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# Classifying Spaces for Polarized Mixed Hodge Structures and for Brieskorn Lattices

CLAUS HERTLING
*Mathematisches Institut der Universität Bonn, Beringstraße 3, 53115 Bonn, Germany*
*e-mail: hertling@math.uni-bonn.de*

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**Abstract.** Classifying spaces and moduli spaces are constructed for two invariants of isolated hypersurface singularities, for the polarized mixed Hodge structure on the middle cohomology of the Milnor fibre, and for the Brieskorn lattice as a subspace of the Gauß–Manin connection. The relations between them, period mappings for $\mu$-constant families of singularities, and Torelli theorems are discussed.

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**Key words:** polarized mixed Hodge structure, classifying space, hypersurface singularity, Gauß–Manin connection, Brieskorn lattice, Torelli theorem.

## 1. Introduction

There are many ways to look at isolated hypersurface singularities, and many different objects which are associated to singularities. It is natural to ask which properties of the singularities they reflect and how they vary. Many are invariants of the $\mu$-homotopy type singularity, like the Milnor lattice, the Coxeter Dynkin diagrams, the topological type for $n \neq 2$, but also the spectral pairs. Some others, like the Tjurina number and Bernstein polynomial, are invariants of the contact equivalence class, and can jump within a $\mu$-constant family. Most of the invariants are of a discrete nature.

Here we want to study two nondiscrete invariants of the right equivalence class, which vary continuously within a $\mu$-constant family. They are natural candidates for Torelli type questions. The first is the isomorphism class of the mixed Hodge structure of Steenbrink, the second comes from the Brieskorn lattice as a subspace of the Gauß–Manin connection. We will give precise descriptions of these invariants, define and analyse classifying spaces, discuss period mappings, and report on known Torelli theorems.

If $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function germ with an isolated singularity, then the middle cohomology $H^n(X_\infty, \mathbb{C})$ of the Milnor fibre carries a mixed Hodge structure [St]. Within a $\mu$-constant family, the weight filtration and the Hodge numbers are constant, and the Hodge filtration is varying holomorphically. It turns out that the mixed Hodge structure is sensitive to some of the analytic

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moduli in the $\mu$-constant stratum, but not to all of them. For example, in the case of semiquasihomogeneous singularities, the mixed Hodge structure depends only on the quasihomogeneous part of the singularity.

A better invariant for Torelli type questions is the Brieskorn lattice $H_0'' = \Omega^{n+1}_{\mathbb{C}^{n+1},0}/\,\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0}$ [Br]. M. Saito showed that it varies holomorphically within the $\mu$-constant stratum, and that it is sensitive to all of the analytic moduli [SM1] [SM2]. So it satisfies some infinitesimal Torelli theorem. The author defined an equivalence class BL of Brieskorn lattices, which is an invariant of the right equivalence class of a singularity, and proved global Torelli theorems for several $\mu$-homotopy classes of singularities [He1] [He2] [He3].

Following Varchenko [Va1] (cf. [Ph2] [SchSt] [SM1]), the Hodge filtration can be described in terms of the Gauß–Manin connection and the Brieskorn lattice. This shows that the Brieskorn lattice can be seen as an extension of the mixed Hodge structure. For the analysis of Brieskorn lattice and invariant BL and for the definition of a classifying space, it is necessary first to consider the mixed Hodge structure.

For the classifying space of mixed Hodge structures, it is important to realize that the mixed Hodge structure of Steenbrink is polarized. This is more or less well known. It is reviewed in Section 3. There an explicit description is also given, which is less well known, of the polarizing form $S$ in terms of variation or Seifert form.

There are several possibilities to define a polarized mixed Hodge structure (PMHS). Steenbrink's mixed Hodge structure is a PMHS in the sense of Schmid ([Schm] Sect. 6). In Section 2 such PMHS's are defined and discussed from a general viewpoint. Section 2 is of interest independently of the application to singularities in the following chapters. A classifying space $D_{\mathrm{PMHS}}$ for the PMHS's is constructed. It is a fibre bundle over a classifying space $D_{\mathrm{prim}}$ for pure polarized Hodge structures. A discrete group $G_{\mathbb{Z}}$ acts on $D_{\mathrm{PMHS}}$. The quotient $D_{\mathrm{PMHS}}/G_{\mathbb{Z}}$ is the moduli space for the isomorphism classes of PMHS's. The main result of Section 2 is the following.

