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Compositio Math. **153** (2017), 667–677.

doi:10.1112/S0010437X16008186

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

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# Equivariant $K$-theory of Grassmannians II: the Knutson–Vakil conjecture

Oliver Pechenik and Alexander Yong

Abstract

In 2005, Knutson–Vakil conjectured a *puzzle* rule for equivariant $K$-theory of Grassmannians. We resolve this conjecture. After giving a correction, we establish a modified rule by combinatorially connecting it to the authors' recently proved tableau rule for the same Schubert calculus problem.

## 1. Introduction

Knutson–Vakil [CV09, § 5] conjectured a combinatorial rule for the structure coefficients of the torus-equivariant $K$-theory ring of a Grassmannian. The structure coefficients are with respect to the basis of Schubert structure sheaves. Their rule extends *puzzles*, combinatorial objects founded in work of Knutson and Tao [KT03] and in their collaboration with Woodward [KTW04]. The various puzzle rules play a prominent role in modern Schubert calculus; see, e.g., [BKT03, Vak06, CV09], recent developments [Knu10, KP11, BKPT16, Buc15] and the references therein.

This paper is a sequel to [PY15], where we gave the first proved tableau rules for these structure coefficients, including a conjecture of Thomas and the second author [TY12]. Here we use these results to prove a mild correction of the puzzle conjecture.

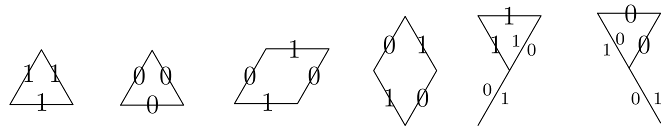
### 1.1 The puzzle conjecture

Let $X = \mathrm{Gr}_k(\mathbb{C}^n)$ denote the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^n$. The general linear group $\mathsf{GL}_n$ acts transitively on $X$ by change of basis. The Borel subgroup $\mathsf{B} \subset \mathsf{GL_n}$ of invertible lower triangular matrices acts on $X$ with finitely many orbits, i.e., the *Schubert cells* $X_\lambda^\circ$. These orbits are indexed by $\{0,1\}$-sequences $\lambda$ of length $n$ with $k$-many 1's. The *Schubert varieties* are the Zariski closures $X_\lambda := \overline{X_\lambda^\circ}$. The $X_\lambda$ are stable under the action of the maximal torus $\mathsf{T} \subset \mathsf{B}$ of invertible diagonal matrices. Therefore, their structure sheaves $\mathcal{O}_{X_\lambda}$ admit classes in $K_{\mathsf{T}}(X)$, the Grothendieck ring of $\mathsf{T}$-equivariant vector bundles over $X$. Now $K_{\mathsf{T}}(X)$ is a $K_{\mathsf{T}}(\mathrm{pt})$-module and the $\binom{n}{k}$ Schubert classes form a module basis. One may make a standard identification $K_{\mathsf{T}}(\mathrm{pt}) \cong \mathbb{Z}[t_i^{\pm 1} : 1 \leqslant i \leqslant n]$. The structure coefficients $K_{\lambda,\mu}^{\nu} \in K_{\mathsf{T}}(\mathrm{pt})$ are defined by

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{\nu} K_{\lambda,\mu}^{\nu} [\mathcal{O}_{X_\nu}].$$

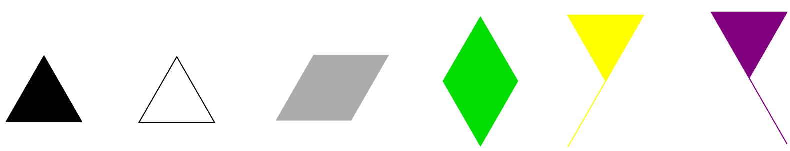
Consider the $n$-length equilateral triangle oriented as $\Delta$. A *puzzle* is a filling of $\Delta$ with the following *puzzle pieces*:

Received 13 September 2015, accepted in final form 19 July 2016, published online 9 March 2017.
*2010 Mathematics Subject Classification* 14M15 (primary), 05E05, 05E15 (secondary).
*Keywords:* Grassmannian, equivariant $K$-theory, puzzle, Schubert calculus, Littlewood–Richardson rule.



The double-labeled edges are *gashed*. A *filling* requires that the common (non-gashed) edges of adjacent puzzle pieces share the same label. Two gashed edges may not be overlayed. The pieces on either side of a gash must have the indicated labels. The first three may be rotated but the fourth (*equivariant piece*) may not [KT03]. We call the remainder *KV-pieces*; these may not be rotated. The fifth piece may only be placed if the equivariant piece is attached to its left. There is a 'non-local' requirement [CV09, § 5] for using the sixth piece: it 'may only be placed (when completing the puzzle from top to bottom and left to right as usual) if the edges to its right are a (possibly empty) series of horizontal 0's followed by a 1'. A *KV-puzzle* is a puzzle filling of $\Delta$.

