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The lower radical construction for non-associative rings: Examples and counterexamples

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Some sufficient conditions are presented for the lower radical construction in a variety of algebras to terminate at the step corresponding to the first infinite ordinal. An example is also presented, in a variety satisfying some non-trivial identities, of a lower radical construction terminating in four steps.

Introduction

Two of the most important questions in radical theory are

(1) Are semi-simple classes hereditary?

and

(2) How many steps are needed in the (Kurosh) lower radical construction?

Affirmative answers to (1) have been obtained for associative rings [*1*], alternative rings [*1*], and groups [*6*], while in these settings it has been shown that the lower radical construction stops at the ωth step, where ω is the first infinite ordinal [*13*], [*5*], [*11*]. On the other hand, for arbitrary rings, semi-simple classes are rarely hereditary (see [*3*] for an indication of just how rarely), while the lower radical construction need never stop [*9*].

Having hereditary semi-simple classes and never requiring more than ω

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steps in the lower radical construction are "nice" properties for a universal class. The meagre evidence set out above might lead one to conjecture that these two properties go together. Some contrary evidence will be adduced in §2.

In recent years there have been a number of investigations of hereditary semi-simple classes with various varieties of rings or algebras used as universal classes; see, for example, [8], [12], [3], [2]. In the last of these, two conditions on a variety of algebras over a field (conditions satisfied by the class of associative rings) were studied, which jointly imply that semi-simple classes are hereditary. In §1 of this paper we look at the termination of the lower radical construction from a similar point of view.

In §2 we present an example of a lower radical construction which terminates at the fourth step. The universal class is the class of (non-associative) rings R for which there is a finite series

$$I \triangleleft J \triangleleft R$$

with associative factors, and the class on which the construction is carried out is the class of all fields (necessarily associative).

Throughout this paper we shall deal with algebras (not necessarily associative) over a commutative, associative ring Ω with identity, sometimes specializing to the case where Ω is the ring Z of integers. The symbol $\triangleleft$ indicates an ideal. A *zeroalgebra* is one satisfying the identity $xy = 0$. A *universal class* of Ω-algebras (a setting for radical theory) is a hereditary, homomorphically closed class; in this paper, all universal classes will be varieties.

We finally recall the Kurosh Lower Radical Construction (see [6] and [13]). Let M be a non-empty homomorphically closed class of Ω-algebras (in some universal class W). Let $M_1 = M$;

$$M_\beta = \{A \mid \text{every non-zero homomorphic image of } A \text{ has a non-zero ideal in } M_\alpha \text{ for some } \alpha < \beta\} ,$$

where β is an ordinal greater than 1 . Then $L(M)$, the lower radical class generated by M , is $\bigcup_\beta M_\beta$.

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1. Some universal classes with ω-step lower radical constructions

In this section we shall consider some examples of lower radical constructions which terminate in ω steps. Our universal classes will be varieties $\mathcal{W}$ of Ω-algebras satisfying various combinations of properties (V1)-(V4) listed below.

For any algebra A, we define

$$A^{(0)} = A,\ A^{(1)} = AA,\ \ldots,\ A^{(n+1)} = A^{(n)}A^{(n)},\ \ldots,\ A^{(\alpha+1)} = A^{(\alpha)}A^{(\alpha)}$$

and $A^{(\beta)} = \bigcap_{\alpha<\beta} A^{(\alpha)}$ if β is a limit ordinal. A is *solvable* if $A^{(n)} = 0$ for some finite n.

A^n, as usual, is the linear span of all products of length n of elements of A. If $I \triangleleft J \triangleleft A$, we denote by I^* the ideal of A generated by I.

The conditions with which we shall be concerned are the following:

(V1) if $I \triangleleft J \triangleleft A \in \mathcal{W}$, then $(I^*/I)^{(\alpha)} = 0$ for some ordinal α;

(V2) if $I \triangleleft J \triangleleft A \in \mathcal{W}$, then every non-zero homomorphic image of I^*/I has a non-zero nilpotent ideal;

(V3) if $I \triangleleft J \triangleleft A \in \mathcal{W}$, then $(I^*/I)^{(n)} = 0$ for some finite n;

(V4) if $I \triangleleft A \in \mathcal{W}$, then $I^s \triangleleft A$ for some integer $s > 1$.