THEOREM 1.1 (2.5+2.6). *The space $D_{\mathrm{PMHS}}$ is a complex manifold and a homogeneous space with respect to some real Lie group. The fibration $D_{\mathrm{PMHS}} \to D_{\mathrm{prim}}$ is holomorphic and locally trivial, the fibres are isomorphic to $\mathbb{C}^{N_{\mathrm{PMHS}}}$ for some $N_{\mathrm{PMHS}} \in \mathbb{N}$. The group $G_{\mathbb{Z}}$ acts properly discontinuously on $D_{\mathrm{PMHS}}$.*

In the case of singularities, this group $G_{\mathbb{Z}}$ is the group of all automorphisms of the Milnor lattice, which respect the Seifert form. Then they automatically also respect monodromy and intersection form.

In Section 4 the properties of the Gauß–Manin connection and the Brieskorn lattice are reviewed. The presentation is as short and elementary as possible. Other, more detailed expositions can be found in [He1] [He2] [SM1] [SM2] [SchSt] [Ka] (Sects. 1, 2 in [Ka], see [SM2] for a critical discussion of the statements in the following chapters of [Ka]). Just as the Brieskorn lattice can be seen as an extension of the mixed Hodge structure, there is an extension of the polarizing form $S$ to the Gauß–Manin connection. In Section 4 this extension is defined and identified with K. Saito's higher residue pairing. This gives a more concise description of this pairing and the relation to intersection form and Seifert form than can be found in the literature.

Section 5 is the center piece and the most technical part of this paper. There a classifying space $D_{\mathrm{BL}}$ for Brieskorn lattices is constructed. It is a fibre bundle over $D_{\mathrm{PMHS}}$.

THEOREM 1.2 (5.3–5.5). *$D_{\mathrm{BL}}$ is a complex manifold. The bundle $D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$ is a holomorphic locally trivial bundle with fibres isomorphic to $\mathbb{C}^{N_{\mathrm{BL}}}$ for some $N_{\mathrm{BL}} < \frac{1}{4}\mu^2$. There is a canonical $\mathbb{C}^*$-action with negative weights on the fibres of this bundle. The group $G_{\mathbb{Z}}$ acts on $D_{\mathrm{BL}}$ properly discontinuously, respecting the fibration and the $\mathbb{C}^*$-action.*

The quotient $D_{\mathrm{BL}}/G_{\mathbb{Z}}$ is the moduli space for the invariant BL. Although there is some similarity between the fibrations $D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$ and $D_{\mathrm{PMHS}} \to D_{\mathrm{prim}}$, the proofs are totally different. There is no transitive natural group action on $D_{\mathrm{BL}}$ present. The analysis of $D_{\mathrm{BL}}$ uses and extends the construction in [SM1] Section 3, which leads there to the main result of that paper, the existence of bases of $H_0''$ with very special properties. So one can see Section 5 as a continuation of the analysis of the structure of the Brieskorn lattice, which M. Saito had undertaken in [SM1] Section 3.

Section 6 contains a discussion of results from [SM1] [SM2] [He1] [He2] [He3] on the period mappings from a $\mu$-constant family to $D_{\mathrm{BL}}$ and $D_{\mathrm{PMHS}}$. For example, in the case of a quasihomogeneous singularity, the $\mu$-constant stratum is locally a fibre bundle $S = S^0 \times S^- = S^0 \times \mathbb{C}^{\dim S^-} \to S^0$ with a $\mathbb{C}^*$-action with negative weights on the fibres $S^-$. Then the period mapping $S \to D_{\mathrm{BL}}$ together with $S^0 \to D_{\mathrm{PMHS}}$ is an embedding of bundles, which preserves the fibres and the $\mathbb{C}^*$-action.

The main part of Section 6 is a short discussion of the most important features of the period mappings and the global Torelli theorems, which the author had obtained. They include the unimodal and bimodal singularities [He1] [He2], the Brieskorn–Pham singularities with coprime exponents, and the semiquasihomogeneous singularities with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ [He3]. This last class is especially nice, because there the results imply a global Torelli theorem for cubics in $\mathbb{P}^3$ in terms of some pure polarized Hodge structure. This is remarkable, as the Hodge structures on the cohomology of the cubics themselves are trivial.

## 2. Classifying Spaces for Polarized Mixed Hodge Structures

What is a polarized mixed Hodge structure? There are several possible definitions. The simplest is to require that any quotient $\mathrm{Gr}_l^W$ of the weight filtration is equipped



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with a bilinear form which gives a polarization of the pure Hodge structure on $\mathrm{Gr}^W_l$. This is often called a graded polarized mixed Hodge structure. But Schmid's limit mixed Hodge structure [Schm] (Thm. 6.16) motivates another definition, which is given in Definition 2.2. This is also the correct one for isolated hypersurface singularities.