For simplicity, we will illustrate these six puzzle pieces by the following respective shaded versions (colour online):



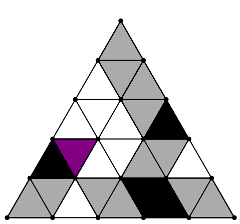
Let $\Delta_{\lambda,\mu,\nu}$ be $\Delta$ with the boundary given by:

- $\lambda$ as read $\nearrow$ along the left-hand side;
- $\mu$ as read $\searrow$ along the right-hand side; and
- $\nu$ as read $\rightarrow$ along the bottom side.

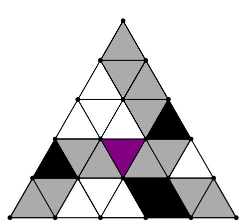
The *weight* $\mathrm{wt}(P)$ of a *KV-puzzle* $P$ is a product of the following factors. Each KV-piece contributes a factor of $-1$. For each equivariant piece one draws a $\searrow$ diagonal arrow from the center of the piece to the $\nu$-side of $\Delta$; let $a$ be the unit segment of the $\nu$-boundary, as counted from the right. Similarly, one determines $b$ by drawing a $\swarrow$ antidiagonal arrow. The equivariant piece contributes a factor of $1 - t_a/t_b$.

Conjecture 1.1 (The Knutson–Vakil puzzle conjecture). $K_{\lambda,\mu}^{\nu} = \sum_P \mathrm{wt}(P)$, where the sum is over all KV-puzzles of $\Delta_{\lambda,\mu,\nu}$.

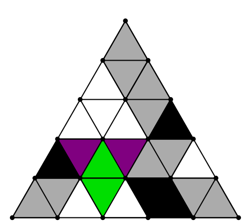
We consider the structure coefficient $K_{01001,00101}^{10010}$ for $\mathrm{Gr}_2(\mathbb{C}^5)$. The reader can check that there are six KV-puzzles $P_1, P_2, \ldots, P_6$ with the indicated weights.



$\mathrm{wt}(P_1) = -1$



$\mathrm{wt}(P_2) = -1$



$\mathrm{wt}(P_3) = (-1)^2(1 - \frac{t_3}{t_4})$



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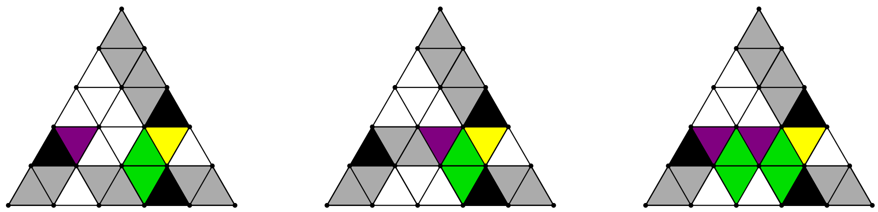


$\mathrm{wt}(P_4) = (-1)^2(1 - \frac{t_2}{t_3})$ $\mathrm{wt}(P_5) = (-1)^2(1 - \frac{t_2}{t_3})$ $\mathrm{wt}(P_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$

Using double Grothendieck polynomials [LS82] (see also [FL94] and references therein), one computes $K_{01001,00101}^{10010} = -(t_2/t_4) = \mathrm{wt}(P_2) + \mathrm{wt}(P_3) + \mathrm{wt}(P_5) + \mathrm{wt}(P_6)$. This gives a counterexample to Conjecture 1.1. Actually, this subset of four puzzles is explained by the rule of Theorem 1.2 below.

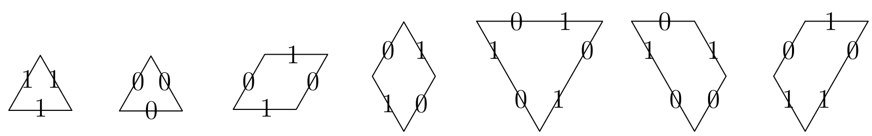
### 1.2 A modified puzzle rule

We define a *modified KV-puzzle* to be a KV-puzzle with the non-local condition on the second KV-piece replaced by the requirement that the second KV-piece only appears in the combination pieces  or .

THEOREM 1.2. $K_{\lambda,\mu}^{\nu} = \sum_P \mathrm{wt}(P)$, *where the sum is over all modified KV-puzzles of* $\Delta_{\lambda,\mu,\nu}$.

We have a few remarks. First, the rule of Theorem 1.2 is 'positive' in the sense of Anderson *et al.* [AGM11]; cf. the discussion in [PY15, § 1.4]. Second, it is a natural objective to interpret Theorem 1.2 via geometric degeneration; see [CV09, Knu10]. Third, the first author has found a tableau formulation similar to that of [PY15] to complement the puzzle rule of [Knu10] for the *different* Schubert calculus problem in $K_{\mathsf{T}}(\mathsf{X})$ of multiplying a class of a Schubert variety by that of an opposite Schubert variety; further discussion may appear elsewhere. Fourth, we observe that the rule of Theorem 1.2 may be easily reformulated to avoid gashed edges and restricted placement rules, as in the following result.

