIGROUPS OF COMPOSITION OPERATORS IN BERGMAN SPACES

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Semigroups consisting of composition operators are considered on weighted Bergman spaces. They are strongly continuous, and their infinitesimal generators are identified. One specific semigroup is used to obtain information on an averaging operator.

1. Introduction.

Recall that for $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of the functions f analytic on the unit disk $\mathbb{D}$ such that

(1.1) $$\|f\|^p_{p,\alpha} = \frac{1}{\pi}\int_0^1 M_p(r,f)(1-r^2)^\alpha\, rdr < \infty ,$$

where $M_p(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta$. If $1 \le p < \infty$ then A^p_α is a Banach space, if $p = 2$ a Hilbert space. If $\phi: \mathbb{D} \to \mathbb{D}$ is analytic, it is well-known that the composition operator $T_\phi(f) = f \circ \phi$ is bounded on A^p_α. Suppose $\{\phi_t: t \ge 0\}$ is a one-parameter semigroup of analytic functions mapping $\mathbb{D}$ into itself (that is $\phi_t \circ \phi_s = \phi_{t+s}$ for $t, s \ge 0$, $\phi_0(z) \equiv z$, and $\phi_t(z)$ is continuous in (t,z) on $[0,\infty) \times \mathbb{D}$).

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The purpose of this paper is to study the semigroups of composition operators $\{T_t : t \geq 0\}$ induced by $\{\phi_t\}$ on A^p_α by

(1.2) $$T_t(f) = f \circ \phi_t , \quad f \in A^p_\alpha , \ t \geq 0 .$$

Such semigroups were studied on Hardy spaces by Berkson and Porta [1] and by the author in [14]. Here we will show that $\{T_t\}$ is continuous in the strong but not in the uniform operator topology of A^p_α. We shall also identify the corresponding infinitesimal generator $\Gamma_{p,\alpha}$ and its point spectrum. Further, an averaging operator on A^p_α related to such semigroups will be considered.

2. Preliminaries.

If $\{\phi_t\}$ is a semigroup of analytic functions on D, the limit $G(z) = \lim_{t\to 0} \frac{\partial\phi_t(z)}{\partial t}$ gives the infinitesimal generator of $\{\phi_t\}$. $G(z)$ has the unique representation $G(z) = F(z)(\bar{b}z-1)(z-b)$ where $|b| \leq 1$ and $F(z)$ is analytic with $\mathrm{Re} F \geq 0$ on D. The distinguished point b in the representation of G is called the Denjoy-Wolff point of $\{\phi_t\}$ (DW point). Except for the case when $\{\phi_t\}$ consists of elliptic Möbius transformations of D, we have $\lim_{t\to\infty} \phi_t(z) = b$ for each $z \in D$. If $|b| < 1$ then b is a common fixed point for ϕ_t, $t > 0$ [1]. To each semigroup $\{\phi_t\}$ there corresponds a unique analytic univalent function $h: D \to C$ with $h(0) = 0$, $h'(0) = 1$ such that

(2.1) $$h(\phi_t(z)) = h(z) + G(0)t, \quad z \in D, \ t \geq 0$$

if the DW point b of $\{\phi_t\}$ has modulus one; and

(2.2) $$(h \circ \gamma_b)(\phi_t(z)) = e^{ct}(h \circ \gamma_b)(z), \quad z \in D, \ t \geq 0$$

where $c = G'(b)$ and $\gamma_b(z) = (z-b)/(1-\bar{b}z)$ if $|b| < 1$ ([3][14]). The function h in (2.1) is given by

(2.3) $$h(z) = \int_0^z \frac{G(0)}{G(\zeta)}\, d\zeta , \quad z \in D$$

while the function $h \circ \gamma_b$ in (2.2) is defined as the unique solution

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$y(z)$ of

(2.4) $$y'(z)G(z) - G'(b)y(z) = 0, \quad y'(b) = 1/(1-|b|^2) .$$

The trivial semigroup, $\phi_t(z) \equiv z$ for each $t \geq 0$, corresponds to the generator $G \equiv 0$.