The main purpose of this section is to describe a classifying space $D_{\mathrm{PMHS}}$ and a moduli space $D_{\mathrm{PMHS}}/G_{\mathbb{Z}}$ for these polarized mixed Hodge structures. Similar results for graded polarized mixed Hodge structures are indicated in [Us] and [SSU], but without proof. So this section might be of interest independently of the following sections. In this paper the space $D_{\mathrm{PMHS}}$ is used only for the definition of an even bigger classifying space $D_{\mathrm{BL}}$ for Brieskorn lattices.

Definition 2.2 is based on the structure which is given in the following lemma from [Schm] (Lemma 6.4, cf. also [Gr] 255–256).

LEMMA 2.1. *Let $m \in \mathbb{N}, H_{\mathbb{Q}}$ a finite-dimensional $\mathbb{Q}$ vector space, $S$ a nondegenerate bilinear form on $H_{\mathbb{Q}}, S\colon H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$, which is symmetric for even $m$ and skewsymmetric for odd $m$ ('$(-1)^m$-symmetric'), and $N\colon H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ a nilpotent endomorphism with $N^{m+1} = 0$, which is an infinitesimal isometry, i.e. $S(Na, b) + S(a, Nb) = 0$ for $a, b \in H_{\mathbb{Q}}$.*

(a) *There exists a unique increasing filtration $0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = H_{\mathbb{Q}}$ such that $N(W_l) \subset W_{l-2}$ and such that $N^l\colon \mathrm{Gr}^W_{m+l} \to \mathrm{Gr}^W_{m-l}$ is an isomorphism.*
(b) *$S(W_l, W_{l'}) = 0$ if $l + l' < 2m$.*
(c) *A nondegenerate $(-1)^{m+l}$-symmetric bilinear form $S_l$ is well-defined on $\mathrm{Gr}^W_{m+l}$ for $l \geqslant 0$ by the requirement: $S_l(a, b) = S(\tilde{a}, N^l\tilde{b})$ if $\tilde{a}, \tilde{b} \in W_{m+l}$ represent $a, b \in \mathrm{Gr}^W_{m+l}$.*
(d) *The primitive subspace $P_{m+l}(H_{\mathbb{Q}})$ of $\mathrm{Gr}^W_{m+l}$ is defined by*

$$P_{m+l} = \ker(N^{l+1}\colon \mathrm{Gr}^W_{m+l} \to \mathrm{Gr}^W_{m-l-2}),$$

*if $l \geqslant 0$ and $P_{m+l} = 0$ if $l < 0$. Then*

$$\mathrm{Gr}^W_{m+l} = \bigoplus_{i \geqslant 0} N^i P_{m+l+2i},$$

*and this decomposition is orthogonal with respect to $S_l$ if $l \geqslant 0$.*

DEFINITION 2.2. A polarized mixed Hodge structure of weight $m$ (abbreviation: PMHS) is given by the following data: a lattice $H_{\mathbb{Z}}$ with $H_{\mathbb{Z}} \subset H_{\mathbb{Q}} \subset H_{\mathbb{R}} \subset H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$, a bilinear form $S$ on $H_{\mathbb{Q}}$ and an endomorphism $N$ of $H_{\mathbb{Q}}$ such that $m$, $H_{\mathbb{Q}}$, $S$, $N$, $W_\bullet$, $S_l$, $P_{m+l}$ satisfy all properties in Lemma 2.1, and a decreasing Hodge filtration $F^\bullet$ on $H_{\mathbb{C}}$ with the properties

(i) $F^\bullet\mathrm{Gr}^W_k$ gives a pure Hodge structure of weight $k$ on $\mathrm{Gr}^W_k$, i.e. $\mathrm{Gr}^W_k = F^p\mathrm{Gr}^W_k \oplus \overline{F^{k+1-p}\mathrm{Gr}^W_k}$,



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(ii) $N(F^p) \subset F^{p-1}$, i.e. $N$ is a $(-1,-1)$-morphism of mixed Hodge structures,
(iii) $S(F^p, F^{m+1-p}) = 0$,
(iv) the pure Hodge structure $F^\bullet P_{m+l}$ of weight $m+l$ on $P_{m+l}$ is polarized by $S_l$, i.e.

$(\alpha)$ $S_l(F^p P_{m+l}, F^{m+l+1-p} P_{m+l}) = 0$, and

$(\beta)$ $i^{2p-m-l} S_l(u, \bar{u}) > 0$ if $u \in F^p P_{m+l} \cap \overline{F^{m+l-p} P_{m+l}}$, $u \neq 0$.

*Remarks.* (a) $P_{m+l}$ carries a pure Hodge structure, because it is the kernel of the morphism $N^{l+1}\colon \mathrm{Gr}^W_{m+l} \to \mathrm{Gr}^W_{m-l-2}$ of pure Hodge structures. The strictness of the $(-1,-1)$-morphism $N$ also implies

$$F^p \mathrm{Gr}^w_k = \bigoplus_{j \geqslant 0} F^p N^j P_{k+2j} \quad \text{and} \quad F^p N^j P_{k+2j} = N^j F^{p+j} P_{k+2j}.$$

(b) In Lemma 2.1 the number $m$ could be replaced by some bigger number, but in Definition 2.2 the weight $m$ is essential for condition (iii). Also the assumption that $S$ is $(-1)^m$-symmetric is not important in Lemma 2.1, but essential for (iv) $(\beta)$.