COROLLARY 1.3. $K_{\lambda,\mu}^{\nu} = \sum_P \mathrm{wt}(P)$, *where the sum is over all tilings of* $\Delta_{\lambda,\mu,\nu}$ *by*



*(where only the first three may be rotated and the pieces are given the appropriate weights).*

While this latter formulation is arguably simpler and involves fewer puzzles, we focus here on the modified KV-puzzles of Theorem 1.2 to emphasize the close connection to Conjecture 1.1.

To prove Theorem 1.2, we first give a variant of the main theorem of [PY15]; see § 2. In § 3, we then give a weight-preserving bijection between modified KV-puzzles and the objects of the rule of § 2.

## 2. A tableau rule for $K_{\lambda,\mu}^{\nu}$

We need to briefly recall the definitions of [PY15, §§ 1.2–1.3]; there the Schubert varieties $X_\lambda$ are indexed by Young diagrams $\lambda$ contained in a $k \times (n-k)$ rectangle. (Throughout, we orient Young diagrams and tableaux according to the English convention.)

An *edge-labeled genomic tableau* is a filling of the boxes and horizontal edges of a skew diagram $\nu/\lambda$ with subscripted labels $i_j$, where $i$ is a positive integer and the $j$ that appear for each $i$ form an initial interval of positive integers. Each box of $\nu/\lambda$ contains one label, whereas the horizontal edges weakly between the southern border of $\lambda$ and the northern border of $\nu$ are filled by (possibly empty) sets of labels. A genomic edge-labeled tableau $T$ is *semistandard* if:

(S.1) the box labels of each row strictly increase lexicographically from left to right;

(S.2) ignoring subscripts, each label is strictly less than any label strictly south in its column;

(S.3) ignoring subscripts, the labels appearing on a given edge are distinct;

(S.4) if $i_j$ appears strictly west of $i_k$, then $j \leqslant k$.

Index the rows of $\nu$ from the top starting at 1. We say that a label $i_j$ is *too high* if it appears weakly above the north edge of row $i$. We refer to the collection of all $i_j$ (for fixed $i, j$) as a *gene* of *family* $i$. The *content* of $T$ is the composition $(\alpha_1, \alpha_2, \dots)$, where $\alpha_i$ is greatest so that $i_{\alpha_i}$ is a gene of $T$.

Recall that in the classical tableau theory, a semistandard tableau $S$ is *ballot* if, reading the labels down columns from right to left, we obtain a word $W$ with the following property: for each $i$, every initial segment of $W$ contains at least as many $i$'s as $(i+1)$'s. Given an edge-labeled genomic tableau $T$, choose one label from each gene and delete all others; now delete all subscripts. We say that $T$ is *ballot* if, regardless of our choices from genes, the resulting tableau (possibly containing holes) is necessarily ballot in the above classical sense. (In the case of multiple labels on a edge, read them from least to greatest.)

We now diverge slightly from the treatment of [PY15], borrowing notation from [TY12]. Given a box $\mathsf{x}$ in an edge-labeled genomic tableau $T$, we say that $\mathsf{x}$ is *starrable* if it contains $i_j$, is in row $> i$ and $i_{j+1}$ is not a box label to its immediate right. Let $\mathtt{StarBallotGen}_\mu(\nu/\lambda)$ be the set of all ballot semistandard edge-labeled genomic tableaux of shape $\nu/\lambda$ and content $\mu$ with no label too high, where the label of each starrable box may freely be marked by $\star$ or not. The tableau $T$ illustrated in Figure 2 is an element of $\mathtt{StarBallotGen}_{(10,5,3)}((15,8,5)/(12,2,1))$. There are three starrable boxes in $T$, in only one of which the label has been starred.

Let $\mathsf{Man}(\mathsf{x})$ denote the length of any $\{\uparrow, \rightarrow\}$-lattice path from the southwest corner of $k \times (n-k)$ to the northwest corner of $\mathsf{x}$. For $\mathsf{x}$ in row $r$ containing $i_j^\star$, set $\mathtt{starfactor}(\mathsf{x}) := 1 - (t_{\mathsf{Man}(\mathsf{x})+1})/(t_{r-i+\mu_i-j+1+\mathsf{Man}(\mathsf{x})})$. For an edge label $\ell = i_j$ in the southern edge of $\mathsf{x}$ in row $r$, set $\mathtt{edgefactor} := 1 - (t_{\mathsf{Man}(\mathsf{x})})/(t_{r-i+\mu_i-j+1+\mathsf{Man}(\mathsf{x})})$. Finally for $T \in \mathtt{StarBallotGen}_\mu(\nu/\lambda)$, define

$$\widehat{\mathrm{wt}}(T) := (-1)^{\hat{d}(T)} \times \prod_{\ell} \mathtt{edgefactor}(\ell) \times \prod_{\mathsf{x}} \mathtt{starfactor}(\mathsf{x});$$

here the products are respectively over edge labels $\ell$ and boxes $\mathsf{x}$ containing starred labels, while $\hat{d}(T) := \#(\text{labels in } T) + \#(\star\text{'s in } T) - |\mu|$. Let

$$\hat{L}_{\lambda,\mu}^{\nu} := \sum_{T} \widehat{\mathrm{wt}}(T),$$

where the sum is over all $T \in \mathtt{StarBallotGen}_\mu(\nu/\lambda)$.