3. Strong continuity.

We shall need an estimate, given in the next lemma, on the norm of a composition operator on A^p_α. The case $p=2$ can be found in [2] or [5]. A similar result for Hardy spaces was studied in [11] and [12].

LEMMA 1. *Let $\phi: D \to D$ be analytic and nonconstant. Then the norm of the operator $T_\phi: f \to f \circ \phi$ on A^p_α satisfies:*

(3.1) $$\|T_\phi\| \leq C \left(\frac{\|\phi\|_\infty + |\phi(0)|}{\|\phi\|_\infty - |\phi(0)|} \right)^{\frac{\alpha+2}{p}}$$

where $C \equiv 1$ if $\alpha \geq 0$ and $C = (\|\phi\|_\infty + |\phi(0)|)^{\alpha/p}(\|\phi\|_\infty + 3|\phi(0)|)^{-\alpha/p}$ if $-1 < \alpha < 0$.

Proof. We will use a harmonic majorant technique of Nordgren. Put $c = |\phi(0)|$, $d = \|\phi\|_\infty$ and fix $0 < r < 1$. By a standard application of the Schwarz-Pick Lemma on the map $\phi_1 = d^{-1}\phi$ we have $|\phi(z)| \leq (dc+d^2r)/(d+cr)$ for $|z| \leq r$. But $(c+dr)/(d+dr) \leq ((d-c)r + 2c)/(d+c)$ for all $0 < r < 1$, so $|\phi(z)| \leq dR \leq R$ where $R = R(r) = ((d-c)r + 2c)/(d+c)$. If $f \in A^p_\alpha$ let $u(z)$ be the harmonic extension of $|f(Re^{i\theta})|^p$ on $|z| \leq dR$. $u(z)$ is continuous on $|z| \leq dR$ and majorises $|f(z)|^p$ there, so $|f(\phi(z))|^p \leq u(\phi(z))$ for $|z| \leq r$. It follows that

(3.2) $$M_p(r,f\circ\phi) = \int_0^{2\pi} |f(\phi(re^{i\theta}))|^p \, d\theta \leq \int_0^{2\pi} (U\circ\phi)(re^{i\theta})d\theta = 2\pi u(\phi(0)) .$$

By Harnack's inequality we have

(3.3) $$u(\phi(0)) \leq \frac{dR+c}{dR-c}\, u(0) = \frac{dR+c}{dR-c}\, \frac{1}{2\pi} \int_0^{2\pi} |f(dRe^{i\theta})|^p \, d\theta .$$

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Combining (3.2) and (3.3) and observing that $M_p(r,f)$ is increasing in r we obtain

$$(3.4)\qquad M_p(r,\ f\circ\phi) \le \frac{dR+c}{dR-c} M_p(R,f)\ .$$

Now multiply both sides of (3.4) by $(1-r^2)^\alpha r$ and integrate with respect to r from 0 to 1. The change of variable $u = ((d-c)r+2c)/(d+c)$ in the second integral gives:

$$(3.5)\qquad \int_0^1 \frac{dR+c}{dR-c} M_p(R,f)(1-r^2)^\alpha\ r\ dr =$$

$$= \left(\frac{d+c}{(d-c)^2}\right)^{\alpha+1} \int_{2c/d+c}^{1} \frac{ud+c}{ud-c}\ \frac{u(d+c)-2c}{u} \left(\frac{u(d+c)+d-3c}{1+u}\right)^\alpha (1-u^2)^\alpha\ M_p(u,f)u\ du\ .$$

