ON TENSOR PRODUCT GRAPHS

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Abstract

The tensor product \( G \oplus H \) of graphs \( G \) and \( H \) is the graph with point set \( V(G) \times V(H) \) where \((u_1, v_1) \text{ adj } (u_2, v_2)\) if, and only if, \( u_1 \text{ adj } u_2 \) and \( v_1 \text{ adj } v_2 \). We obtain a characterization of graphs of the form \( G \oplus H \) where \( G \) or \( H \) is \( K_2 \).

1. Notation and preliminary results

As usual, let \( K_p \) denote the complete graph on \( p \) points, \( C_n \) a cycle of length \( n \). For a connected graph \( G \), \( nG \) is the graph with \( n \) components each being isomorphic to \( G \).

REMARK. The tensor product \( G \oplus H \) is also called conjunction (Harary (1969)), and Kronecker product (Weichsel 1963)).

Weichsel (1963) has proved Theorem 1 and Corollary 1.1.

THEOREM 1. (Weichsel). For connected graphs \( G \) and \( H \) the product \( G \oplus H \) is connected if and only if either \( G \) or \( H \) contains an odd cycle.

COROLLARY 1.1. (Weichsel). If \( G \) and \( H \) are connected graphs with no odd cycles then \( G \oplus H \) has exactly two components.

We now prove

COROLLARY 1.2. For a connected graph \( G \) with no odd cycles, \( G \oplus K_2 = 2G \).

PROOF. Let \( \{a_i\} \) be the point set of \( G \) and \( K_2 \) be the line \( b_1b_2 \). By Corollary 1.1, \( G \oplus K_2 \) has exactly two components, say \( G_1 \) and \( G_2 \). If for some point \( a_{i_0} \) in \( G \), \( (a_{i_0}, b_1) \in G_1 \) then \( (a_{i_0}, b_2) \in G_2 \). For, if there is a path \( (a_{i_0}, b_1)(a_{i_1}, b_2)\) then \( k \) is even and \( G \) has the odd cycle \( a_{i_0}a_{i_1}a_{i_2} \cdots a_{i_k}a_{i_0} \), provided the points \( a_{i_r}, r = 0, 1, \ldots, k \) are all distinct. Suppose \( a_{i_r} = a_{i_s} \) for some \( r \) and \( s \), and \( r < s \). Let \( s \) be the smallest such integer. Clearly, if \( r \) is even (odd) then \( s \) is odd (even) and \( G \) has an odd cycle \( a_{i_0}a_{i_{r+1}} \cdots a_{i_{s-1}}a_{i_r} \). Now, the function \( f: G \to G_1 \) defined by

268
f(a_i) = (a_i, b_1) if (a_i, b_1) ∈ G_1
   = (a_i, b_2) if (a_i, b_1) ∉ G_1

is clearly an isomorphism. Similarly, G ≅ G_2.

**Theorem 2.** For a graph G, G = H ⊗ K_2 if and only if the following conditions I–IV are true.

I. G has an even number of points and lines.
II. G has no odd cycles.
III. If G is connected, the following should hold.
   (a) G has a cycle, say
       \[ C_{2n} : x_1x_2 \cdots x_{2n}x_1 \] where n > 1 is odd.
   (b) Let G_1 be the graph obtained from G by removing all lines of C_{2n}. The components of G_1 should be of the following two types only.

   **Type I.** Components E such that for 1 ≤ r ≤ n the point x_r of C_{2n} belongs to E if and only if the point x_{n+r} belongs to a component E' (≠ E) isomorphic to E.

   **Type II.** Components F such that for 1 ≤ r ≤ n the point x_r of C_{2n} belongs to F if and only if x_{n+r} ∈ F. Further, F should satisfy I, II and III. That is, F should have an even number of points and lines, and should contain a cycle C_{2m} where m > 1 is odd, etc.

IV. Suppose G is disconnected. Then the components of G which are not of the form mentioned in III should be in isomorphic pairs.

**Proof.** Let G = H ⊗ K_2, V(H) = \{a_i\} and K_2 be the line b_1b_2. Clearly condition I is true. We observe that points of G are labeled alternately with the elements of the sets \(V_1 = \{a_i\} \times \{b_1\}\) and \(V_2 = \{a_i\} \times \{b_2\}\). This will not be possible if G has odd cycles, and thus II is true.

Let G be connected. Then by Theorem 1, H is connected and contains an odd cycle say, \(C_n = a_1a_2, \ldots, a_na_1\). This implies that G contains the cycle,

\[ C_{2n} = (a_1, b_1)(a_2, b_2)(a_3, b_1) \cdots (a_n, b_1)(a_1, b_2) \cdots (a_n, b_2)(a_1, b_1). \]

Consider the graph G_1 obtained from G by removing all lines of C_{2n}. In a relabeling of the points of C_{2n} as \(x_1x_2 \cdots x_{2n}x_1\) we find that if \((a_r, b_1)\) is labeled as \(x_r\) then \((a_r, b_2)\) is labeled as \(x_{n+r}\) (we make use of this fact in our proof without any further reference). Let E be a component of G_1 and \(x_r = (a_r, b_1)\) be a point of C_{2n} such that \(x_r \in E\) and \(x_{n+r} = (a_r, b_2)\) belongs to a different component E' of G_1. We now show that (i) a point \((a_r, b_1)\) (not necessarily on C_{2n}) belongs to E if and only if \((a_r, b_2)\) belongs to E', and (ii) E ≅ E'.