We need a reformulation of [PY15, Theorem 1.3]; the proof is a simple application of the 'inclusion–exclusion' identity $\prod_{i\in[m]} a_i = \sum_{S\subseteq[m]} (-1)^{|S|} \prod_{i\in S}(1-a_i)$.

Theorem 2.1. $K_{\lambda,\mu}^{\nu} = \hat{L}_{\lambda,\mu}^{\nu}$.

*Example* 2.2. Let $k = 2, n = 5$ and $\lambda = (2, 0)$, $\mu = (1, 0)$ and $\nu = (3, 1)$. The four tableaux contributing to $\hat{L}^{\nu}_{\lambda,\mu}$ are


$\widehat{\mathrm{wt}}(T_2) = -1$ $\quad\widehat{\mathrm{wt}}(T_3) = (-1)^2(1 - \frac{t_3}{t_4})$ $\quad\widehat{\mathrm{wt}}(T_5) = (-1)^2(1 - \frac{t_2}{t_3})$ $\quad\widehat{\mathrm{wt}}(T_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$

Our indexing of these tableaux alludes to the precise connection to the four puzzles $P_2, P_3, P_5$ and $P_6$ of § 1.1, as explained in the next section. □

## 3. Proof of Theorem 1.2: bijecting the tableau and puzzle rules

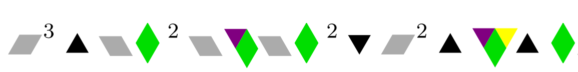
### 3.1 Description of the bijection

To relate the modified KV-puzzle rule of Theorem 1.2 with the tableau rule of Theorem 2.1, we give a variant of Tao's 'proof without words' [Vak06] (and its modification by Purbhoo [Pur08]) that bijects cohomological puzzles (using the first three pieces) and a tableau Littlewood–Richardson rule. An extension of this proof for equivariant puzzles (i.e., fillings that additionally use the equivariant piece) was given by Kreiman [Kre10]; we also incorporate elements of his bijection in our analysis.

Figure 1 gives a 'generic' example of a (modified) KV-puzzle $P$. We will define a *track* $\pi_i$ from the $i$th 1 (from the left) on the $\nu$-boundary of $\Delta_{\lambda,\mu,\nu}$ to the $i$th 1 (from the top) on the $\mu$-boundary. To do this, we describe the *flow* through the (oriented, non-KV) puzzle pieces that use a 1 and four *combination pieces* (possible ways one can use the KV-pieces under the rules for a modified KV-puzzle):

(A.1) : go northeast
(A.2) : go north then northeast
(A.3) : go left to right
(A.4) : go northeast
(A.5) : go in through the north \ of the , come out northeast from the gash into the southwest \ of the and pass northeast through this rhombus
(A.6) : come in through the left side and out the top
(A.7) : come in through the southwest side of the and out the top of the
(A.8) : come in through the north \ of the , out the gash into the \ of the , out the — of into the bottom of the and out its top
(A.9) : come into the north \ of the , out the gash into the southwest \ of the and out the northeast \ into the left side of the and then go out the — of that triangle.

Thinking of the (combination) pieces in (A.1)–(A.9) as letters of an alphabet, we can encode the northernmost track in $P$ (from Figure 1) as the word




Recall, if $\kappa$ is a letter/word in some alphabet, then the *Kleene star* is $\kappa^* := \{\emptyset, \kappa, \kappa\kappa, \ldots\}$.