The quantity $q(u) = (ud+c)(u(d+c)-2c)/(ud-c)u$ inside the integral has derivative $q'(u) = -2c^2(u^2d - 2ud+c)/(u^2d-uc)^2$ and the zeros of $q'(u)$ are $1 \pm \sqrt{1-c/d}$. Since the interval $[2c/d+c,\ 1]$ is contained in the interval with endpoints $1 \pm \sqrt{1-c/d}$, $q'(u)$ does not change sign over $[2c/d+c,\ 1]$. Since $q'(1)$ is positive, $q(u)$ increases on $[2c/d+c,\ 1]$ and is bounded above by $q(1) = d+c$. Also, by calculus,

$$\frac{(d+c)(d-c)}{d+3c} \le \frac{u(d+c)\ +\ d-3c}{1+u} \le d-c$$

for $u \in [2c/d+c,\ 1]$. From these and (3.4), (3.5) we obtain

$$\int_0^1 (1-r^2)^\alpha\ M_p(r,\ f\circ\phi)\ rdr \le \left(\frac{d+c}{d-c}\right)^{\alpha+2} \int_0^1 (1-u^2)^\alpha\ M_p(u,f)\ udu$$

for $\alpha \ge 0$, while

$$\int_0^1 (1-r^2)^\alpha\ M_p(r,\ f\circ\phi)\ rdr \le \left(\frac{d+c}{d+3c}\right)^\alpha \left(\frac{d+c}{d-c}\right)^{\alpha+2} \int_0^1 (1-u^2)^\alpha\ M_p(u,f)\ udu$$

for $-1 < \alpha < 0$. The conclusion follows.

THEOREM 1. *Suppose* $1 \le p < \infty$, $-1 < \alpha < \infty$ *and* $\{\phi_t\}$ *is a given semigroup with generator* G. *Then*

(i) The induced semigroup $\{T_t\}$ *defined in (1.2) is strongly continuous on* A^p_α.

(ii) The infinitesimal generator $\Gamma_{p,\alpha}$ *of* $\{T_t\}$ *has domain*

$\mathcal{D}(\Gamma_{p,\alpha}) = \{f \in A^p_\alpha : Gf' \in A^p_\alpha\}$ *and* $\Gamma_{p,\alpha}(f) = Gf'$ *for each* $f \in \mathcal{D}(\Gamma_{p,\alpha})$.

(iii) If $\{T_t\}$ *is continuous in the uniform operator topology then* $\{\phi_t\}$ *is trivial.*

Proof. (i) Let $f \in A^p_\alpha$, $f_\rho(z) = f(\rho z)$ for $0 < \rho < 1$ and $\varepsilon > 0$. An argument similar to the one employed in [7, Theorem 3(ii)] shows that $\|f_\rho - f\|_{p,\alpha} \to 0$ as $\rho \to 1$. Thus $\|f_\rho - f\|_{p,\alpha} < \varepsilon/2$ for ρ sufficinetly near 1. The power series of f_ρ converges to f_ρ uniformly on the closed disk $\overline{D}$ so there is a polynomial P such that $\|f_\rho - P\|_{p,\alpha} < \varepsilon/2$. It follows that the polynomials are dense in A^p_α. For the strong continuity we need to show that for every $f \in A^p_\alpha$, $\lim_{t\to 0} \|T_t(f) - f\|_{p,\alpha} = 0$. From Lemma 1 we see that $\{\|T_t\|\}$ is uniformly bounded on every bounded interval of t. The triangle inequality and the fact that polynomials are dense in A^p_α show that it suffices to prove $\lim_{t\to 0} \|T_t(P) - P\|_{p,\alpha} = 0$ for each polynomial P. Equivalently we will show that for each $n \geq 0$ $\lim_{t\to 0} \|\phi^n_t - \chi_n\|_{p,\alpha} = 0$ where $\chi_n(z) = z^n$. Fix $0 < r < 1$, then ϕ^n_t converges uniformly to χ_n as $t \to 0$ on the circle $|z| = r$. Thus $\lim_{t\to 0} M_p(r, \phi^n_t - \chi_n) = 0$. Now $(1-r^2)^\alpha r\, M_p(r, \phi^n_t - \chi_n) \leq 2^{p+1}\pi(1-r^2)^\alpha r$ for each $t \geq 0$ and $r \in (0,1)$. Since $\alpha > -1$ the right hand side is integrable on $(0,1)$ and we can apply the Lebesgue dominated convergence theorem to obtain $\lim_{t\to 0} \|\phi^n_t - \chi_n\|_{p,\alpha} = 0$.

