A nonlinear complementarity problem
in mathematical programming
in Hilbert space

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In this paper we prove the following existence and uniqueness theorem for the nonlinear complementarity problem by using the Banach contraction principle. If $T : K \to H$ is strongly monotone and lipschitzian with $k^2 < 2c < k^2 + 1$, then there is a unique $y \in K$, such that $Ty \in K^*$ and $(Ty, y) = 0$ where $H$ is a Hilbert space, $K$ is a closed convex cone in $H$, and $K^*$ the polar cone.

1. Introduction and statement of the theorem

Let $H$ be a real Hilbert space and let $K$ be a closed convex cone in $H$ with the vertex at 0. The polar of $K$ is the cone $K^*$, defined by $K^* = \{ y \in H : (x, y) \geq 0 \text{ for every } x \in K \}$.

A mapping $T : H \to H$ is said to be monotone on $K$ if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in K$ and strictly monotone if strict inequality holds whenever $x \neq y$. $T$ is called strongly monotone if there is a constant $c > 0$ such that $(Tx - Ty, x - y) \geq c\|x - y\|^2$. $T$ is said to be lipschitzian if there is a constant $k > 0$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in H$ whenever $x \neq y$, and a contraction if $0 < k < 1$.

The purpose of this note is to prove the following existence and uniqueness theorem for the nonlinear complementarity problem.

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233
THEOREM. Let \( T : K \to H \) be strongly monotone and lipschitzian with \( k^2 < 2c < k^2 + 1 \). Then there is a unique \( y_0 \) such that

\[ y_0 \in K , \ Ty_0 \in K^* , \text{ and } \langle Ty_0 , y_0 \rangle = 0 . \]

2. Proof of the theorem

Since \( K \) is a nonempty closed convex set in \( H \), for every \( y \in K \) there is a unique \( x \in K \) closest to \( y - Ty \); that is,

\[ \|x-y+Ty\| \leq \|z-y+Ty\| \]

for every \( z \in K \). Let the correspondence \( y \mapsto x \) be denoted by \( \theta \). Let \( z \) be any element of \( K \) and let \( 0 \leq \lambda \leq 1 \). Since \( K \) is convex, \( (1-\lambda)x + \lambda z \in K \). Define a function \( h : [0, 1] \to \mathbb{R}^+ \) by the rule

\[ h(\lambda) = \|y - Ty - (1-\lambda)x - \lambda z\|^2 . \]

Then \( h \) is a twice continuously differentiable function of \( \lambda \) and

\[ h'(\lambda) = 2\langle y - Ty - \lambda z - (1-\lambda)x , x - z \rangle . \]

Since \( x \) is the unique element closest to \( y - Ty \), we must have \( h'(0) \geq 0 \), and therefore

\[ (y - Ty - x , x - z) \geq 0 \]

for every \( z \in K \). Let \( y_1 \) and \( y_2 \) be two elements of \( K \) and \( y_1 \neq y_2 \). Let \( \theta(y_1) = x_1 \) and \( \theta(y_2) = x_2 \). Putting \( y = y_1 \) and \( z = \theta(y_2) \) in (2.1) we get

\[ (y_1 - Ty_1 - \theta(y_1) , \theta(y_1) - \theta(y_2)) \geq 0 . \]

Again, putting \( y = y_2 \) and \( z = \theta(y_1) \) in (2.1), we get

\[ (y_2 - Ty_2 - \theta(y_2) , \theta(y_2) - \theta(y_1)) \geq 0 . \]

From (2.2) and (2.3) we have

\[ (y_1 - Ty_1 - \theta(y_1) - y_2 + Ty_2 + \theta(y_2) , \theta(y_1) - \theta(y_2)) \geq 0 . \]

Hence
Therefore,

\[
\|\theta(y_1) - \theta(y_2)\|^2 \leq \|y_1 - Ty_1 + Ty_2, \theta(y_1) - \theta(y_2)\| \leq \|y_1 - Ty_1 + Ty_2\|\|\theta(y_1) - \theta(y_2)\|.
\]

Thus

\[
(2.4) \quad \|\theta(y_1) - \theta(y_2)\| \leq \|Ty_1 - Ty_2 + y_1 + y_2\|.
\]

Since \( T \) is strongly monotone and lipschitzian, it follows from the inequality (2.4) that

\[
\|\theta(y_1) - \theta(y_2)\|^2 \leq \|Ty_1 - Ty_2 + y_1 + y_2\|^2 = (Ty_1 - Ty_2 + y_1 + y_2, Ty_1 - Ty_2 + y_1 + y_2)
\]

\[
= \|Ty_1 - Ty_2\|^2 + \|y_1 - y_2\|^2 - 2(Ty_1 - Ty_2, y_1 - y_2)
\]

\[
\leq (k^2 + 1 - 2\alpha)\|y_1 - y_2\|^2.
\]

Since \( k^2 < 2\alpha < k^2 + 1 \), we have \( 0 < k^2 + 1 - 2\alpha < 1 \). Putting

\[
\alpha^2 = k^2 + 1 - 2\alpha \quad \text{in the above inequality we obtain}
\]

\[
\|\theta(y_1) - \theta(y_2)\| \leq \alpha\|y_1 - y_2\|
\]

where \( 0 < \alpha < 1 \). Thus \( \theta \) is a contraction. Now applying the Banach contraction principle (see, for example, [1]) we conclude that \( \theta \) has a unique fixed point, say \( y_0 \). Now putting \( y = y_0 \) in (2.1) we get

\[
(2.5) \quad \{Ty_0, z - y_0\} \geq 0
\]

for every \( z \in K \). Since \( 0 \in K \) we have from (2.5) that \( \{Ty_0, y_0\} \leq 0 \).

Again since \( K \) is a convex cone, \( 2y_0 \in K \) and therefore putting \( z = 2y_0 \) in (2.5) we get \( \{Ty_0, y_0\} \geq 0 \). Thus \( \{Ty_0, y_0\} = 0 \) and \( \{Ty_0, z\} \geq 0 \) for every \( z \in K \), showing that \( Ty_0 \in K^* \). Therefore \( y_0 \) is the unique solution to the complementarity problem (1.1) and this completes the proof.
Reference


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