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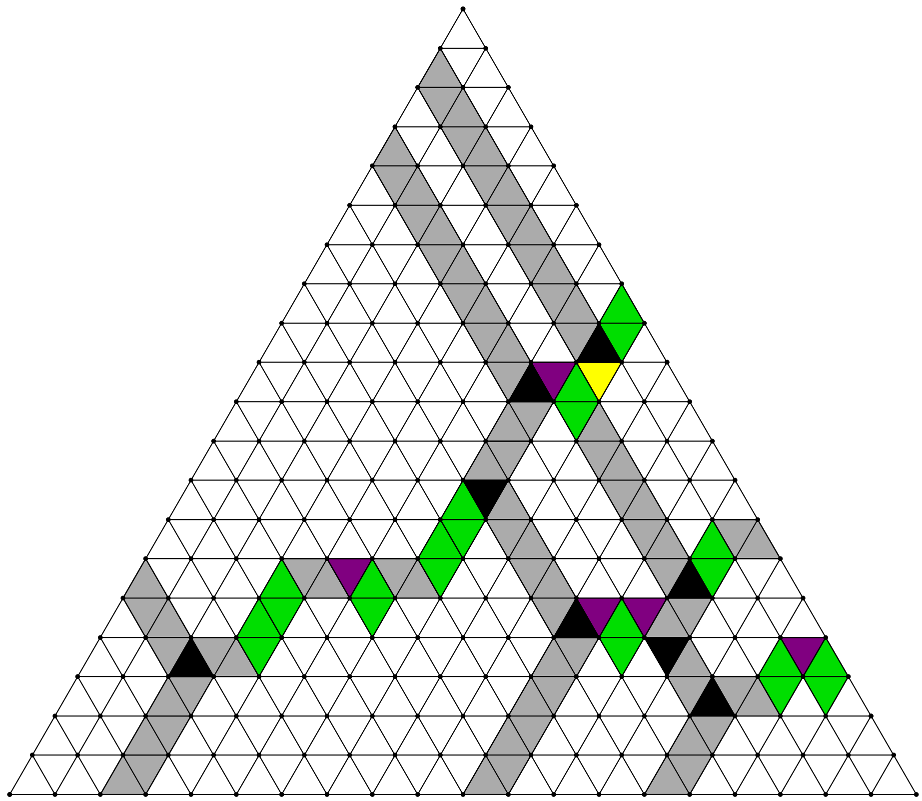

FIGURE 1. A 'generic' modified KV-puzzle $P$ $(k = 3, n = 20)$.

PROPOSITION 3.1 (Decomposition of $\pi_i$). *The list of (combination) pieces that appear in $\pi_i$, as read from southwest to northeast, is a word from the following formal grammar:*

$$\texttt{boxes}[\texttt{edges startrow boxes}]^* \ \texttt{edges}, \tag{3.1}$$

*where*

boxes := $^*$ $\blacktriangle$

edges := [ $\cup$  $\cup$ ]$^*$

startrow := $\blacktriangledown$ $\cup$  $\cup$  $\cup$ 

*Proof.* This is by inspection of the rules for modified KV-puzzles. $\square$

The remaining filling of the puzzle is forced, which we explain in two steps. First there is the *NWray* of each $\blacktriangle$, i.e., the (possibly empty) path of upward-pointing rhombi  growing from the / of this $\blacktriangle$.

LEMMA 3.2. *The NWray of $\blacktriangle$ ends either at the $\lambda$-boundary of $\Delta$ or with a piece from* `startrow`. *In the latter case, the shared edge is the south-then-easternmost edge of the (combination) piece.*

*Proof.* The north / of  is labeled 1. By inspection, the only (combination) pieces that can connect to this edge are  and those from `startrow` (at the stated shared edge). $\square$

Second, pieces of the puzzle not in a track or NWray are 0-triangles (depicted white).



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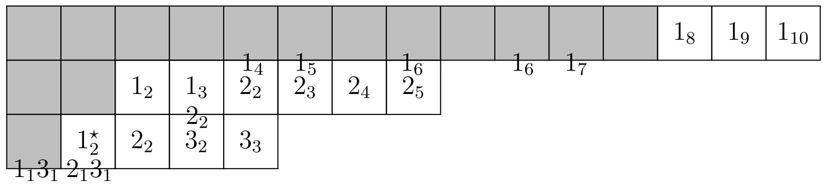


FIGURE 2. The tableau $T := \phi(P)$ corresponding to the modified KV-puzzle $P$ of Figure 1.

We correspond Young diagrams to $\{0,1\}$-sequences. Trace the $\{\leftarrow,\downarrow\}$-lattice path defined by the southern boundary of $\lambda$ (as placed in the northwest corner of $k\times(n-k)$) starting from the northeast corner of $k\times(n-k)$ towards the southeast corner of $k\times(n-k)$. Record each $\leftarrow$ step with '0' and each $\downarrow$ step with '1'.

We now convert $P$ into (we claim) an edge-labeled starred genomic tableau $T := \phi(P)$ of shape $\nu/\lambda$ with content $\mu$. The placement of the labels of family $i$ is governed by the decomposition (3.1) of $\pi_i$. The initial sequence of $k$ 's indicates the leftmost possible placement of box labels $i_{\mu_i}, i_{\mu_i-1},\ldots,i_{\mu_i-k+1}$ (from right to left) in row $i$ of $T$. Continuing to read the sequence, one interprets:

(B.1)  $\leftrightarrow$ 'place (unstarred) box label of next smaller gene'
(B.2) ▲ $\leftrightarrow$ 'end placing box labels in current row'
(B.3)  $\leftrightarrow$ 'skip to the next column left'
(B.4)  $\leftrightarrow$ 'place lower edge label of the next smaller gene'
(B.5)  $\leftrightarrow$ 'place lower edge label of the same gene last used'
(B.6) ▼ $\leftrightarrow$ 'go to next row'
(B.7)  $\leftrightarrow$ 'go to next row and place $\star$-ed box label of the next smaller gene'
(B.8)  $\leftrightarrow$ 'go to next row and place (unstarred) box label of the same gene last used'
(B.9)  $\leftrightarrow$ 'go to next row and place $\star$-ed box label of the same gene last used'.