(ii) The proof of this part is similar to the one in [1, Theorem 3.7] so we will only provide an outline. Let $\mathcal{D}_1 = \{f \in A^p_\alpha : Gf' \in A^p_\alpha\}$ and Γ_1 be the operator defined on $\mathcal{D}_1$ by $\Gamma_1(f) = Gf'$. The pointwise limit $\lim_{t\to 0}((f \circ \phi_t)(z) - f(z))/t$ exists for each $z \in \mathbb{D}$ and equals $G(z)\, f'(z)$.

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Since convergence in A^p_α implies in particular pointwise convergence, it follows that $\mathcal{D}(\Gamma_{p,\alpha}) \subseteq \mathcal{D}_1$ and Γ_1 extends $\Gamma_{p,\alpha}$. The proof can be completed by showing that there exists a $\lambda > 0$ such that $\lambda - \Gamma_{p,\alpha}$ is onto A^p_α while $\lambda - \Gamma_1$ is one-to-one. This then implies that $\Gamma_1 = \Gamma_{p,\alpha}$. The details for the existence of such a λ are as in [1, Theorem 3.7].

(iii) If $\{T_t\}$ is continuous in the uniform operator topology then $\Gamma_{p,\alpha}$ is bounded on A^p_α. For $n \geq 0$ let χ_n be as in part (ii). Then $\Gamma_{p,\alpha}(\chi_n) = nG\,\chi_{n-1}$, and taking $n=1$ we see that $G \in A^p_\alpha$. For $n \geq 1$ we have $n\,\|G\,\chi_{n-1}\|_{p,\alpha} \leq \|\Gamma_{p,\alpha}\|\;\|\chi_n\|_{p,\alpha}$, that is

$$(3.6)\quad n^p \int_0^1 (1-r^2)^\alpha r^{(n-1)p+1} M_p(r,G)\,dr \leq 2\pi \|\Gamma_{p,\alpha}\|^p \int_0^1 (1-r^2)^\alpha r^{np+1}\,dr\,.$$

Let $\delta \in (0,1)$ be such that $\int_0^\delta (1-r^2)^\alpha\,dr = \frac{1}{2}\int_0^1 (1-r^2)^\alpha\,dr$. Since $M_p(r,G)$ is increasing in r we have

$$\begin{aligned} n^p M_p(\delta,G) \int_\delta^1 (1-r^2)^\alpha r^{(n-1)p+1}\,dr &\leq n^p \int_\delta^1 (1-r^2)^\alpha r^{(n-1)p+1} M_p(r,G)\,dr \\ (3.7)\qquad &\leq n^p \int_0^1 (1-r^2)^\alpha r^{(n-1)p+1} M_p(r,G)\,dr \\ &= n^p\,\pi\,\|G\,\chi_{n-1}\|^p_{p,\alpha}\,. \end{aligned}$$

Also $\int_0^\delta (1-r^2)^\alpha r^{np+1}\,dr \leq \int_\delta^1 (1-r^2)^\alpha r^{np+1}\,dr$ so

$$\int_0^1 (1-r^2)^\alpha r^{np+1}\,dr \leq 2\int_\delta^1 (1-r^2)^\alpha r^{np+1}\,dr \leq 2\int_\delta^1 (1-r^2)^\alpha r^{(n-1)p+1}\,dr\,,$$

that is,

$$(3.8)\qquad \|\chi_n\|_{p,\alpha} \leq 4\int_\delta^1 (1-r^2)^\alpha r^{(n-1)p+1}\,dr\,.$$

Compining (3.6), (3.7) and (3.8) we obtain $n^p\,M(\delta,G) \leq 4\pi\,\|\Gamma_{p,\alpha}\|^p$ for each $n = 1,2,\ldots$. It follows that $M(\delta,G) = 0$ so $G \equiv 0$.

Denote by $P\sigma(\Gamma_{p,\alpha})$ the point spectrum of $\Gamma_{p,\alpha}$.