(c) Condition (iii) implies (iv) $(\alpha)$, but in general it is not equivalent to (iv) $(\alpha)$. One can easily see the following. Under the assumption of all conditions except for (iii) and (iv), condition (iv) $(\alpha)$ for all $p$ and $l$ is equivalent to

$$S(F^p \cap W_{m+l}, F^{m+1-p} \cap W_{m-l}) = 0 \quad \text{for all } p \text{ and } l.$$

(d) The definition of a polarized mixed Hodge structure in [CaKa] differs from Definition 2.2 only by the omission of condition (iii).

Lemma 2.3 shows how Deligne's Hodge decomposition $I^{p,q}$ for a mixed Hodge structure as in Definition 2.2 fits together with the polarizing form $S$.

LEMMA 2.3. *For a* PMHS *as in Definition* 2.2 *let*

$$I^{p,q} := (F^p \cap W_{p+q}) \cap \left( \overline{F}^q \cap W_{p+q} + \sum_{j>0} \overline{F}^{q-j} \cap W_{p+q-j-1} \right).$$

*Then*

(a) $F^p = \bigoplus_{i,q\colon i \geqslant p} I^{i,q}, W_l = \bigoplus_{p+q \leqslant l} I^{p,q}, N(I^{p,q}) \subset I^{p-1,q-1}$.
(b) $S(I^{p,q}, I^{r,s}) = 0$ *for* $(r,s) \neq (m-p, m-q)$.

*For* $p+q \geqslant m$ *let* $I_0^{p,q}$ *be the primitive subspace of* $I^{p,q}$,

$$I_0^{p,q} = \ker(N^{p+q-m+1}\colon I^{p,q} \to I^{m-q-1,m-p-1}).$$

*Then*

(c) $I^{p,q} = \bigoplus_{j\geqslant 0} N^j I_0^{p+j,q+j}$.
(d) $S(N^i I_0^{p,q}, N^j I_0^{r,s}) = 0$ *for* $(r,s,i+j) \neq (q,p,p+q-m)$.

*Proof.* (a) [De2] (Lemma 1.2.8).
(b) From the definition we have

$$S(W_l, W_{l'}) = 0 \quad \text{if } l+l' < 2m,$$
$$S(F^p, F^r) = 0 \quad \text{if } p+r > m,$$
$$S(\overline{F}^q, \overline{F}^s) = 0 \quad \text{if } q+s > m.$$

One has to show $S(I^{p,q}, I^{r,s}) = 0$ in the following four cases.

*Case* 1: $p+r > m$.
*Case* 2: $p+r \leqslant m,\ q+s < m$.
*Case* 3: $p+r < m,\ q+s = m$.
*Case* 4: $p+r \leqslant m,\ q+s > m$.

In case 1, it follows from $I^{p,q} \subset F^p,\ I^{r,s} \subset F^r$. In cases 2 and 3, it follows from $I^{p,q} \subset W_{p+q}, I^{r,s} \subset W_{r+s}$. Case 4 is left: Let $p+r \leqslant m,\ q+s > m$.

$$S(\overline{F}^q \cap W_{p+q}, \overline{F}^s \cap W_{r+s}) = 0 \quad \text{because of } q+s > m,$$
$$S(\overline{F}^q \cap W_{p+q}, \overline{F}^{s-j} \cap W_{r+s-j-1}) = 0 \quad \text{and}$$
$$S(\overline{F}^{q-j} \cap W_{p+q-j-1}, \overline{F}^s \cap W_{r+s}) = 0$$

because of $q+s-j > m$ or $p+q+r+s-j-1 < 2m$,

$$S(\overline{F}^{q-i} \cap W_{p+q-i-1}, \overline{F}^{s-j} \cap W_{r+s-j-1}) = 0$$

because of $q+s-i-j > m$ or $p+q+r+s-i-j-2 < 2m$, thus $S(I^{p,q}, I^{r,s}) = 0$.

(c) For $p+q \geqslant m$ the mapping $N^{p+q-m}\colon I^{p,q} \to I^{m-q,m-p}$ is an isomorphism because of (a) and Lemma 2.1. This implies (c).

