ON THE NON-VANISHING OF POINCARÉ SERIES

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(Received 18th August 1987)

R. A. Rankin [2] and J. Lehner [1] considered the non-vanishing of Poincaré series for the classical modular matrix group and for an arbitrary fuchsian group, respectively.

In this paper we consider the non-vanishing of Poincaré series for the congruence group

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}; \quad N \geq 1.$$

For \(k > 2, k \equiv 0 \pmod{2}\), let \(\mathcal{M}_k^0(\Gamma)\) be the space of cusp forms for \(\Gamma\) of weight \(k\). Let \(\mu_k\) be the dimension of \(\mathcal{M}_k^0(\Gamma)\). Let

$$P_m(z, k) = \sum_{\gamma \in \Gamma \backslash \Gamma} (j(\gamma, z))^{-k} e(myz),$$

where

$$j(\gamma, z) = cz + d \quad \text{if} \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

and

$$e(z) = e^{2\pi i z}$$

be the Poincaré series of weight \(k\) attached to \(\Gamma\). The space \(\mathcal{M}_k^0(\Gamma)\) is spanned by \(P_m(z, k)\). Since \(\mathcal{M}_k^0(\Gamma)\) is finite dimensional, there must be many linear relations between \(P_m(z, k)\). Very little is known about these relations. In particular one does not know which \(P_m(z, k)\) do not vanish identically.

In the case of full modular group \(\Gamma = \Gamma_0(1)\), when \(k = 4, 6, 8, 10\) and \(14\), \(\mathcal{M}_k^0(\Gamma)\) has dimension zero; so that \(P_m(z, k)\) vanishes identically for all positive integers \(m\). We have \(\mu_k > 0\) for \(k = 12\) and all \(k \geq 16\). Indeed by Theorem 6.1.2 in [3] we have for \(k \geq 4\),

$$\mu_k = \begin{cases} \left[ \frac{k}{12} \right] & \text{if } k \not\equiv 2 \pmod{12}, \\
\left[ \frac{k}{12} \right] - 1 & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$
Clearly, since $P_m(z, k)$ (for $1 \leq m \leq \mu_k$) span the space $\mathcal{M}_k^0(\Gamma)$, we have $P_m(z, k) \not\equiv 0$ for $1 \leq m \leq \mu_k$. Rankin [2] was able to show that many more Poincaré series do not vanish.

In this paper we extend the arguments of Rankin to establish:

**Theorem 1.** For $\Gamma = \Gamma_0(N); N \geq 1$, we have $P_m(z, k) \equiv 0$ if

$$m(m, N)\alpha^2(m) \leq \frac{1}{2^{15} \pi^5} \left( \frac{N}{\tau(N) \log 2N} \right)^2$$

where

$$\alpha(m) = \sum_{d|m} \frac{\tau(d)}{\sqrt{d}}$$

and $\tau(N)$ is the number of positive divisors of $N$.

**Remarks.** Stripped of factors of lower order, Theorem 1 states essentially that

$$m(m, N) \leq K(N/\log N)^2,$$

where $K$ is an explicitly defined numerical constant and $K < 1$. Thus, for small values of $N$, where the right hand side is less than 1, this tells us nothing. Even for large $N$ it is vacuous in some cases, e.g., when $N$ divides $m$, as it then gives $m/N < K(\log N)^{-2}$. However, in other cases it will give information. For example, whenever

$$N/\log N > K^{-1/2}$$

it tells us that, for all $k > 2$, the first Poincaré series does not vanish.

Note also that, unlike the results of Rankin and Lehner, the upper bound does not depend on the weight $k$. However, in Theorem 2 and Theorem 3, which follow, the upper bound does depend on the weight $k$.

Let $S(m, m; c)$ be the Kloosterman sum defined

$$S(m, m; c) = \sum_{d \equiv 1(\mod c)}^* e \left( m \frac{d + \bar{d}}{c} \right); d\bar{d} \equiv 1(\mod c).$$

Let $J_{k-1}(y)$ be the Bessel function of order $k - 1$.