Let \((a_r, b_1) \in E\). Then in E there is a path

\((a_i, b_1)(a_i, b_2)(a_i, b_1) \cdots (a_{i_k}, b_2)(a_r, b_1).\)
This implies that there is a path 

\((a_i, b_2) (a_i, b_1) (a_i, b_2) \cdots (a_n, b_1) (a_r, b_2)\)

and hence \((a_i, b_2) \in E'\), since \((a_r, b_2) \in E'\). Likewise, \((a_j, b_1) \in E'\) implies \((a_j, b_2) \in E\).

Clearly, the function \(f: E \to E'\) defined by

\[ f(a_i, b_1) = (a_i, b_2) \] if \((a_i, b_1) \in E\)
\[ f(a_i, b_2) = (a_i, b_1) \] if \((a_i, b_2) \in E\)

is an isomorphism.

Suppose now \(F\) is a component of \(G_1\) and \(x_r = (a_r, b_1)\) is a point of \(C_{2n}\) such that both \(x_r\) and \(x_{n+r} = (a_r, b_2)\) belong to \(F\). We show that in general, a point \((a_i, b_1)\) (not necessarily on \(C_{2n}\)) belongs to \(F\) if and only if \((a_i, b_2) \in F\). Let \((a_i, b_1) \in F\). Then there is a path

\[(a_r, b_1)(a_i, b_2) \cdots (a_n, b_2)(a_1, b_1).\]

This implies that there is a path

\[(a_r, b_2)(a_i, b_1) \cdots (a_n, b_1)(a_i, b_2)\]

and since \((a_r, b_2) \in F\), \((a_i, b_2) \in F\). Likewise, \((a_j, b_2) \in F\) implies \((a_j, b_1) \in F\). It follows now that \(F = H_1 \oplus K_2\) where \(H_1\) is the subgraph of \(H\) induced by the points \(V(H_1) = \{a_i: (a_i, b_1) \in F\}\), and hence \(F\) should satisfy I, II and III. This proves that \(G_1\) can have only two types of components as mentioned in III. If \(G\) is disconnected, a component \(G_i\) of \(G\) which is not of the form mentioned in III corresponds to a component \(H_i\) of \(H\) which does not have an odd cycle and so \(H_1 \oplus K_2 = 2H_i = 2G_i\) by Corollary 1.2. Thus \(G\) should have such components in pairs and such components are even in number.

Conversely, let the conditions I–IV hold good for a graph \(G\) having \(2m\) points. Let \(A = \{a_1, a_2, \ldots, a_m\}\) and \(B = \{b_1, b_2\}\). We label the points of \(G\) with \(2m\) elements of \(A \times B\) such that

(i) no two elements of \(A \times \{b_1\}\) or \(A \times \{b_2\}\) are adjacent, and
(ii) the function \(f: G \to G\) defined by

\[ f(a_i, b_1) = (a_i, b_2) \]
\[ f(a_i, b_2) = (a_i, b_1) \]

is an isomorphism. The given conditions I–IV ensure that this is indeed possible. Suppose now \(G\) is connected. By hypothesis it contains a cycle \(C_{2n}\) where \(n\) is odd. We label the points of this cycle successively with the \(2n\) elements

\[(a_1, b_1)(a_2, b_2)(a_3, b_1) \cdots (a_n, b_1)(a_1, b_2) \cdots (a_n, b_2).\]

The above isomorphism takes the cycle into itself and the subgraph \(G_1\) described
in III into itself. In case $G_1$ is disconnected the isomorphism maps a component of type I into an isomorphic component of the same type, while a component of type II is mapped onto itself. The labeling of the points of $G$ is illustrated in Figure 1, where for convenience we write $i$ for $a_i$ and $i'$ for $b_i$. In this labeling we observe that $(a_i, b_1)$ is adjacent to $(a_j, b_2)$ if and only if $(a_i, b_2)$ is adjacent to $(a_j, b_1)$. Now, consider the graph $H$ constructed on the points set $A$ as follows. In $H$, $a_i$ adj $a_j$ if and only if $(a_i, b_1)$ adj $(a_j, b_2)$. It follows now that $G = H \oplus K_2$. The graph $G$ in Figure 1 is the tensor product of $H$ and $K_2$ in Figure 3. If $G$ is disconnected and if a component $G_i$ of $G$ is of the form mentioned in III, then $G_i = H_i \oplus K_2$ for some graph $H_i$ as above. If a component $G_j$ of $G$ is not of this form then by hypothesis they are in isomorphic pairs and $G_j \oplus K_2 = 2G_j$ by Corollary 1.2. This completes the proof of the Theorem.

\[\text{Fig. 1}\]
For any two graphs $G_1$ and $G_2$ we have

$$G_1 \oplus G_2 = G_1 \oplus \bigcup_i e_i = \bigcup_i G_1 \oplus e_i$$

where $\{e_i\}$ is the set of lines of $G_2$. Each product graph $G_1 \oplus e_i$ is isomorphic to $G_1 \times K_2$. Also, the graphs $G_1 \oplus e_i$ are line disjoint and the number of common
points (if any) between any two of these graphs is equal to that in $G_1$. These observations lead to

**Theorem 3.** A necessary condition for a graph $G$ to be the tensor product of two graphs is that $G$ is the line disjoint union of a number of graphs of the form $H \oplus K_2$ for some graph $H$, and the number of common points (if any) between any two of these graphs must be that in $H$.

For example, the graph $G$ in Figure 4 is the line disjoint union of $K_{1,3} \otimes K_2$ and $K_{1,3} \oplus K_2$ as illustrated. The number of common points between these two graphs is that in $K_{1,3}$, namely 4 and $G = K_{1,3} \otimes P_3$, where $P_3$ is a path of length 2.

![Figure 4](image)

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**References**