Varieties satisfying (V3) (respectively (V4)) are called *Andrunakievich varieties* (respectively *s-varieties*). In [2] it was shown that the n and s in (V3), (V4) can be chosen independently of the algebra A. Zwier [14] initiated the study of s-varieties; they include the varieties of associative algebras (a 2-variety) alternative algebras (a 2-variety) and Jordan algebras (a 3-variety). In [2] characterizations of semi-simple classes were obtained for $\mathcal{W}$ satisfying (V3) and (V4) when Ω is a field. Less complete results were obtained for more general Ω. This pattern repeats itself here: life is much simpler with algebras over a field because there is "essentially one" zeroalgebra.

The following result appears in [10]; we include a proof for completeness.

LEMMA 1.1. *Let* M *be a homomorphically closed class. If* $I \lhd A$, $I \in M_k$, *and* $A/I \in M_l$, *then* $A \in M_t$, *where* $t = \max\{k+1, l\}$.

Proof. Let $A/K \neq 0$. If $I \not\subseteq K$, then we have

$$0 \neq I/I \cap K \cong (I+K)/K \lhd A/K$$

with $I/I \cap K \in M_k$, while if $I \subseteq K$, then $A/K \cong (A/I)/(K/I) \in M_l$. Thus A/K certainly has a non-zero ideal in some M_r with $r < t$, so that $A \in M_t$. //

The next two propositions provide most of the information needed for the principal results of this section.

PROPOSITION 1.2. *Let* W *be a variety of* Ω*-algebras satisfying* (V2) *and* (V4). *Let* M *be a non-void homomorphically closed subclass of* W *containing all zeroalgebras in* W. *If* $I \lhd J \lhd A \in W$ *and* $I \in M$, *then* $I^* \in M_{s+1}$.

Proof. Consider a non-zero homomorphic image I^*/K of I^*. If $I \not\subseteq K$, then

$$I/I \cap K \cong (I+K)/K \lhd I^*/K ,$$

and $I/I \cap K \in M = M_1$. If $I \subseteq K$, then I^*/K, as a homomorphic image of I^*/I, has a non-zero nilpotent ideal X (by (V2)). By (V4), we can define a sequence $X_{(0)}, X_{(1)}, X_{(2)}, \ldots$ of ideals of I^*/K by

$$X_{(0)} = X,\ X_{(1)} = X^s,\ X_{(2)} = X^s_{(1)},\ \ldots .$$

Since X is nilpotent, there is an m such that $X_{(m)} = 0 \neq X_{(m-1)}$. Let $X_{(m-1)} = L/K$. Then $0 \neq L/K \lhd I^*/K$ and $(L/K)^s = 0$. Thus $(L/K)^{(s)} \subseteq (L/K)^{2^s} = 0$ and we have a series

$$\begin{array}{ccccccccc} 0 = & (L/K)^{(s)} & \lhd & (L/K)^{(s-1)} & \lhd & \cdots \lhd & (L/K)^{(1)} & \lhd & (L/K) \\ & \| & & \| & & & \| & & \\ & (L/K)^{(s-1)}(L/K)^{(s-1)} & & (L/K)^{(s-2)}(L/K)^{(s-2)} & & & (L/K)(L/K) & & \end{array}$$

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with zeroalgebra factors. By Lemma 1.2,

$$(L/K)^{(s-1)},\ (L/K)^{(s-2)}/(L/K)^{(s-1)} \in M_1 ,$$

so $(L/K)^{(s-2)} \in M_2$. Similarly

$$(L/K)^{(s-3)} \in M_3,\ \ldots,\ L/K = (L/K)^{(s-s)} \in M_s .$$

We have now shown that every non-zero homomorphic image of I^* has a non-zero ideal in M_k for some $k < s+1$; that is that $I^* \in M_{s+1}$. //

When Ω is a field we get a stronger statement.