THEOREM 2. *Suppose* $1 \leq p < \infty$, $-1 < \alpha < \infty$ *and let* $\{\phi_t\}$ *be a semigroup with generator* G, *associated univalent function* h, *and* DW

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point b.

(i) *If* $|b| < 1$ *then* $P\sigma(\Gamma_{p,\alpha}) \subseteq \{kG'(b): k = 0,1,2, \ldots\}$. *The point* $kG'(b)$ *is in* $P\sigma(\Gamma_{p,\alpha})$ *if and only if* $h(z)^k \in A^p_\alpha$.

(ii) *If* $|b| = 1$ *then* $P\sigma(\Gamma_{p,\alpha}) = \{\lambda G(0): \exp(\lambda h(z)) \in A^p_\alpha\}$.

Proof. (i) From (2.4), $G = G'(b)(h \circ \gamma_b)/(h \circ \gamma_b)'$. Suppose f is analytic, $f \not\equiv 0$ and $\lambda \in \mathbb{C}$ are such that $\Gamma_{p,\alpha}(f) = \lambda f$. Then $f'/f = \lambda(h \circ \gamma_b)'/(G'(b)(h \circ \gamma_b))$. Integrating f'/f over $|z| = r$ where $|b| < r < 1$ is chosen so that f has no zeros on $|z| = r$, and applying the argument principle, we find that $\lambda/G'(b) = k$, a nonnegative integer, so $\lambda = kG'(b)$. The nonzero analytic solutions of $[(h \circ \gamma_b)/(h \circ \gamma_b)']f' = kf$ are $f(z) = c(h \circ \gamma_b)^k(z)$, $c \neq 0$. The conclusion follows by observing that $(h \circ \gamma_b)^k \in A^p_\alpha$ if and only if $h^k \in A^p_\alpha$.

(ii) In this case $G = G(0)/h'$. If $f(z) = \exp(\lambda h(z)) \in A^p_\alpha$ then $\Gamma_{p,\alpha}(f) = \lambda G(0)f$ so $\lambda G(0) \in P\sigma(\Gamma_{p,\alpha})$. Conversely if $\lambda G(0) \in P\sigma(\Gamma_{p,\alpha})$ is an eigenvalue and $f \not\equiv 0$ an eigenvector corresponding to $\lambda G(0)$, then $f'(z) = \lambda h'(z)f(z)$. It follows that $f(z) = c\exp(\lambda h(z))$, $c \neq 0$, and the conclusion follows.

4. An averaging operator on A^p_α.

We will now consider a specific semigroup, namely

$$\phi_t(z) = e^{-t}z + 1-e^{-t}, \quad z \in \mathbb{D},\ t \geq 0, \tag{4.1}$$

and will obtain information about the operator A defined on A^p by

$$A(f)(z) = \frac{1}{z-1}\int_1^z f(\zeta)\, d\zeta. \tag{4.2}$$

It is necessary to consider (4.2) formally until made precise later. The operator A has been studied on Hardy spaces in connection with the Cesàro operator ([4][13]). It has also been studied on other spaces of analytic functions including the Bergman space $A^2 = A^2_0$ ([8][10]).

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Let $\{T_t\}$ be the semigroup of composition operators induced by $\{\phi_t\}$ as in (4.1). The generator of $\{\phi_t\}$ is $G(z) = \frac{\partial(e^{-t}z + 1-e^{-t})}{\partial t}\Big|_{t=0} = -z+1$, so the infinitesimal generator of $\{T_t\}$ is given by

$$\Gamma_{p,\alpha}(f)(z) = (1-z)\, f'(z)\ . \tag{4.3}$$

LEMMA 2. *The spectrum of* $\Gamma_{p,\alpha}$ *is* $\sigma(\Gamma_{p,\alpha}) = \{z: \text{Re} z \le \frac{\alpha+2}{p}\}$.

