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CONTACT AND PANSU DIFFERENTIABLE MAPS ON CARNOT GROUPS

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Abstract

The equivalence between contact and Pansu differentiable maps on Carnot groups is established within the class of maps that are C^1 with respect to the ambient Euclidean structure.

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1. Introduction

A Carnot group G is a connected, simply connected, stratified nilpotent Lie group, equipped with a left-invariant sub-Riemannian metric, defined on a left-invariant sub-bundle of the tangent bundle. The sub-bundle is called the horizontal bundle and the metric is called the Carnot–Carathéodory metric. Diffeomorphisms which preserve the horizontal bundle are called contact maps.

In [5], Pansu introduced his notion of the derivative for mappings between Carnot groups and in a weak sense and showed that Pansu differentiable maps are contact maps almost everywhere. Pansu stopped short of proving the converse statement that contact maps are Pansu differentiable which appears to be considerably more difficult. In a forthcoming paper [3], Magnani establishes the converse without assumptions of smoothness.

The assumption that the maps in question are C^1 with respect to the ambient Euclidean structure facilitates an elementary approach, which in some sense is the familiar proof that a map defined on an open set Ω is $C^1(\Omega)$ if and only if the partial derivatives exist and are continuous on Ω. Reading this familiar theorem in an analogous way in the Carnot group setting, we replace partial derivatives with horizontal partial derivatives and total derivative with Pansu derivative.

A Carnot group G can always be modelled on its Lie algebra $\mathfrak{g}$ using the multiplication $\star$ arising from the Baker–Campbell–Hausdorff formula. We denote this

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model by $(\mathfrak{g}, \star)$ and note that it shows that G carries an ambient Euclidean structure imparted from $\mathfrak{g}$ by the exponential map. In this setting the main result of this article reads as the following result.

THEOREM 1. *Let $\Omega \subseteq (\mathfrak{g}, \star)$ be an open set and let $f : \Omega \to (\mathfrak{g}, \star)$ be $C^1(\Omega)$ with respect to the ambient Euclidean structure of $\mathfrak{g}$. Then f is a contact map if and only if it is Pansu differentiable at every $X \in \Omega$.*

2. Carnot groups

A nilpotent Lie algebra $\mathfrak{g}$ is said to admit an n-step *stratification* if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,$$

such that $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$, where $j = 1, \ldots, n-1$, and $\mathfrak{g}_n$ is contained in the centre $Z(\mathfrak{g})$. A connected, simply connected nilpotent Lie group G, with stratified Lie algebra $\mathfrak{g}$, equipped with an inner product $\langle\, , \rangle_{\mathfrak{g}}$, such that $\mathfrak{g}_i \perp \mathfrak{g}_j$ when $i \neq j$, is called a Carnot group.

A left-invariant vector field $X \in \Gamma(TG)$ has the form $g \mapsto (\tau_g)_*(V)$, where $V \in \mathfrak{g} \equiv T_e G$, and it follows that the left-invariant vector fields inherit the stratification of $\mathfrak{g}$. In particular, if L_i is the sub-bundle of TG defined by $L_i(g) = (\tau_g)_*(\mathfrak{g}_i)$, then $L_{i+1}(g) = [L_1, L_i]\,(g)$ where $i = 1, \ldots, n-1$. The inner product of $\mathfrak{g}$ induces an inner product on $T_g G$ by setting

$$\langle V, W \rangle_g = \langle (\tau_{g^{-1}})_*(V), (\tau_{g^{-1}})_*(W) \rangle_{\mathfrak{g}},$$

and it follows that $L_i(g) \perp L_j(g)$ when $i \neq j$. The horizontal tangent space at $g \in G$ is the subspace $L_1(g) \subseteq T_g G$, and a curve $\gamma : I \to G$ is said to be horizontal if $\dot{\gamma}(t) \in L_1(\gamma(t))$ for all $t \in I$. If $\mathrm{H}(g_1, g_2)$ denotes the set of horizontal curves joining g_1 to g_2, then the *Carnot–Carathéodory* distance is

$$d(g_1, g_2) = \inf_{\gamma \in \mathrm{H}(g_1, g_2)} \int \|\dot{\gamma}(t)\|\, dt,$$

where $\|\dot{\gamma}(t)\| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}}$. The theorem of Chow, see [1], implies that G is path connected via horizontal curves, and that d is a metric. By definition d is left-invariant, that is, $d(\tau_g(g_1), \tau_g(g_2)) = d(g_1, g_2)$.