Applying $\phi$ to the puzzle $P$ of Figure 1 gives the tableau $T$ of Figure 2. Here $\lambda = 0^5 1 0^{10} 1010$, corresponding to the inner shape $(12,2,1)$ (which is shaded in grey). Since $\mu = 0^7 1 0^5 1 0^2 1 0^3$, the content of $T$ is $(10,5,3)$. Finally, since $\nu = 0^2 1 0^7 1 0^3 1 0^5$, the outer shape of $T$ is $(15,8,5)$. As another example, $\phi$ connects the puzzles $P_2, P_3, P_5$ and $P_6$ of § 1 respectively with the tableaux $T_2, T_3, T_5$ and $T_6$ of Example 2.2.

Conversely, given $T \in \texttt{StarBallotGen}_\mu(\nu/\lambda)$, construct a word $\sigma_i$ using the correspondences (B.1)–(B.9), for $1 \leqslant i \leqslant k$. That is, read the occurrences (possibly zero) of family $i$ in $T$ from right to left and from the $i$th row down. We note about (B.6) in the degenerate case that there are no labels of family $i$ in the next row: use ▲ after reading the leftmost box in that row of $\nu/\lambda$ (i.e., the one without any family-$i$ box entries) without a label of family $< i$.

LEMMA 3.3. *Each $\sigma_i$ is of the form (3.1).*

*Proof.* Since $T$ is semistandard, in any row, all box labels of family $i$ are contiguous and strictly right of any (lower) edge labels of that family on that row. The lemma follows. □

We describe a claimed filling $P := \psi(T)$ of $\Delta_{\lambda,\mu,\nu}$. There are $k$ 1's on each side of $\Delta_{\lambda,\mu,\nu}$; to the $i$th 1 from the left on the $\nu$-boundary of $\Delta_{\lambda,\mu,\nu}$, place puzzle pieces in the order indicated by $\sigma_i$.



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That is, attach the next (combination) piece using the northernmost $\backslash$ edge on its west side, if it exists. Otherwise attach at the piece's unique southern edge. We attach at the unique — or $\backslash$ edge of the thus far constructed track. Fill in the order $i = 1, 2, 3, \ldots, k$. Now stack $\blacklozenge$'s northwest of each $\blacktriangle$ until (we claim) we reach one of the pieces of (A.6)–(A.9) at the southernmost / edge or the $\lambda$-boundary of $\Delta_{\lambda,\mu,\nu}$. Complete using white triangles.

Sections 3.2–3.4 prove that $\phi$ and $\psi$ are well-defined and weight-preserving maps between

$$\mathcal{P} := \{\text{modified KV-puzzles of } \Delta_{\lambda,\mu,\nu}\} \quad \text{and} \quad \mathcal{T} := \texttt{StarBallotGen}_{\mu}(\nu/\lambda).$$

Semistandardness (specifically (S.4)) implies that knowing the locations of labels of family $i$, and which labels are repeated or $\star$-ed, uniquely determines the gene(s) in each location. The injectivity of $\phi$ and $\psi$ is easy from this. Moreover, by construction (cf. Lemma 3.3), the two maps are mutually reversing. Thus, Theorem 1.2 follows from Theorem 2.1. $\square$

### 3.2 Well definedness of $\phi : \mathcal{P} \to \mathcal{T}$

Let $P \in \mathcal{P}$ be a modified KV-puzzle for $\Delta_{\lambda,\mu,\nu}$. For the track $\pi_i$, let $\blacktriangle_{i,j}$ refer to the $j$th $\blacktriangle$ seen along $\pi_i$ (as read from southwest to northeast). Let $\mathbb{S}$ denote any of the (combination) pieces that appear in $\texttt{startrow}$. Similarly, we let $\mathbb{S}_{i,j}$ be the $j$th such piece on $\pi_i$.

Figure 1 illustrates the 'ragged honeycomb' structure of modified KV-puzzles. To formalize this, first note by inspection that the $\pi_i$ do not intersect. Second, we have the following claim.