COROLLARY 1.3. *Let* W *be a variety of algebras over a field satisfying* (V2) *and* (V4). *Let* M *be a non-void homomorphically closed subclass of* W *containing the one-dimensional zeroalgebra. If* $I \lhd J \lhd A \in W$ *and* $I \in M$, *then* $I^* \in M_{s+2}$.

Proof. Let B be a zeroalgebra. Then every non-zero homomorphic image of B has an ideal which is a one-dimensional zeroalgebra. Thus $B \in M_2$, so M_2 satisfies the hypotheses of Proposition 1.3. If now $I \in M$ and $I \lhd J \lhd A$, then $I \in M_2$, so $I^* \in (M_2)_{s+1} = M_{s+2}$. //

PROPOSITION 1.4. *Let* W *be a variety of* Ω*-algebras satisfying* (V1) *and* (V4), M *a homomorphically closed subclass of* W *containing no zero-algebras (not equal to* 0 *). If* $I \lhd J \lhd A \in W$ *and* $I \in M$, *then* $I \lhd A$.

Proof. Since I/I^2 is a zeroalgebra in M , I must be idempotent. The proof now runs like that of Proposition 3.5 in [2]. //

Combining Corollary 1.3 with the field case of Proposition 1.4, we get

PROPOSITION 1.5. *Let* Ω *be a field,* W *a variety of* Ω*-algebras satisfying* (V1), (V2), *and* (V4). *If* $I \lhd J \lhd A \in W$ *and* $I \in M$, *then* $I^* \in M_{s+2}$.

To see this we need only observe that a homomorphically closed class of zeroalgebras in this case either contains a one-dimensional zeroalgebra or contains no zeroalgebras. //

Let A be a non-zero solvable algebra in a variety $\mathcal{W}$ satisfying (V3) and (V4). Then $A^s \subseteq A^2$, that is, in the notation used above, $A_{(1)} \subseteq A^{(1)}$. By induction, $A_{(n)} \subseteq A^{(n)}$ for each n. But $A^{(n)} = 0$ eventually, so, for some n, we have

$$0 \neq A_{(n-1)} \triangleleft A \ ; \quad \left(A_{(n-1)}\right)^s = A_{(n)} = 0 \ .$$

Thus A has a non-zero nilpotent ideal. It follows that $\mathcal{W}$ satisfies (V2) (and, of course, (V1)). Combining these observations with the preceding results, we get

PROPOSITION 1.6. *Let $\mathcal{W}$ be a variety of Ω-algebras satisfying* (V3) *and* (V4). *Let M be a homomorphically closed subclass of $\mathcal{W}$ containing all zeroalgebras or none. If $I \triangleleft J \triangleleft A \in \mathcal{W}$ and $I \in \mathsf{M}$, then $I^* \in \mathsf{M}_{s+1}$.* //

COROLLARY 1.7. *Let $\mathcal{W}$ be a variety of algebras over a field, satisfying* (V3) *and* (V4). *If M is a homomorphically closed subclass of $\mathcal{W}$, $I \triangleleft J \triangleleft A$ and $I \in \mathsf{M}$, then $I^* \in \mathsf{M}_{s+2}$.* //

For a non-void homomorphically closed subclass M of a variety $\mathcal{W}$, let

$Y(\mathsf{M}) = \{A \in \mathcal{W} \mid$ every non-zero homomorphic image of A has a non-zero accessible subalgebra in $\mathsf{M}\}$.

Then $Y(\mathsf{M})$ is a radical class [7] and hence $L(\mathsf{M}) \subseteq Y(\mathsf{M})$.

We can now prove our termination theorem.