For simply connected nilpotent Lie groups, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Moreover, the exponential map is an isomorphism $(\mathfrak{g}, \star) \to G$ when we define $X \star Y = \exp^{-1}(\exp(X)\exp(Y))$. The dilation $\delta_t \in \mathrm{Aut}(\mathfrak{g})$ is defined by $\delta_t(X) = \sum_{j=1}^n t^j X_j$, where $X = \sum_{j=1}^n X_j$ and $X_j \in \mathfrak{g}_j$. Dilation of $g \in G$ is defined by $g \mapsto \exp \circ\, \delta_t \circ \exp^{-1}(g)$, and where no confusion arises, we denote dilation on G by $\delta_t(g)$. By definition, the Carnot–Carathéodory distance is homogeneous with respect to dilation, that is, $d(\delta_t(g_1), \delta_t(g_2)) = t\, d(g_1, g_2)$.

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The Baker–Campbell–Hausdorff formula, see [7], is the explicit expression

$$X \star Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{0<p_i+q_i \\ 1\le i\le n}} \frac{1}{C_{p,q}} T(X^{p_1}, Y^{q_1}, \dots, X^{p_n}, Y^{q_n}),$$

where

$$C_{p,q} = p_1!\, q_1! \, \cdots \, p_n!\, q_n! \sum_{i=1}^{n} p_i + q_i$$

and

$$\begin{aligned} &T(X^{p_1}, Y^{q_1}, \dots, X^{p_n}, Y^{q_n}) \\ &\quad = \begin{cases} (\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \cdots (\operatorname{ad} X)^{p_n} (\operatorname{ad} Y)^{q_n - 1} Y & \text{if } q_n \ge 1, \\ (\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \cdots (\operatorname{ad} X)^{p_n - 1} X & \text{if } q_n = 0. \end{cases} \end{aligned} \tag{2.1}$$

The expansion to order 4 takes the form

$$\begin{aligned} X \star Y &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) \\ &\quad + \frac{1}{48}([Y, [X, [Y, X]]] - [X, [Y, [X, Y]]]) + \cdots . \end{aligned}$$

By construction, the pair $(\mathfrak{g}, \star)$ is a Lie group with Lie algebra $\mathfrak{g}$ such that $\operatorname{Aut}(\mathfrak{g}, \star) = \operatorname{Aut}(\mathfrak{g})$. Furthermore, any Carnot group G with Lie algebra $\mathfrak{g}$ is group isomorphic to $(\mathfrak{g}, \star)$ via a stratification-preserving isomorphism, and when an inner product $\langle\ ,\ \rangle_{\mathfrak{g}}$ is given, the isomorphism becomes an isometry when we define

$$\langle V, W \rangle_X = \langle (\tau_{X^{-1}})_*(V), (\tau_{X^{-1}})_*(W) \rangle_{\mathfrak{g}}.$$

It follows from these observations that the theory of Carnot groups can be developed in the context of the model $(\mathfrak{g}, \star)$.

Choosing an orthonormal basis, say $\mathcal{B}$, identifies $\mathfrak{g}$ with $\mathbb{R}^{\dim \mathfrak{g}}$ and $X \star Y$ becomes polynomial in the coordinates X and Y of degree up to $n-1$. The triple $(\mathfrak{g}, \star, \mathcal{B})$ is said to be a *normal model of the first kind.*

Let

$$\{e_{i,\alpha} \mid i = 1, \dots, n,\ \alpha = 1, \dots d_i = \dim \mathfrak{g}_i\},$$

denote a basis of $\mathfrak{g}$ such that

$$\mathfrak{g}_i = \operatorname{span}\{e_{i,\alpha} \mid \alpha = 1, \dots, d_i = \dim \mathfrak{g}_i\},$$

and let

$$\{\lambda_{i,\beta} \mid i = 1, \dots, n,\ \beta = 1, \dots, d_i = \dim \mathfrak{g}_i\} \subset \mathfrak{g}^*$$

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denote the corresponding dual basis such that

$$\lambda_{i,\beta}(e_{j,\alpha}) = \begin{cases} 1 & \text{if } i = j \text{ and } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The vector fields $X_{i,\alpha}$, defined by $X_{i,\alpha}(X) = (\tau_X)_*(e_{i,\alpha})$, form a basis for the left-invariant vector fields of $(\mathfrak{g}, \star)$, and the corresponding dual forms on $\mathfrak{g}$ are $\theta_{i,\alpha}|_X = ({\tau_X}^{-1})^* \lambda_{i,\alpha}$.