Claim 3.4. *There is a bijective correspondence between the* 1*'s on the* $\lambda$*-boundary and the* $\blacktriangle$*'s in* $\pi_1$*. Specifically, the* $j$*th* 1 *on the* $\lambda$*-boundary is the terminus of the NWray of* $\blacktriangle_{1,j}$*. Similarly, there is a bijective correspondence between* $\blacktriangle_{i+1,j}$ *and* $\mathbb{S}_{i,j}$ *in that the former's NWray terminates at the southernmost / edge of the latter.*

*Proof.* This follows by combining Proposition 3.1 and Lemma 3.2. $\square$

Define $\mathcal{L}_i$ to be the *left sequence* of $\pi_i$: start at the southwest corner of $\Delta_{\lambda,\mu,\nu}$ and read the $\{\rightarrow, \nearrow\}$-lattice path that starts along the $\nu$-boundary and travels up the left-hand boundary of $\pi_i$. The $\{0,1\}$-sequence records the labels of the edges seen. Similarly, define $\mathcal{R}_i$ to be the *right sequence* of $\pi_i$ by traveling up the right-hand side of $\pi_i$ *but only reading the* $\rightarrow$ *and* $\nearrow$ *edges.* (In Figure 1, $\mathcal{L}_1 = 0^5 10^{10} 1010 (= \lambda)$ while $\mathcal{R}_1 = 0^2 10^{11} 10^2 10^2$.)

In view of Claim 3.4, the following is 'graphically' clear by considering the $n$ diagonal strips through $P$.

Claim 3.5. $\mathcal{L}_1 = \lambda$, $\mathcal{L}_{i+1} = \mathcal{R}_i$ *for* $1 \leqslant i \leqslant k-1$ *and* $R_k = \nu$.

Let $T^{(i)}$ be the tableau after adding labels of family $1, 2, \ldots, i$. We declare $T^{(0)}$ to be the empty tableau of shape $\lambda/\lambda$. Let $\nu^{(i)}$ be the outer shape of $T^{(i)}$ (interpreted as the $\{0,1\}$-sequence for its lattice path).

Claim 3.6. $\mathcal{L}_i = \nu^{(i-1)}$ *and* $\mathcal{R}_i = \nu^{(i)}$.

*Proof.* Both assertions follow by inspection of the correspondences (B.1)–(B.9). (Also, the second follows from the first, by Claim 3.5.) $\square$

It is straightforward from Claims 3.5 and 3.6 that $T = \phi(P)$ is semistandard in the sense of (S.1)–(S.4) of [PY15]. By Proposition 3.1, no label of $T$ is $\star$-ed unless it is the rightmost box label of its family in a row ($> i$). Since labels of family $i$ are placed in the boxes of row $i$ or below, no label of $T$ can be too high. Since $\mathcal{R}_k = \nu$, the shape of $T$ is $\nu/\lambda$.



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CLAIM 3.7. *$T$ has content $\mu$.*

*Proof.* Let $\beta$ be the content of $T$. Then $\beta_i$ is the number of (distinct) genes of family $i$ that appear in $T$, which, in terms of $P$, is the number of ▰ and ◆ in $\pi_i$ minus the number of purple KV-pieces ◥ in $\pi_i$. Thus, the vertical height $h_i$ of $\pi_i$ (at its right end *point*) is $\beta_i + \# \blacktriangle$. However, $h_i$ equals the number of line segments strictly below the $i$th 1 on the $\mu$-boundary; i.e., $h_i = n - i - (n - k - \mu_i) = (k - i) + \mu_i$. By Claims 3.4 and 3.1, $\# \blacktriangle = (k - i)$, hence $\beta = \mu$, as desired. $\square$

Finally, we have the following claim.

CLAIM 3.8. *$T$ is ballot.*

*Proof.* The *height* of a (combination) piece is the distance of any northernmost point to the $\nu$-boundary as measured along any (anti)diagonal. The height $h$ of $\blacktriangle_{i+1,j}$ equals the number of ▰'s, $\blacktriangle$'s and ◆'s that appear weakly before $\blacktriangle_{i+1,j}$ in $\pi_{i+1}$ minus the number of ◥'s before $\blacktriangle_{i+1,j}$ in $\pi_{i+1}$. There are exactly $j$ such $\blacktriangle$'s, while the number of ▰'s and ◆'s is the number of labels used and the number of ◥'s is the number of these labels that are repeats. That is, $h = j +$ (#distinct genes of family $i+1$ in row $j+1$ and above), where we do *not* include labels on the lower edges of row $j+1$. Similarly, the height $h'$ of $\mathbb{S}_{i,j}$ is given by $h' = j +$ (#distinct genes of family $i$ in row $j$ and above), where we include labels on the lower edges of row $j$. If $\mathbb{S}_{i,j} = \blacktriangledown$, then, by Claim 3.4, $h' - h \geqslant 0$, and ballotness follows, since the $h' - j$ genes of family $i$ appearing in row $j$ and above appear entirely in those rows. Otherwise $\mathbb{S}_{i,j}$ is a combination piece, and $h' - h \geqslant 1$ by Claim 3.4; ballotness follows, since of the $h' - j$ genes of family $i$ that appear in row $j$ and above, all but at most one appear entirely in those rows. $\square$

### 3.3 Well definedness of $\psi : \mathcal{T} \to \mathcal{P}$

Let $T \in \mathcal{T}$ be a starred ballot genomic tableau of shape $\nu/\lambda$ and content $\mu$. Let $P = \psi(T)$. Let $\pi_i$ be the track associated to $\sigma_i$. As in § 3.2, we define the $\{0,1\}$-sequences $\mathcal{L}_i$ and $\mathcal{R}_i$ associated to $\pi_i$. Here $T^{(i)}$ is *defined* as the subtableau of $T$ using the labels of family $1, 2, \ldots, i$. Hence, $T^{(0)}$ is the empty tableau of shape $\lambda/\lambda$. Let $\nu^{(i)}$ be the outer shape of $T^{(i)}$.