(d) $(r,s,i+j) = (q,p,p+q-m)$ and $(r-j,s-j,r+s) = (m-p+i,m-q+i,p+q)$ are equivalent. If $(r-j,s-j) \neq (m-p+i,m-q+i)$ then $S(N^i I_0^{p,q}, N^j I_0^{r,s}) = 0$ because of (a) and (b). So suppose $(r-j,s-j) = (m-p+i,m-q+i)$ and $r+s \neq p+q$. Then

either $\quad i+j \geqslant p+q-m+1, \quad$ thus $\ N^{i+j} I_0^{p,q} = 0$,
or $\quad i+j \geqslant r+s-m+1, \quad$ thus $\ N^{i+j} I_0^{r,s} = 0$.

$N$ is an infinitesimal isometry, therefore $S(N^i I_0^{p,q}, N^j I_0^{r,s}) = 0$. □

*Remarks.* (a) M. Saito shows [SM1](Lemma 2.8)

$$\bigoplus_q I^{p,q} = F^p \cap \left( \sum_q \overline{F}^q \cap W_{p+q} \right).$$



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(b) If we are not interested in the Hodge filtration, but just look for a decomposition of the weight filtration which harmonizes with $S$ and $N$, then Lemma 2.3 gives subspaces

$$B_l = \bigoplus_{p+q=l} I_0^{p,q} \quad \text{and} \quad V_l = \bigoplus_{j\geqslant 0} N^j B_{l+2j}$$

for $l \in \{0, 1, .., 2m\}$. In general these are not defined over $H_{\mathbb{Q}}$, but they satisfy all other properties in Lemma 2.3.

It is a tedious exercise in linear algebra to show that there exist such subspaces over $H_{\mathbb{Q}}$. One can start with a Jordan base of $H_{\mathbb{Q}}$ for $N$, apply Lemma 2.1 and refine the Jordan base inductively. The details of the proof of Lemma 2.3 are left to the reader.

LEMMA 2.4. *Let* $m,\ H_{\mathbb{Q}} \subset H_{\mathbb{C}},\ S,\ N,\ W_\bullet, P_\bullet \subset \mathrm{Gr}_\bullet^W$ *be as in Lemma* 2.1. *There exist subspaces* $B_l, V_l \subset H_{\mathbb{C}}$*, which are defined over* $H_{\mathbb{Q}}$*, and which give a decomposition of the weight filtration with the following properties.*

$$V_l = \bigoplus_{j\geqslant 0} N^j B_{l+2j}, \quad W_l = \bigoplus_{k\leqslant l} V_k,$$
$$N(V_l) \subset V_{l-2}, \quad S(V_l, V_{l'}) = 0 \quad \textit{if } l + l' \neq 2m,$$
$$S(N^i B_k, N^j B_l) = 0 \quad \textit{if } (k, k-m) \neq (l, i+j),$$
$$\textit{the mapping} \quad B_l \to B_l + W_{l-1}/W_{l-1} \subset \mathrm{Gr}_l^W$$

*is an isomorphism onto* $P_l$.

In the following, a classifying space $D_{\mathrm{PMHS}}$ for PMHS as in Definition 2.2 is defined and studied, together with many other spaces and groups.

Let $m,\ H_{\mathbb{Q}} \subset H_{\mathbb{C}},\ S,\ N,\ W_\bullet, P_\bullet \subset \mathrm{GR}_\bullet^W,\ S_l$ be as in Definition 2.2. Also let $B_l$ and $V_l$ be a fixed decomposition of the weight filtration as in Lemma 2.3. Finally, let $F_0^\bullet$ be a fixed Hodge filtration of a PMHS as in Definition 2.2, which shall serve as a reference filtration. $F_0^\bullet$ (can and) shall be chosen such that $F_0^p = \bigoplus_{l,j} F_0^p \cap N^j B_{l+2j}$, i.e. it respects the given decomposition of the weight filtration.

$$f_l^p := \dim F_0^p P_l,$$
$$\check{D}_l := \{\text{filtrations } F^\bullet P_l \text{ on } P_l \mid \text{ for all } p \ \ \dim F^p P_l = f_l^p,$$
$$S_{l-m}(F^p P_l, F^{l-p+1} P_l) = 0\},$$
$$D_l := \{\text{filtrations } F^\bullet P_l \text{ on } P_l \mid \text{ for all } p \ \ \dim F^p P_l = f_l^p,$$
$$F^\bullet P_l \text{ gives a polarized Hodge structure of weight } l \text{ on } P_l\},$$
$$\check{D}_{\mathrm{prim}} := \prod_l \check{D}_l, \qquad D_{\mathrm{prim}} := \prod_l D_l,$$