If $f : \Omega_1 \to \Omega_2$ is a diffeomorphism between open sets $\Omega_1, \Omega_2 \subseteq \mathfrak{g}$, and $V \in T_X\mathfrak{g}$, where $X \in \Omega_1$, then

$$\begin{aligned} f_*(V_X) &= \sum_i \sum_\alpha \sum_j \sum_\beta \lambda_{i,\alpha}(f_*(e_{j,\beta}))\lambda_{j,\beta}(V_X) e_{i,\alpha} \\ &= \sum_i \sum_\alpha \sum_j \sum_\beta (\theta_{i,\alpha})_{f(X)}(f_*(X_{j,\beta}(X)))\theta_{j,\beta}(V_X) X_{i,\alpha}(f(X)) \\ &= \sum_i \sum_\alpha \sum_j \sum_\beta \theta_{i,\alpha}(f_* X_{j,\beta})\theta_{j,\beta}(V_X) X_{i,\alpha}(f(X)). \end{aligned}$$

We use the notation Jf and Df to denote the matrices with block form

$$Jf_{i,j} = (\lambda_{i,\alpha}(f_* e_{j,\beta}))_{\alpha,\beta} \quad \text{and} \quad Df_{i,j} = (\theta_{i,\alpha}(f_* X_{j,\beta}))_{\alpha,\beta}.$$

Note that the substitutions

$$(X_{i,\alpha})\,(X) = {\tau_X}_*(e_{i,\alpha})$$

and

$$(\theta_{i,\alpha})_X = (\tau_X^{-1})^* \lambda_{i,\alpha}$$

show that

$$Df(X) = J(\tau_{f(X)}^{-1} \circ f \circ \tau_X)\,(0).$$

3. Contact maps

A local C^1 diffeomorphism $f : G \to G$ that preserves horizontal curves is called a contact map. If f is a contact map, then $f_*(L_1(g)) = L_1(f(g))$; moreover, the contact maps of G correspond with contact maps of $(\mathfrak{g}, \star)$ via the exponential map. The trivial examples are left translations and dilations.

Let f be a contact map of $\Omega \subseteq (\mathfrak{g}, \star)$; then $\theta_{i,1}(f_* X_{1,\beta}) = 0$ when $i > 1$ and $1 \le \beta \le d_1$. However, more is true – in fact $\theta_{i,\alpha}(f_* X_{j,\beta}) = 0$ when $1 \le j < i$, and $Df(X)$ is block upper triangular. If f is C^2, this fact follows from the invariance of the Lie bracket, that is,

$$(f_*[V, W])_{f(p)} = [f_* V, f_* W]_{f(p)},$$

where V and W are vector fields. If f is C^1, the Pansu differentiability of f will force $Df(X)$ to block upper triangular, which demonstrates some of the significance of the Pansu derivative.

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4. Pansu differentiability

Let f be a map of some open set $\Omega \subseteq (\mathfrak{g}, \star)$ into $(\mathfrak{g}, \star)$, and let

$$\psi_X = \tau_{f(X)}^{-1} \circ f \circ \tau_X.$$

Then f is said to be Pansu differentiable at $X \in \Omega$, if the limit

$$\lim_{t\to 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z)$$

converges locally uniformly with respect to $Z \in (\mathfrak{g}, \star)$, and the map

$$\phi_X(Z) = \lim_{t\to 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z),$$

called the Pansu derivative of f at X, is an element of $\mathrm{Aut}(\mathfrak{g}, \star)$. We say that f is Pansu differentiable on Ω if it is Pansu differentiable at every $X \in \Omega$. The topology of the convergence is the metric topology induced by the Euclidean norm, the Carnot–Carathéodory distance, or the gauge metric of Nagel *et al.* [4].