CLAIM 3.9 (Cf. Claim 3.6). *$\mathcal{L}_i = \nu^{(i-1)}$ and $\mathcal{R}_i = \nu^{(i)}$.*

*Proof.* By inspection of the correspondences (B.1)–(B.9). $\square$

By the lattice path definition, each $\nu^{(j)}$ is a length-$n$ sequence. So, $\pi_i$ is a track that (by definition) starts at the south border of $\Delta$ and terminates at the east border of $\Delta$. Also, define $\blacktriangledown_{i,j}$ and $\mathbb{S}_{i,j}$ as before.

CLAIM 3.10. *$\mathbb{S}_{i,j}$ and $\blacktriangle_{i+1,j}$ share a diagonal with the former strictly northwest of the latter.*

*Proof.* The 1's in $\mathcal{L}_{i+1}$ result solely from the $\blacktriangle$'s appearing in $\pi_{i+1}$ while the 1's appearing in $\mathcal{R}_i$ result solely from the $\mathbb{S}$ (combination) pieces. Thus, that the pieces share a diagonal follows from Claim 3.9. For the 'northwest' assertion, repeat Claim 3.8's argument but reverse the logic of the final sentence: since by assumption $T$ is ballot, it follows that $h' \geqslant h$. $\square$

Since Claims 3.9 and 3.10 combine to imply that the $\pi_i$ are non-intersecting, attaching NWrays to each $\blacktriangle$ and filling with white 0-triangles as prescribed, we have a filling $P$ of $\Delta_{\widetilde{\lambda},\widetilde{\mu},\nu}$ satisfying the modified KV-puzzle rule. It remains to check the $\lambda$- and $\mu$-boundaries.

Claim 3.11. $\widetilde{\lambda} = \lambda$.

*Proof.* Graphically, $\widetilde{\lambda} = \mathcal{L}_1$. On the other hand, by Claim 3.9, we know that $\mathcal{L}_1 = \lambda$. □

Claim 3.12. $\widetilde{\mu} = \mu$.

*Proof.* This is given by reversing the logic of the proof of Claim 3.7; here we are given the content of $T$ and are determining the heights of the tracks $\pi_i$. □

### 3.4 Weight preservation

We wish to show the following result.

Claim 3.13. *$\phi$ is weight preserving, i.e.,* $\mathrm{wt}(P) = \widehat{\mathrm{wt}}(T)$.

*Proof.* The $\pm 1$ sign associated to $P$ and $T$ is the same since each usage of a KV-piece in $P$ corresponds to a $\star$-ed label or a repetition of a gene in $T$.

Now consider the weight $1 - t_a/t_b$ assigned to an equivariant piece $p$ in $P$. Here $a$ is the ordinal (counted from the right) of the line segment $s$ on the $\nu$-boundary hit by the diagonal 'right leg' emanating from $p$. Then $b$ equals $a + h - 1$, where $h$ is the height of the piece $p$. Suppose that $p$ lies in track $\pi_i$, and corresponds either to $i_j$ on the lower edge of box $\mathsf{x}$ in row $r$ or to $i_j^\star \in \mathsf{x}$ in row $r$. Consider the edge $e$ on the left boundary of $\pi_i$ that is on the same diagonal as $s$. If $p$ is not attached to the first KV-piece, so it corresponds to an edge label, then $e$'s index from the right in the string $\mathcal{L}_i$ equals $\mathsf{Man}(\mathsf{x})$. Otherwise $e$'s index from the right in the string $\mathcal{L}_i$ equals $\mathsf{Man}(\mathsf{x}) + 1$.

Note that $h$ equals the number of ▰'s, $\blacktriangle$'s and ♦'s appearing weakly before $p$ in $\pi_i$ minus the number of ▼'s appearing before $p$ in $\pi_i$. The number of such $\blacktriangle$'s equals $1 + r - i$ if $p$ corresponds to an edge label and equals $r - i$ if $p$ corresponds to a starred label. The number of such ▰'s and ♦'s minus the number of such ▼'s equals $\mu_i - j + 1$. Weight preservation follows. □

Acknowledgements

We thank Hugh Thomas for enlightening conversations during our work on [PY15] and also thank Allen Knutson and Ravi Vakil for helpful correspondence. We thank two anonymous reviewers for helpful comments. O. P. was supported by an Illinois Distinguished Fellowship, an NSF Graduate Research Fellowship and NSF MCTP grant DMS 0838434. A. Y. was supported by NSF grants.

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https://doi.org/10.1112/S0010437X16008186 Published online by Cambridge University Press

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