**Lemma 1.** (A. Weil cf. [4]). We have

$$|S(m, m; c)| \leq (m, c)^{1/2} c^{1/2} \tau(c).$$

**Lemma 2.** (cf. [5]). We have
Proof (of Theorem 1).

By the argument presented in Section 2 of [2], and in Chapter 5 of [3], we have

\[ P_m(z, k) \equiv 0 \quad \text{if} \quad |S_m| < \frac{1}{2\pi} \]

where

\[ S_m = \sum_{r=1}^{\infty} (rN)^{-1} S(m, m; rN) J_{k-1} \left( \frac{4\pi m}{rN} \right). \]

Clearly, by Lemma 1 and Lemma 2, we have

\[ |S_m| \leq \frac{(m, N)^{1/2} \tau(N)}{N^{1/2}} \sum_{r=1}^{\infty} \frac{(m, r)^{1/2} \tau(r)}{r^{1/2}} \min \left\{ 1, \frac{2\pi m}{rN} \right\} \]

\[ \leq \frac{(m, N)^{1/2} \tau(N)}{N^{1/2}} \sum_{d|m} d \sum_{r=1}^{\infty} \frac{\tau(r)}{r^{1/2}} \min \left\{ 1, \frac{2\pi m}{r dN} \right\} \]

\[ \leq \frac{(m, N)^{1/2} \tau(N)}{N^{1/2}} \sum_{d|m} \frac{1}{d} \sum_{r_1}^{\infty} \sum_{r_2}^{\infty} \frac{1}{(r_1 r_2)^{1/2}} \min \left\{ 1, \frac{2\pi m}{r_1 r_2 dN} \right\}. \]

Let

\[ R = \left( \frac{2\pi m}{dN} \right). \]

Then

\[ \sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{1}{(r_1 r_2)^{1/2}} \min \left\{ 1, \frac{R}{r_1 r_2} \right\} = 2 \sum_{r=1}^{\infty} \frac{1}{r^{1/2}} \min \left\{ 1, \frac{R}{r} \right\} + \sum_{r_1=2}^{\infty} \sum_{r_2=2}^{\infty} \frac{1}{(r_1 r_2)^{1/2}} \min \left\{ 1, \frac{R}{r_1 r_2} \right\} \]

\[ = 2S_1 + S_2. \]

Case I: \( R > 1. \)

\[ S_1 \leq 1 + \int_{1}^{R} t^{-1/2} \, dt + \int_{R}^{\infty} t^{-3/2} \, dt; \text{so that} \]

\[ 1 + \int_{1}^{R} t^{-1/2} \, dt + \int_{R}^{\infty} t^{-3/2} \, dt. \]
$S_1 \leq 4R^{1/2} - 1 \leq 4R^{1/2}(1 + \log(R + 1))$.

$$S_2 \leq \int_1^R \left( \int_1^{R/t_1} (t_1 t_2)^{-1/2} dt_2 \right) dt_1 + R \int_1^R \left( \int_1^R (t_1 t_2)^{-3/2} dt_2 \right) dt_1$$

$$+ R \int_1^\infty \left( \int_1^R (t_1 t_2)^{-3/2} dt_2 \right) dt_1; \text{ so that}$$

$$S_2 \leq 4R^{1/2}(1 + \log(R + 1)).$$

**Case II:** $0 < R \leq 1$.

$$S_1 \leq R + R \int_1^\infty t^{-3/2} dt; \text{ so that}$$

$$S_1 \leq R + 2R \leq 3R^{1/2}(1 + \log(R + 1)); \text{ since } R \leq 1.$$ 

$$S_2 \leq \int_1^\infty \left( \int_1^R \frac{R}{(t_1 t_2)^{3/2}} dt_2 \right) dt_1; \text{ so that}$$

$$S_2 \leq 4R \leq 4R^{1/2}(1 + \log(R + 1)); \text{ since } R \leq 1.$$ 

By combining both cases with the earlier calculations, the proof is completed. \(\square\)

**Theorem 2.** Let $\Gamma = \Gamma_0(N); N \geq 1$. There exist positive constants $k_0$ and $B$ (both independent of $N$), where $B > 4 \log 2$ such that, for all $k \geq k_0$ and all positive integers $m$ such that

$$k \leq m \leq k^2 \exp(-B \log k/\log \log k),$$

$P_m(z, k) \neq 0$.

**Proof.** Let $Q^* = (4\pi m/vN)$.

$$|S_m| \leq \sum_{1 \leq q < Q^*} \left| \frac{S(m, m; qN)}{qN} \right| J_{k-1} \left( \frac{4\pi m}{qN} \right) + \sum_{q \equiv Q^*} \left| \frac{S(m, m; qN)}{qN} \right| J_{k-1} \left( \frac{4\pi m}{qN} \right).$$

$$|S_m| \leq S'_m + S''_m.$$

Clearly,
where $Q$ is defined in [2]. Hence by exactly the same argument presented in [2], we have

$$S_m' \leq A_6M(m)\{\sigma^6m^{1/2}\sigma_{-1/2}(m) + (4\pi)^{1/2}\sigma^2\sigma_0(m)\}.$$  