THEOREM 1.8. *Let $\mathcal{W}$ be a variety of Ω-algebras, M a non-void homomorphically closed subclass of $\mathcal{W}$. Then $L(\mathsf{M}) = \mathsf{M}_\omega$ if one of the following sets of conditions holds:*

(i) $\mathcal{W}$ *satisfies* (V2) *and* (V4) *and* M *contains all zeroalgebras;*

(ii) $\mathcal{W}$ *satisfies* (V1) *and* (V4) *and* M *contains no zeroalgebras;*

(iii) $\mathcal{W}$ *satisfies* (V3) *and* (V4) *and* M *contains all zeroalgebras or none;*

(iv) $\mathcal{W}$ *satisfies* (V2) *and* (V4), M *contains a zeroalgebra, and* Ω *is a field;*

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(v) W *satisfies* (V1), (V2), *and* (V4), *and* Ω *is a field;*

(vi) W *satisfies* (V3) *and* (V4), *and* Ω *is a field.*

In (ii) and in (iii), (iv), and (v), where M *consists of idempotent algebras, we have* $L(M) = M_2$.

Proof. Reference to the earlier results of this section establishes that in *(i)-(vi)*, if $I \lhd J \lhd A \in W$ and $I \in M$, then $I^* \in M_{s+2}$. If $R \in L(M)$, then $R \in Y(M)$; so for every non-zero R/L we have a finite series

$$0 \neq M_n \lhd M_{n-1} \lhd \ldots \lhd M_2 \lhd M_1 = R/L$$

with $M_n \in M$. Consider $M_n \lhd M_{n-1} \lhd M_{n-2}$. We have $M_n^* \in M_{s+2}$. Arguing by induction, we show that R/L has a non-zero ideal in M_k for some finite k . Thus $R \in M_\omega$.

This last assertion of the theorem follows from Proposition 1.4. //

As noted above, the varieties of associative, alternative, and Jordan algebras satisfy (V4). By Andrunakievich's Lemma $\left((I^*/I)^3 = 0\right)$, the associative algebras satisfy (V1), (V2), (V3). Results in [4] show that the alternative algebras satisfy (V1) and (V2). If $\frac{1}{2} \in \Omega$, then the Jordan algebras satisfy (V2). To see this we use the following result.

THEOREM 1.9 (Slin'ko [12], Theorem 1). *If* A *is a Jordan algebra over a ring* Ω *containing* $\frac{1}{2}$ *, if* $I \lhd J \lhd A$ *, and if* J/I *has no nilpotent ideals, then* $I \lhd A$.

PROPOSITION 1.10. *The variety of Jordan algebras over a ring* Ω *containing* $\frac{1}{2}$ *satisfies* (V2).

Proof. Let $I \lhd J \lhd A$. Any homomorphic image of I^*/I is isomorphic to I^*/K for some $K \lhd I^*$ with $I \subseteq K$. If I^*/K has no nilpotent ideals, then by Theorem 1.9, $K \lhd A$. But then (since $I \subseteq K$) $I^* = K$. //

Varieties satisfying (V3) and (V4) are discussed in [2].

The main drawback, of course, in the approach we have used in this section, is that "complete" results can be stated only for algebras over a field (see [2]). If some way can be found to take account of the additive structure of algebras, the method may produce new and simpler proofs of known results (for example, for alternative rings) as well as new information.

2. A 4-step termination

It is not known whether, in the universal class of associative rings, there exists a class M with $L(M) = M_k \neq M_{k-1}$ for $3 < k < \omega$; in fact no example appears to have been given in any universal class of a class with this property. We present such an example here. Let A denote the variety of associative rings ($\mathbb{Z}$-algebras). Recall that for varieties V, W, $V \circ W$ is the variety of algebras which are extensions of V-algebras by W-algebras.

THEOREM 2.1. *Let* F *be the class of (associative) fields. Then in* $(\mathsf{A} \circ \mathsf{A}) \circ \mathsf{A}$ *, we have*

$$L(\mathsf{F}) = \mathsf{F}_4 \neq \mathsf{F}_3 .$$

Proof. Let K be a finite prime field and let A be the K-algebra with basis $\{e, f, g\}$ and multiplication table

	e	f	g
e	e	0	f
f	0	f	0
g	0	e	g

Let $0 \neq I \lhd A$. If $ae + bf + cg \in I$, then

$$bf = (bf+ce)f = \big((ae+bf+cg)f\big)f \in I .$$

Hence $ae + cg \in I$, so

$$cg = g(ae+cg) \in I ,$$

and thus $ae \in I$. If $c \neq 0$, then (since K is a finite prime field) $g \in I$. But then $e = gf \in I$ and $f = eg \in I$, so $I = A$. Thus, if $I \neq A$, we have $I \subseteq \langle e, f\rangle$, the subspace spanned by e and f .