5. Proof of Theorem 1

First we show that Pansu differentiability of f implies that f is a contact map. Suppose that f is Pansu differentiable at $X \in \Omega$. Since

$$\frac{1}{t^i}\lambda_{i,\alpha} \circ \psi_X(t^j e_{j,\beta}) = \lambda_{i,\alpha} \circ \delta_{1/t} \circ \psi_X \circ \delta_t(e_{j,\beta}),$$

it follows that

$$\lim_{t\to 0} \frac{1}{t^i}\lambda_{i,\alpha} \circ \psi_X(t^j e_{j,\beta}) = \lambda_{i,\alpha} \circ \phi_X(e_{j,\beta}).$$

In particular, if f is C^1 in a neighbourhood of X, then $Df(X) = J\psi_X(0)$ is block upper triangular, and ϕ_X is given by the diagonal part of $Df(X)$. Moreover, f is a contact map since $Df(X)_{i,1} = 0$ when $i > 1$, that is, f preserves horizontal curves.

In the rest of this section we first outline the proof that a C^1 contact map

$$f : \Omega \to (\mathfrak{g}, \star)$$

is Pansu differentiable at every $X \in \Omega$, and then provide the details. The proof uses [2, Lemma 1.40], which states that there exist a constant $C > 0$, an integer m, and a map $g : \{1, \dots, m\} \to \{1, \dots, d_1\}$, such that every $W \in (\mathfrak{g}, \star)$ has the form

$$W = w_1 e_{1,g(1)} \star \cdots \star w_m e_{1,g(m)}, \tag{5.1}$$

where $|w_i| \leq C|W|^{1/n}$.

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The proof proceeds first by observing that for every C^1 horizontal curve γ such that $\gamma(0) = 0$,

$$\lim_{t\to 0} \delta_{1/t} \circ \gamma(t) = \gamma_1'(0).$$

Next we observe that since f preserves horizontal curves, the curve $\gamma(t) = \psi_X(tZ)$ is horizontal when $Z \in \mathfrak{g}_1$, and $\gamma(0) = 0$. It follows that

$$\lim_{t\to 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z) = \gamma_1'(0),$$

and the smoothness of f implies that the convergence is uniform when

$$Z \in D_R = \{Z \in \mathfrak{g}_1 \ : \ |Z| \le R\}.$$

Next we use Pansu's decomposition [5], that is, if $Y, Z \in \mathfrak{g}$ then

$$\begin{aligned}
\delta_{1/t} \circ \psi_X \circ \delta_t(Y \star Z) &= \delta_{1/t}(f(X)^{-1} \star f(X \star \delta_t(Y) \star \delta_t(Z))) \\
&= \delta_{1/t}(f(X)^{-1} \star f(X \star \delta_t(Y)) \star f(X \star \delta_t(Y))^{-1} \star f(X \star \delta_t(Y) \star \delta_t(Z))) \\
&= (\delta_{1/t} \circ \tau_{f(X)}^{-1} \circ f \circ \tau_X \circ \delta_t(Y)) \star \ (\delta_{1/t} \circ \tau_{f(X\star\delta_t(Y))}^{-1} \circ f \circ \tau_{X\star\delta_t(Y)} \circ \delta_t(Z)).
\end{aligned}$$

Now assume that the limit

$$\lim_{t\to 0} \delta_{1/t} \circ \tau_{f(X)}^{-1} \circ f \circ \tau_X \circ \delta_t(Y)$$

converges uniformly when Y is an element of the $(j-1)$-fold product

$$D_R^{j-1} = D_R \star \cdots \star D_R.$$

Assume further that $Z \in D_R$, and that

$$\lim_{t\to 0} \delta_{1/t} \circ \tau_{f(X\star\delta_t(Y))}^{-1} \circ f \circ \tau_{X\star\delta_t(Y)} \circ \delta_t(Z) = \lim_{t\to 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z),$$

and the convergence is uniform when $Y \in D_R^{j-1}$ and $Z \in D_R$. Then Pansu's decomposition shows that

$$\lim_{t\to 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Y \star Z) = \phi_X(Y) \star \phi_X(Z),$$

and the limit converges uniformly when $Y \in D_R^{j-1}$ and $Z \in D_R$. By induction on j and (5.1), it then follows that

$$\lim_{t\to 0} \delta_{1/t} \circ \psi_X \circ \delta_t(W) = w_1\phi_X(e_{1,g(1)}) \star \cdots \star w_m\phi_X(e_{1,g(m)}),$$

uniformly when $|W| \le R$, and consequently $\phi_X \in \operatorname{Aut}(\mathfrak{g}, \star)$.

Now we come to the details.