Proof. From Lemma 1 we have

$$\|T_t\| \le C(t)(2e^t-1)^{\frac{\alpha+2}{p}}$$

where $C(t) = 1$ if $\alpha \ge 0$ and $C(t) = ((2-e^{-t})/(4-3e^{-t}))^{\frac{\alpha}{p}}$ if $-1 < \alpha < 0$. In any case $C(t)$ is bounded by a constant independent of t. Thus the type ω_0 of $\{T_t\}$ satisfies

$$\omega_0 = \lim_{t\to\infty} \frac{\log \|T_t\|}{t} \le \frac{\alpha+2}{p}\ .$$

It follows that $\sigma(\Gamma_{p,\alpha}) \subseteq \{z: \text{Re} z \le (\alpha+2)/p\}$ (see [6, Theorem VIII.1.11]). On the other hand if $\text{Re}\lambda < (\alpha+2)/p$ then $g_\lambda(z) = (1-z)^\lambda \in A^p_\alpha$ and we have $\Gamma_{p,\alpha}(g_\lambda) = \lambda g_\lambda$ so λ is an eigenvalue of $\Gamma_{p,\alpha}$. This together with the fact that $\sigma(\Gamma_{p,\alpha})$ is a closed set finish the proof.

LEMMA 3. *If* $\alpha+2 < p$ *and* $g \in A^p_\alpha$ *then for any* $z \in D$ *the integral* $\int_1^z g(\zeta)\, d\zeta$ *is finite.*

Proof. If $g \in A^p_\alpha$ then the estimate $|g(z)| \le C(p,\alpha)\, \|g\|_{p,\alpha} (1-|z|)^{-(\alpha+2)/p}$ holds. The path of integration is taken for simplicity to be the segment from 1 to z. If ζ is a point on this segment then there is a constant k such that $1 - |\zeta| \ge k|1-\zeta|$ as $\zeta \to 1$. It follows that $|g(\zeta)| \le C(p,\alpha)\, k^{-1}\, \|g\|_{p,\alpha}\, |1-\zeta|^{-(\alpha+2/p}$, and so for $\alpha+2 < p$ $|g(\zeta)|$ is integrable on the segment from 1 to z, completing the proof.

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Consider now the semigroup $\{S_t\}$ where $S_t = e^{-t} T_t$, $t \geq 0$. The generator of $\{S_t\}$ is given by

$$(4.4) \qquad \Delta_{p,\alpha}(f)(z) = (1-z)f'(z) - f(z) ,$$

and by virtue of Lemma 2, we have

$$(4.5) \qquad \sigma(\Delta_{p,\alpha}) = \{z\colon \operatorname{Re} z \leq -1 + \frac{\alpha+2}{p}\} .$$

Assume now that $\alpha+2 < p$. Then $0 \notin \sigma(\Delta_{p,\alpha})$ so the resolvent operator $R(0, \Delta_{p,\alpha}) = (-\Delta_{p,\alpha})^{-1}$ is bounded on A^p_α. As a consequence of Lemma 3 the operator A in (4.2) is well defined in this case, and an easy computation shows that

$$(4.6) \qquad R(0, \Delta_{p,\alpha}) = A \ (\alpha+2 < p) .$$

Applying the spectral mapping theorem we find from (4.6) and (4.5)

$$(4.7) \qquad \sigma(A) = \{z\colon \left| z - \frac{p}{2(p-2-\alpha)} \right| \leq \frac{p}{2(p-2-\alpha)}\}$$

and that each point in the interior of $\sigma(A)$ is an eigenvalue.

On the other hand if $\alpha+2 \geq p$ then $0 \in \sigma(\Delta_{p,\alpha})$. In this case, for $g \in A^p_\alpha$, the differential equation $(1-z)y'(z) - y(z) = g(z)$ either does not have analytic solution of the form (4.2) or if such a solution exists, it is not necessarily in A^p_α. We have proved the following:

THEOREM 3. *The operator* A *given by (4.2) is bounded on* A^p_α *if and only if* $\alpha+2 < p$. *In this case the spectrum* $\sigma(A)$ *is given by (4.7). Each point in the interior of* $\sigma(A)$ *is an eigenvalue.*

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