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$$\begin{aligned}
\check D_{\text{PMHS}} &:= \{\text{filtrations } F^\bullet \text{ on } H_{\mathbb C} \mid \text{ for all } p \text{ and } l \quad \dim F^pP_l = f_l^p,\\
&\qquad F^pN^jP_l = N^jF^{p+j}P_l,\ F^p\mathrm{Gr}_l^W = \textstyle\bigoplus_{j\geqslant 0} F^pN^jP_{l+2j},\\
&\qquad N(F^p)\subset F^{p-1},\ S(F^p,F^{m+1-p})=0\},\\
\check\pi\colon \check D_{\text{PMHS}} &\to \check D_{\text{prim}} \quad \text{is the canonical projection},\\
D_{\text{PMHS}} &:= \{\text{filtrations } F^\bullet \text{ on } H_{\mathbb C} \mid \text{ for all } p \text{ and } l \quad \dim F^pP_l = f_l^p,\\
&\qquad F^\bullet \text{ is the Hodge filtration of a PMHS of weight } m \text{ on } H_{\mathbb C}\},\\
\pi\colon D_{\text{PMHS}} &\to D_{\text{prim}} \quad \text{is the canonical projection},\\
G_{\mathbb Z} &:= \{\text{autom. } g\colon H_{\mathbb Z}\to H_{\mathbb Z} \mid g\circ N = N\circ g,\ S\circ(g\times g) = S\},\\
&\quad \text{analogously } G_{\mathbb R},\ G_{\mathbb C},\\
G'_{\mathbb C} &:= \{g\in G_{\mathbb C}\mid \mathrm{Gr}^W g = \mathrm{id}\},\quad G'_{\mathbb R} := G'_{\mathbb C}\cap G_{\mathbb R},\\
G''_{\mathbb C} &:= \{g\in G_{\mathbb C}\mid \text{ for all } l \ \ g(B_l) = B_l\},\quad G''_{\mathbb R} := G''_{\mathbb C}\cap G_{\mathbb R},\\
G^l_{\mathbb C} &:= \{\text{autom. } g\colon B_l\to B_l \mid S_{l-m}\circ(g\times g) = S_{l-m}\},\\
G^l_{\mathbb R} &:= \{\text{autom. } g\colon B_l\cap H_{\mathbb R}\to B_l\cap H_{\mathbb R} \mid S_{l-m}\circ(g\times g) = S_{l-m}\},\\
B_{\mathbb C} &:= \mathrm{Stab}(F_0^\bullet) = \{g\in G_{\mathbb C}\mid g(F_0^\bullet) = F_0^\bullet\},\\
B'_{\mathbb C} &:= G'_{\mathbb C}\cap B_{\mathbb C},\ B''_{\mathbb C} := G''_{\mathbb C}\cap B_{\mathbb C},\ B''_{\mathbb R} := G''_{\mathbb R}\cap B_{\mathbb C}.
\end{aligned}$$

The next eight statements are well known or follow from the definitions.

(A) $G''_{\mathbb C}$ is canonically isomorphic to $\Pi_l G^l_{\mathbb C}$ and semisimple. $G'_{\mathbb C}$ is unipotent and a normal subgroup of $G_{\mathbb C}$. The semidirect product $G_{\mathbb C} = G'_{\mathbb C} \rtimes G''_{\mathbb C}$ is a Levi decomposition of $G_{\mathbb C}$.

(B) $B_{\mathbb C} = B'_{\mathbb C}\rtimes B''_{\mathbb C}$ because of the special choice of the reference filtration $F_0^\bullet$.

(C) $\check D_l$ and $D_l$ are the classifying spaces for Hodge filtrations, which are defined in [Schm] and [Gr]. $G^l_{\mathbb C}$ acts on $\check D_l$, and $G^l_{\mathbb R}$ acts on $D_l$. Both actions are transitive. $\check D_l$ is a projective manifold, $D_l$ is a complex manifold and an open subset of $\check D_l$.

(D) $\check D_{\text{prim}}$ is a projective manifold and a complex homogeneous space,

$$\check D_{\text{prim}} = \prod \check D_l \cong G''_{\mathbb C}/B''_{\mathbb C} \cong G_{\mathbb C}/G'_{\mathbb C}\rtimes B''_{\mathbb C}.$$

(E) $D_{\text{prim}}$ is a complex manifold and a real homogeneous space, $D_{\text{prim}}$ is an open subset of $\check D_{\text{prim}}$,

$$D_{\text{prim}} = \prod D_l \cong G''_{\mathbb R}/B''_{\mathbb R} \cong G'_{\mathbb C}\rtimes G''_{\mathbb R}/G'_{\mathbb C}\rtimes B''_{\mathbb R}.$$

(F) The canonical projection $G_{\mathbb C}/B_{\mathbb C}\to G_{\mathbb C}/G'_{\mathbb C}\rtimes B''_{\mathbb C}\cong \check D_{\text{prim}}$, is a locally trivial holomorphic fibre bundle with fibres isomorphic to $G'_{\mathbb C}/B'_{\mathbb C}$. As $G'_{\mathbb C}$ is unipotent, the fibres are isomorphic (as complex manifolds) to $\mathbb C^{N_{\text{PMHS}}}$ for some $N_{\text{PMHS}}\in\mathbb N$, [Bo] (11.13).