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LEMMA 2. *If γ is a C^1 horizontal curve such that $\gamma(0) = 0$, then*

$$\lim_{s\to 0} \delta_{1/s} \circ \gamma(s) = \gamma_1'(0).$$

PROOF. By definition, γ is horizontal if and only if

$$\gamma'(s) = (\tau_{\gamma(s)})_*|_0(v(s)), \tag{5.2}$$

for some $v(s) \in \mathfrak{g}_1$. If $V \in \mathfrak{g} = T_0\mathfrak{g}$, then (2.1) shows that there are constants C_k such that

$$(\tau_X)_*\big|_0(V) = \sum_{k=0}^{n-1} C_k(\mathrm{ad}\, X)^k(V). \tag{5.3}$$

For example, $C_0 = 1$, $C_1 = 1/2$, $C_2 = 1/12$, and $C_3 = 0$. Together, (5.2) and (5.3) show that

$$\begin{aligned}\gamma'(s) &= \sum_{k=0}^{n-1} C_k(\mathrm{ad}\,\gamma(s))^k(v(s)) \\ &= v(s) + \sum_{k=1}^{n-1} C_k(\mathrm{ad}\,\gamma(s))^k(v(s)),\end{aligned}$$

which implies that $v(s) = \gamma_1'(s)$. It follows that

$$\gamma'(s) = \sum_{k=0}^{n-1} C_k(\mathrm{ad}\,\gamma(s))^k(\gamma_1'(s)).$$

Since $(\mathrm{ad}\,\gamma(s))^k(\gamma_1'(s))$ is a weighted sum of terms

$$[\gamma_{\ell_k}(s), [\ldots[\gamma_{\ell_1}(s), \gamma_1'(s)]\ldots]],$$

and

$$\lambda_j([\gamma_{\ell_k}(s), [\ldots[\gamma_{\ell_1}(s), \gamma_1'(s)]\ldots]]) = 0,$$

unless $\ell_k + \cdots + \ell_1 = j - 1$, it follows that

$$\frac{1}{s^{j-1}}\lambda_j((\mathrm{ad}\,\gamma(s))^k(\gamma_1'(s))) = \begin{cases}\lambda_j((\mathrm{ad}\,\delta_{1/s}\circ\gamma(s))^k(\gamma_1'(s))) & \text{if } 1 \le k \le j-1,\\ 0 & \text{otherwise.}\end{cases}$$

In particular,

$$\gamma_j'(s) = \sum_{k=1}^{j-1} C_j\lambda_j((\mathrm{ad}\,(\gamma_1(s) + \cdots + \gamma_{j-1}(s)))^k(\gamma_1'(s))). \tag{5.4}$$

When $j = 2$, (5.4) implies that

$$\gamma_2'(s) = C_1[\gamma_1(s), \gamma_1'(s)],$$

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and it follows that

$$\lim_{t\to 0}\frac{\gamma_2'(t)}{t}=\lim_{t\to 0} C_1\left[\frac{\gamma_1(t)}{t},\gamma_1'(t)\right]=0.$$

Furthermore,

$$\begin{aligned}\left|\frac{\gamma_2(s)}{s^2}\right| &= \left|\frac{C_1}{s^2}\int_0^s[\gamma_1(t),\gamma_1'(t)]\,dt\right|\\ &\le \frac{|C_1|}{s}\int_0^s\left\|\left[\frac{\gamma_1(t)}{t},\gamma_1'(t)\right]\right\|\,dt\\ &= |C_1|\left\|\left[\frac{\gamma_1(c)}{c},\gamma_1'(c)\right]\right\|,\end{aligned} \tag{5.5}$$

where the existence of $c\in(0,s)$ in the the last line is guaranteed by the mean value theorem. It follows that

$$\lim_{s\to 0}\frac{\gamma_2(s)}{s^2}=0.$$

If we assume that $\lim_{t\to 0}(\gamma_\ell(t)/t^\ell)=0$ when $\ell=2,\dots,j-1$, then (5.4) implies that

$$\lim_{t\to 0}\frac{\gamma_j'(t)}{t^{j-1}}=0.$$

Furthermore,

$$\begin{aligned}\left|\frac{\gamma_j(s)}{s^j}\right| &= \left|\frac{1}{s^j}\int_0^s\gamma_j'(t)\,dt\right|\\ &\le \frac{1}{s}\int_0^s\left|\frac{\gamma_j'(t)}{t^{j-1}}\right|\,dt\\ &= \left|\frac{\gamma_j'(c)}{c^{j-1}}\right|,\end{aligned} \tag{5.6}$$