If $ae + bf \in I$, then as above, ae and $bf \in I$. If $a \neq 0$, then $e \in I$, so $f = eg \in I$. Similarly $f, e \in I$ if $b \neq 0$. Since $I \neq \{0\}$

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we have $e, f \in I$ and so $I = \langle e, f\rangle$. Now $\langle e, f\rangle \cong K \oplus K$ and $A/\langle e, f\rangle \cong K$, so $A \in \mathrm{A} \circ \mathrm{A}$.

Let A' be an isomorphic copy of A , with basis $\{e', f', g'\}$ and multiplication table

	e'	f'	g'
e'	e'	0	f'
f'	0	f'	0
g'	0	e'	g' .

We put A and A' together in an algebra B with basis $\{e, f, g, e', f', g', h\}$ and multiplication table

	e	f	g	e'	f'	g'	h
e	e	0	f	0	0	0	e'
f	0	f	0	0	0	0	f'
g	0	e	g	0	0	0	g'
e'	0	0	0	e'	0	f'	g'
f'	0	0	0	0	f'	0	g'
g'	0	0	0	0	e'	g'	0
h	g	g	0	e	f	g	h .

Let I be a non-zero ideal of B and let

$$\alpha = ae + bf + cg + a'e' + b'f' + c'g' + dh$$

be in I . Multiplying on the right by g' , we see that I contains

$$aeg' + bfg' + cgg' + a'e'g' + b'f'g' + c'g'g' + dhg' = a'f' + c'g' + dg \ .$$

Then multiplying on the left by g' , we get

(1) $$a'g'f' + c'g'g' + dg'g = a'e' + c'g' \in I \ .$$

Hence $a'e' = (a'e'+c'g')e' \in I$, so by (1),

(2) $$c'g' \in I \ .$$

Multiplying α on the left by g' , we get

$$b'e' + c'g' = ag'e + bg'f + cg'g + a'g'e' + b'g'f' + c'g'g' + dg'h \in I \ .$$

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Then (2) implies that $b'e' \in I$, so $b'f' = b'e'g' \in I$.

We have thus far shown that $a'e', b'f', c'g' \in I$. A similar argument shows that $ae, bf, cg \in I$, whence also $dh \in I$.

If $d \neq 0$, then $h \in I$, and then (as can be seen from the table, $e = he'$ and so on), $e, f, g, e', f', g' \in I$. Thus if $I \neq B$, then $I \in \langle e, f, g, e', f', g' \rangle = A \oplus A'$. Since $I \neq 0$, the argument used above shows that I contains at least one of e, f, g, e', f', g'.

If $e \in I$, then $f = eg \in I$, $g = he \in I$, $e' = eh \in I$, $f' = e'g' \in I$, $g' = e'h \in I$, so $I = A \oplus A'$. If $f \in I$, then $e = gf \in I$, so $I = A \oplus A'$. If $g \in I$, then $e = gf \in I$, so again $I = A \oplus A'$. Similarly $I = A \oplus A'$ if e', f', or $g' \in I$.

The ideals of B are thus $0, A \oplus A'$, and B. As noted before, the ideals of A are 0, $J = \langle e, f \rangle$, and A, so the ideals of A' are 0, $J' = \langle e', f' \rangle$, and A'. Since $B/A \oplus A' \cong K$, we have $B \in (\mathsf{A} \circ \mathsf{A}) \circ \mathsf{A}$.

Now $J, J' \cong K \oplus K \in \mathsf{F}_2 \backslash \mathsf{F}_1$. Since $A/J \cong K \in \mathsf{F}_1$, we have $A \in \mathsf{F}_3$ by Lemma 1.1. Similarly $A' \in \mathsf{F}_3$. In fact, A and $A' \in \mathsf{F}_3 \backslash \mathsf{F}_2$, since they have no simple ideals.