(G) The restriction of the fibre bundle in (F) to the open subset $D_{\text{prim}}$ of the base $\check D_{\text{prim}}$ is canonically isomorphic to

$$G'_{\mathbb C}\rtimes G''_{\mathbb R}/B'_{\mathbb C}\rtimes B''_{\mathbb R}\to G'_{\mathbb C}\rtimes G''_{\mathbb R}/G'_{\mathbb C}\rtimes B''_{\mathbb R}\cong D_{\text{prim}}.$$



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(H) $D_{\text{PMHS}} = \check{\pi}^{-1}(D_{\text{prim}}) \subset \check{D}_{\text{PMHS}}$.

*Proof.* $D_{\text{PMHS}} \subset \check{D}_{\text{PMHS}}$ holds because $N$ is a strict morphism (Remark (a) after Definition 2.2). If $F^\bullet \in \check{\pi}^{-1}(D_{\text{prim}})$ it remains to show property (i) of Definition 2.2. This follows from $F^\bullet P_l \in D_l$ and the formula for $F^\bullet \text{Gr}_l^W$ in the definition of $\check{D}_{\text{PMHS}}$. □

So $\pi$ is the restriction of $\check{\pi}$ to $D_{\text{PMHS}} = \check{\pi}^{-1}(D_{\text{prim}})$.

PROPOSITION 2.5.

(i) *$G_{\mathbb{C}}$ acts transitively on $\check{D}_{\text{PMHS}}$. The fibre bundle $\check{\pi}$ is isomorphic to the fibre bundle in $(F)$, $G_{\mathbb{C}}/B_{\mathbb{C}} \to G_{\mathbb{C}}/G'_{\mathbb{C}} \cdot B_{\mathbb{C}}$. The group $G'_{\mathbb{C}}$ acts transitively on each fibre of $\check{\pi}$.*

(ii) *The fibre bundle $\pi$ is the restriction of $\check{\pi}$ to $D_{\text{PMHS}} = \check{\pi}^{-1}(D_{\text{prim}})$. It is isomorphic to the fibre bundle in $(G)$, $G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}/B'_{\mathbb{C}} \cdot B''_{\mathbb{R}} \to G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}/G'_{\mathbb{C}} \cdot B''_{\mathbb{R}}$. The group $G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}$ acts transitively on $D_{\text{PMHS}}$. The total space $D_{\text{PMHS}}$ is a complex manifold and a real homogeneous space, the fibres are isomorphic to $\mathbb{C}^{N_{\text{PMHS}}}$ for some $N_{\text{PMHS}} \in \mathbb{N}$.*

*Proof.* It remains to show that $G_{\mathbb{C}}$ acts transitively on $\check{D}_{\text{PMHS}}$. Let $F^\bullet \in \check{D}_{\text{PMHS}}$. As $G''_{\mathbb{C}}$ acts transitively on $\check{D}_{\text{prim}}$, there exists a $g_1 \in G''_{\mathbb{C}}$ such that $\check{\pi}(g_1(F^\bullet)) = \check{\pi}(F_0^\bullet)$ and $g_1(F^\bullet) \in D_{\text{PMHS}}$. For $g_1(F^\bullet)$ and for $F_0^\bullet$ there exist unique Hodge decompositions with the properties in Lemma 2.3. Obviously one can find a $g_2 \in G_{\mathbb{C}}$ which maps one Hodge decomposition to the other. Then $g_2(g_1(F^\bullet)) = F_0^\bullet$. □

PROPOSITION 2.6. *$G_{\mathbb{Z}}$ acts properly discontinuously on $D_{\text{PMHS}}$. Therefore the quotient space $D_{\text{PMHS}}/G_{\mathbb{Z}}$ is a normal complex space and has at most quotient singularities.*

*Proof.* In general, $B_{\mathbb{R}}$ is compact, but $G_{\mathbb{R}}$ does not act transitively on $D_{\text{PMHS}}$; and $G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}$ acts transitively, but $B'_{\mathbb{C}} \cdot B''_{\mathbb{R}}$ is not compact. So the proof is not as easy as in the case of a pure Hodge structure. The central piece of the following proof is the

OBSERVATION. $G'_{\mathbb{R}} \cap gB_{\mathbb{C}}g^{-1} = \{\text{id}\}$ *for any* $g \in G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}$.