where the existence of $c\in(0,s)$ in the the last line is guaranteed by the mean value theorem. It follows that

$$\lim_{s\to 0}\frac{\gamma_j(s)}{s^j}=0,$$

and we conclude that

$$\lim_{s\to 0}\delta_{1/s}\circ\gamma(s)=\gamma_1'(0),$$

as required. □

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For $Z \in \mathfrak{g}_1$, let $\zeta(t) = tZ$; then

$$(\tau_{\zeta(t)})_*\big|_0(\zeta_1'(t)) = \sum_{k=0}^{n-1} C_k(\operatorname{ad} tZ)^k(Z) = Z = \zeta'(t),$$

hence ζ is horizontal. It follows that $\gamma(t) = \psi_X(tZ)$ is a C^1 horizontal curve such that $\gamma(0) = 0$, since ψ_X is a C^1 contact map which fixes 0. Moreover, the previous lemma shows that

$$\lim_{s\to 0} \delta_{1/s} \circ \psi_X \circ \delta_s(Z) = J\psi_X(0)\lambda(Z) = Df(X)\lambda(Z),$$

where $\lambda(Z)$ is the coordinate expression of Z relative to the basis $\{e_{i,\alpha}\}$.

Since $f \in C^1$, it follows that

$$(V, X) \mapsto \|J\psi_X(V) - Df(X)\|_\infty$$

and

$$(V, X) \mapsto \|J\psi_X(V)\|_\infty$$

are both continuous functions of $\mathfrak{g} \times \mathfrak{g}$ into $\mathbb{R}^+$, see [6, p. 188]. For a compact subset $U \subset \mathfrak{g}$, we define

$$N_1(R, U) = \max\{\|J\psi_X(V) - Df(X)\|_\infty \ \ |V| \le R,\ \ X \in U\}$$

and

$$N_2(R, U) = \max\{\|J\psi_X(V)\|_\infty \ \ |V| \le R,\ \ X \in U\}.$$

LEMMA 3. *Let $Z \in D_R$, $X \in U$, and $\gamma(t) = \psi_X(tZ)$. Then*

$$|\gamma_1'(s) - \gamma_1'(0)| \le P(R)N_1(sR, U)$$

and

$$\left|\frac{\gamma_1(s)}{s} - \gamma_1'(0)\right| \le P(R)N_1(sR, U),$$

where $P(R) = R\sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty$.

PROOF. For the first inequality,

$$\begin{aligned}
|\gamma_1'(s) - \gamma_1'(0)| &= \left|\sum_{\mu=1}^{d_1} \lambda_{1,\mu}(J\psi_X(sZ)\lambda(Z) - Df(X)\lambda(Z))e_{1,\mu}\right| \\
&\le \sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty g|J\psi_X(sZ)\lambda(Z) - Df(X)\lambda(Z)| \\
&\le R\sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty \|J\psi_X(sZ) - Df(X)\|_\infty.
\end{aligned}$$

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For the second inequality we apply the mean value theorem, that is, there exists $0 < t_{1,\mu} < s$ such that

$$\begin{aligned}\left|\frac{\gamma_{1,\mu}(s)}{s} - \gamma'_{1,\mu}(0)\right| &= |\gamma'_{1,\mu}(t_{1,\mu}) - \gamma'_{1,\mu}(0)| \\ &= |\lambda_{1,\mu}(J\psi_X(t_{1,\mu}Z)\lambda(Z) - Df(X)\lambda(Z))| \\ &\le \|\lambda_{1,\mu}\|_\infty \, R \, \|J\psi_X(t_{1,\mu}Z) - Df(X)\|_\infty,\end{aligned}$$

and it follows that

$$\left|\frac{\gamma_1(s)}{s} - \gamma'_1(0)\right| \le R \sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty \|J\psi_X(t_{1,\mu}Z) - Df(X)\|_\infty,$$

as claimed. □

LEMMA 4. *Let $Z \in D_R$, $X \in U$, and $\gamma(t) = \psi_X(tZ)$. Then for each $k \ge 2$ there is a positive constant $Q_k(R, U)$ such that*

$$\left|\frac{\gamma_k(s)}{s^k}\right| \le Q_k(R, U) N_1(sR, U).$$

PROOF. From (5.5), there is $c \in (0, s)$ such that

$$\left|\frac{\gamma_2(s)}{s^2}\right| \le |C_1| \left|\left[\frac{\gamma_1(c)}{c}, \gamma'_1(c)\right]\right|.$$