Consider $A \oplus A'$. We have $J, J' \triangleleft A \oplus A'$. Let $A \oplus A'/L \neq 0$. If $J \nsubseteq L$, then

$$0 \neq J/(J \cap L) \cong (J+L)/L \triangleleft A \oplus A'/L ,$$

and $J/(J \cap L) \in \mathsf{F}_2$. Similarly $A \oplus A'/L$ has a non-zero ideal in F_2 if $J' \nsubseteq L$. If L contains both J and J', then $A \oplus A'/L$ is a homomorphic image of $(A \oplus A')/(J \oplus J') \cong K \oplus K \in \mathsf{F}_2$. Thus $A \oplus A' \in \mathsf{F}_3$.

Since $B/A \oplus A' \cong K \in \mathsf{F}_1$, Lemma 1.1 says that $B \in \mathsf{F}_4$. Now $A \oplus A' \notin \mathsf{F}_2$ (since $A, A' \notin \mathsf{F}_2$), and $B \notin \mathsf{F}_2$ (since its non-zero ideals are B and $A \oplus A'$, and neither is simple). Thus B has no non-zero ideals in F_2, whence $B \notin \mathsf{F}_3$ and thus $B \in \mathsf{F}_4 \backslash \mathsf{F}_3$.

We complete the proof by showing that $L(\mathsf{F}) = \mathsf{F}_4$. Let R be in $L(\mathsf{F})$. Then R has an ideal L such that $L \in \mathsf{A} \circ \mathsf{A}$ and $R/L \in \mathsf{A}$, and thus L has an ideal M such that $M \in \mathsf{A}$ and $L/M \in \mathsf{A}$. Now F is

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hereditary, so $L, M \in L(F)$, and also L/M and $R/L \in L(F)$. Now every non-zero homomorphic image of M has a non-zero ideal in F, and since M is associative, this means that $M \in F_2$. In the same way $L/M \in F_2$, so by Lemma 1.1, $L \in F_3$. Since also $R/L \in F_2$, a further application of Lemma 1.1 shows that $R \in F_4$. //

We note that there is some similarity between our construction and that used by Ryabukhin [9] to show that the lower radical construction in the class of all rings need not terminate at all. Our ring B differs, however, from the fourth ring in Ryabukhin's transfinite sequence of rings.

The argument leading up to Theorem 1.8 closely parallels those used in the proof that $L(M) = M_\omega$ for any class M of associative rings (Sulinski, Anderson and Divinsky [13]), alternative rings (Krempa [5]), and also groups (Ščukin [11]). It essentially involves showing

(i) that if T is an accessible subring of R, then the ideal $\overline{T}$ of R generated by T is not "radically" different from T, and

(ii) that then $Y(M) \subseteq L(M)$ (and thus $Y(M) = L(M)$).

The construction of $L(F)$ (in $(A \circ A) \circ A$), which we have just discussed, does not conform to this pattern, even though it terminates after a finite number of steps.

THEOREM 2.2. *In* $(A \circ A) \circ A$, $L(F) \neq Y(F)$, *and* $L(F)$ *does not have a hereditary semi-simple class.*

Proof. It is shown in [3] (Theorem 2.10 (iii)) that a hereditary radical class R of the form $Y(C)$ (called in [3] a *hereditary homomorphic orthogonality radical class*), in $A \circ A$, satisfies the condition

(3) $$R \in R \Leftrightarrow R^{+0} \in R$$

where R^{+0} is the zeroring on the additive group of R. Condition (3) is satisfied also by $(A \circ A) \circ A$ (the proof is quite similar). Since $L(F)$ is hereditary and does not satisfy (3), it can not coincide with $Y(F)$.

The second assertion of the theorem can be obtained from Theorem 2.10

(i) of [3] in the same way. //

Thus - referring back to our questions in the Introduction - a lower radical construction over a "quite small" class can terminate in a finite number of steps and yet produce a radical class with a non-hereditary semi-simple class. How quickly a lower radical construction must terminate when the associated semi-simple class is hereditary remains unknown.

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