Clearly, by the argument presented in [2],

$$S_m' \leq \sum_{q \equiv Q^*} J_v\left(\frac{vQ^*}{q}\right) \leq A_5 \sum_{q \equiv Q^*} f\left(\frac{Q^*}{q}\right) \leq A_5 \left\{ Q^* \int_0^1 x^{-2}F(x)\,dx + \sigma^2 \right\};$$

so that, since $Q^*N = Q$,

$$S_m' \leq A_5\sigma^2 + \frac{A_5\sigma_1^2}{N} + \frac{A_5\sigma_2^2}{N} \leq A_5\sigma^2 + A_5\sigma_1^2 + A_5\sigma_2^2.$$

Hence $|S_m| \leq A_6\sigma^2 M(m)\sigma_{-1/2}(m) + A_4\sigma^2 M_2(m) + A_5\sigma^2 + A_9\sigma_1^2 + A_9\sigma_2^1$, and the result follows by the argument presented in [2] with the observation that $A_5\sigma^2 = o(1)$.  

**Theorem 3.** For $\Gamma = \Gamma_0(N); N \geq 1$, we have $P_m(z, k) \neq 0$ if $k_0(N) \leq k$ and for any $\epsilon > 0$

$$m^{1+\epsilon}(m, N)\sigma^2(m) \ll \left(\frac{Nk}{\tau(N)}\right)^2.$$  

**Proof.** By Lemma 1 we have

$$S_m' \leq \frac{(m, N)^{1/2}\tau(N)}{N^{1/2}} \left( \sum_{1 \leq q \leq Q^*} \frac{(m, q)^{1/2}\tau(q)}{q^{1/2}} \right) \left| J_{k-1}\left(\frac{4\pi m}{qN}\right) \right|.$$  

Clearly,

$$S_m' \leq \frac{(m, N)^{1/2}\tau(N)}{N^{1/2}} \left( \sum_{d | m} \tau(d) \sum_{1 \leq r < (Q^*/d)} \frac{\tau(r)}{r^{1/2}} \right) \left| J_{k-1}\left(\frac{4\pi m}{rdN}\right) \right|$$

$$S_m' \ll \frac{(m, N)^{1/2}\tau(N)m^\epsilon}{N^{1/2}Q^*^{1/2}} \sum_{d | m} \tau(d) d^{1/2}S_d,$$  

where

$$S_d = \sum_{1 \leq r < (Q^*/d)} \left(\frac{Q^*}{rd}\right)^{1/2} \left| J_v\left(\frac{vQ^*}{rd}\right) \right|.$$
By the same argument presented in [2], we have

\[ S_d \ll \left( \frac{m \sigma^9}{Nd} + \sigma^2 \right). \]

Hence

\[ S'_m \ll \frac{(m, N)^{1/2} \tau(N)m^{1/2 + \epsilon}}{N(k-1)} \left( \sum_{d \mid m} \frac{\tau(d)}{d^{1/2}} \right) + (m, N)^{1/2} \tau(N)m^{1/2} \sum_{d \mid m} \tau(d) \left( \frac{d}{m} \right)^{1/2}. \]

But

\[ \sum_{d \mid m} \tau(d) \left( \frac{d}{m} \right)^{1/2} \ll \alpha(m), \]

and since \( d < Q^* \) we have

\[ \sum_{d \mid m} \tau(d) \left( \frac{d}{m} \right)^{1/2} \ll \sum_{d \mid m} \tau(d) \left( \frac{1}{(vN)^{1/2}} \right) \ll \frac{m^{2+e}}{(vN)^{1/2}}. \]

Hence

\[ S'_m \ll \frac{(m, N)^{1/2} \tau(N)m^{1/2 + \epsilon} \alpha(m)}{N(k-1)} + \frac{(m, N)^{1/2} \tau(N)m^{3e}(k-1)^{1/6}}{N^{1/2}(k-1)^{1/2}}. \]

By the argument given in the proof of Theorem 2, we have \( S'_m \ll A_5 \sigma^2 + A_5 m \sigma^{12} \). Hence

\[ S_m \ll \frac{(m, N)^{1/2} \tau(N)m^{1/2 + \epsilon} \alpha(m)}{N(k-1)} + \frac{(m, N)^{1/2} \tau(N)m^{3e}}{N^{1/2}(k-1)^{1/3}} + \frac{1}{(k-1)^{1/3}} + \frac{m}{(k-1)^2}, \]

and the result follows from the hypothesis; since the last three terms are sufficiently small for \( k > k_0(N) \). \qed

**Note added in proof.** All of the results in this paper are also true for the principal congruence groups \( \Gamma(N); N \geq 1 \).

**Acknowledgements.**

I would like to thank Professor Henryk Iwaniec for several helpful conversations concerning this problem, and I would like to express my appreciation to the referee for helpful suggestions concerning exposition.

This work was done during the summer of 1987, and I would like to thank the Institute for Advanced Study for providing me with excellent working conditions.
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