Since there is a constant $M > 0$ such that $|[X, Y]| \le M|X|\,|Y|$,

$$\begin{aligned}\left|\left[\frac{\gamma_1(c)}{c}, \gamma'_1(c)\right]\right| &\le \left|\left[\frac{\gamma_1(c)}{c} - \gamma'_1(0), \gamma'_1(c)\right]\right| + |[\gamma'_1(0), \gamma'_1(c) - \gamma'_1(0)]| \\ &\le M\left(\left|\frac{\gamma_1(c)}{c} - \gamma'_1(0)\right| |\gamma'_1(c)| + |\gamma'_1(0)|\,|\gamma'_1(c) - \gamma'_1(0)|\right).\end{aligned} \tag{5.7}$$

Furthermore, since $c < s \le 1$,

$$\begin{aligned}|\gamma'_1(c)| &= |\lambda_1(J\psi_X(cZ)Z)| \\ &\le P(R)\|J\psi_X(cZ)\|_\infty \\ &\le P(R) N_2(R, U).\end{aligned} \tag{5.8}$$

From (5.8), (5.7), and Lemma 4,

$$\left|\left[\frac{\gamma_1(c)}{c}, \gamma'_1(c)\right]\right| \le 2\, M\, P(R)^2\, N_2(R, U) N_1(cR, U),$$

and $N_1(cR, U) \le N_1(sR, U)$, so

$$\left|\frac{\gamma_2(s)}{s^2}\right| \le 2|C_1|\, M\, P(R)^2\, N_2(R, U) N_1(sR, U),$$

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thus

$$Q_2(R,U) = 2|C_1|\, M\, P(R)^2\, N_2(R,U).$$

Similarly, by (5.6) there is $c \in (0, s)$ such that

$$\left|\frac{\gamma_j(s)}{s^j}\right| \le \left|\frac{\gamma_j'(c)}{c^{j-1}}\right|, \tag{5.9}$$

and by (5.4)

$$\left|\frac{\gamma_j'(c)}{c^{j-1}}\right| \le \sum_{k=1}^{j-1} |C_j| \|\lambda_j\|_\infty \left| \left(\mathrm{ad}\ \frac{\gamma_1(c)}{c} + \cdots + \frac{\gamma_{j-1}(c)}{c^{j-1}} \right)^k (\gamma_1'(c)) \right|. \tag{5.10}$$

We write

$$\begin{aligned} \frac{\gamma_1(c)}{c} + \cdots + \frac{\gamma_{j-1}(c)}{c^{j-1}} &= \left(\frac{\gamma_1(c)}{c} - \gamma_1'(0)\right) + \left(\gamma_1'(0) + \frac{\gamma_2(c)}{c} \cdots + \frac{\gamma_{j-1}(c)}{c^{j-1}}\right) \\ &= A + B_j, \end{aligned}$$

say, and use the inequality $|[X, Y]| \le M|X|\,|Y|$ to obtain

$$\begin{aligned} |(\mathrm{ad}\ \ A + B_j)^k (\gamma_1'(c))| \le |\gamma_1'(c)| M^k \sum_{\ell=0}^{k-1} \binom{k}{\ell} |A|^{k-\ell} |B_j|^\ell \\ + |B_j|^{k-1} |[B_j, \gamma_1'(c)]| M^{k-1}. \end{aligned}$$

Furthermore, if we write $B_j = \gamma_1'(0) + \widetilde{B}_j$, then

$$\begin{aligned} |[B_j, \gamma_1'(c)]| &= |[\gamma_1'(0), \gamma_1'(c)] + [\widetilde{B}_j, \gamma_1'(c)]| \\ &\le |[\gamma_1'(0), \gamma_1'(c)]| + M|\widetilde{B}_j|\, |\gamma_1'(c)| \\ &= \left| \left[\gamma_1'(0) - \frac{\gamma_1'(c)}{c}, \gamma_1'(c) \right] \right| + M|\widetilde{B}_j|\, |\gamma_1'(c)| \\ &\le M|A|\, |\gamma_1'(c)| + M|\widetilde{B}_j|\, |\gamma_1'(c)|, \end{aligned}$$

hence

$$\begin{aligned} |(\mathrm{ad}\ \ A + B_j)^k (\gamma_1'(c))| \le P(R) N_2(R,U) M^k \sum_{\ell=0}^{k-1} \binom{k}{\ell} |A|^{k-\ell} |B_j|^\ell \\ + P(R) N_2(R,U) M^k |B_j|^{k-1} (|A| + |\widetilde{B}_j|). \end{aligned} \tag{5.11}$$

From Lemma 3,

$$|A| \le P(R) N_1(sR, U),$$

and assuming inductively that

$$\left|\frac{\gamma_\ell(s)}{s^\ell}\right| \le Q_\ell(R,U) N_1(sR,U),$$

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for $\ell = 2, \ldots, j-1$, we deduce that

$$|B_j| \leq RN_2(R, U) + N_1(R, U) \sum_{\ell=1}^{j-1} Q_\ell(R, U),$$

and

$$|\widetilde{B}_j| \leq N_1(sR, U) \sum_{\ell=1}^{j-1} Q_\ell(R, U).$$

These estimates, together with (5.11), (5.10), and (5.9), show that there is a constant $Q_j(R, U)$ such that

$$\left|\frac{\gamma_j(s)}{s^j}\right| \leq Q_j(R, U) N_1(sR, U),$$

as stated. □

By Lemmas 3 and 4, the following result holds.

COROLLARY 5. *If $Z \in D_R$, then for each $X \in \mathfrak{g}$, the limit*

$$\lim_{s \to 0} \delta_{1/s} \circ \psi_X \circ \delta_s(Z) = Df(X)\lambda(Z),$$

converges uniformly with respect to Z.

LEMMA 6. *Let X_0, $Y \in \mathfrak{g}$ and $Z \in \mathfrak{g}_1$. Assume that $Y, Z \in \overline{B_R(0)}$. Then*

$$\lim_{s \to 0} \delta_{1/s} \circ \psi_{X_0 \star \delta_s(Y)} \circ \delta_s(Z) = \lim_{s \to 0} \delta_{1/s} \circ \psi_{X_0} \circ \delta_s(Z) = Df(X_0)\lambda(Z),$$

and the convergence is uniform with respect to Z.

PROOF. Let U be the compact set given by

$$U = \{X \mid X = X_0 \star W,\ W \in \overline{B_R(0)}\}.$$

For $X \in U$ let $\gamma^{X,Z}(t) = \psi_X(tZ)$; then

$$\begin{aligned} |\delta_{1/s} \circ \gamma^{X_0,Z}(s) - \delta_{1/s} \circ \gamma^{X,Z}(s)| \leq\ & |\delta_{1/s} \circ \gamma^{X_0,Z}(s) - (\gamma^{X_0,Z})'(0)| \\ & + |\delta_{1/s} \circ \gamma^{X,Z}(s) - (\gamma^{X,Z})'(0)| \\ & + |(\gamma^{X_0,Z})'(0) - (\gamma^{X,Z})'(0)|. \end{aligned} \tag{5.12}$$

By Lemmas 3 and 4,

$$\begin{aligned} |\delta_{1/s} \circ \gamma^{X,Z}(s) - (\gamma^{X,Z})'(0)| &\leq \left|\frac{\gamma_1^{X,Z}(s)}{s} - (\gamma_1^{X,Z})'(0)\right| + \sum_{k=2}^{n} \left|\frac{\gamma_k^{X,Z}(s)}{s^k}\right| \\ &\leq \left(P(R) + \sum_{k=2}^{n} Q_k(R, U)\right) N_1(sR, U), \end{aligned}$$

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and (5.12) gives

$$|\delta_{1/s} \circ \gamma^{X_0,Z}(s) - \delta_{1/s} \circ \gamma^{X,Z}(s)| \leq 2H(R, U)N_1(sR, U) \\ + R\,\|Df(X_0) - Df(X)\|_\infty,$$

where $H(R, U) = P(R) + \sum_{k=2}^{n} Q_k(R, U)$.

Letting $X = X_0 \star \delta_s(Y)$ in the previous estimate shows that

$$\lim_{s\to 0} \delta_{1/s} \circ \gamma^{X_0\star\delta_s(Y),Z}(s) = \gamma^{X_0,Z\prime}(0) = Df(X_0)\lambda(Z),$$

and the convergence is uniform with respect to Y and Z. □

The proof of Theorem 1 is now complete.

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