*Proof.* $g(F_0^\bullet) \in D_{\text{PMHS}}$ for such $g$. Any $h \in G'_{\mathbb{R}} \cap gB_{\mathbb{C}}g^{-1}$ respects $g(F_0^\bullet)$ and $H_{\mathbb{R}} \subset H_{\mathbb{C}}$, so $h$ respects the Hodge decomposition of $g(F_0^\bullet)$ in Lemma 2.3. But $h \in G'_{\mathbb{R}}$ acts as identity on $\text{Gr}_\bullet^W$, so $h = \text{id}$. □

Let $K \subset G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}/B'_{\mathbb{C}} \cdot B''_{\mathbb{R}} \cong D_{\text{PMHS}}$ be a compact subset and $R := \{g \in G_{\mathbb{R}} \mid gK \cap K \neq \emptyset\}$. It is enough to show that $R$ is compact. Then $G_{\mathbb{Z}} \cap R$ is finite, and $G_{\mathbb{Z}}$ acts properly discontinuously on $D_{\text{PMHS}}$.

Let $\psi_1$ and $\psi_2$ be the canonical projections

$$G'_{\mathbb{C}} \cdot G''_{\mathbb{R}} \xrightarrow{\psi_1} G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}/B'_{\mathbb{C}} \xrightarrow{\psi_2} G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}/B'_{\mathbb{C}} \cdot B''_{\mathbb{R}}.$$



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$\psi_2$ is proper, because $B''_{\mathbb{R}}$ is compact. Thus $\psi_2^{-1}(K)$ is compact. The set

$$P := \{(g_1B'_{\mathbb{C}}, g_2B'_{\mathbb{C}}) \in \psi_2^{-1}(K) \times \psi_2^{-1}(K) \mid g_1g_2^{-1} \in G_{\mathbb{R}}\} \\ \subset \psi_2^{-1}(K) \times \psi_2^{-1}(K)$$

is compact and satisfies

$$R = \{g_1g_2^{-1} \mid (g_1B'_{\mathbb{C}}, g_2B'_{\mathbb{C}}) \in P,\ g_1g_2^{-1} \in G_{\mathbb{R}}\}.$$

CLAIM. *If* $g_1, g_2 \in G'_{\mathbb{C}} \cdot G''_{\mathbb{R}}$ *and* $b_1, b_2 \in B'_{\mathbb{C}}$ *are given such that*

$$(g_1B'_{\mathbb{C}}, g_2B'_{\mathbb{C}}) \in P,\ g_1g_2^{-1} \in G_{\mathbb{R}},\ (g_1b_1)(g_2b_2)^{-1} \in G_{\mathbb{R}}.$$

*Then* $g_1g_2^{-1} = (g_1b_1)(g_2b_2)^{-1}$.

*Proof.* There exist (unique) $\gamma \in G'_{\mathbb{C}}, \rho \in G''_{\mathbb{R}}$ such that $g_2 = \gamma \cdot \rho$.

$$\rho^{-1} \cdot (g_1g_2^{-1})^{-1}(g_1b_1)(g_2b_2)^{-1} \cdot \rho \in G_{\mathbb{R}} \quad \text{and} \\ = (\rho^{-1}\gamma\rho)(b_1b_2^{-1})(\rho^{-1}\gamma\rho)^{-1} \quad \in G'_{\mathbb{C}}.$$

The observation above implies $b_1b_2^{-1} = \text{id}$. □

The claim shows that there is a canonical well-defined surjective and continuous mapping $P \to R$. Thus $R$ is compact. □

*Remark.* Often the situation is more complicated than the one which is discussed above. For example, in the case of isolated hypersurface singularities, there is a semisimple automorphism on $H_{\mathbb{Q}}$ which commutes with $N$ and respects $S$. One can modify all the statements and constructions starting from Lemma 2.3 to Proposition 2.6 to include this semisimple automorphism. A more detailed discussion for the case of isolated hypersurface singularities is given in Section 3.

## 3. Hypersurface Singularities and Polarized Mixed Hodge Structures

In Section 3 we will fix notations for the classical topological data of a hypersurface singularity. We will describe the polarizing form for Steenbrink's mixed Hodge structure [St] on the cohomology $H^n(X_\infty, \mathbb{C})$ of the Milnor fibre. This is a canonical, but less well known nondegenerate bilinear form on $H^n(X_\infty, \mathbb{Q})$. The construction of Section 2 will be applied to yield a classifying space $D_{\text{PMHS}}$ for such polarized mixed Hodge structures.

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated singularity at 0 and Milnor number $\mu$. If we choose a sufficiently small open ball $B$ around 0 in $\mathbb{C}^{n+1}$ and a sufficiently small punctured disc $T'$ around 0 in $\mathbb{C}$, then

$$f\colon X' = f^{-1}(T') \cap B \to T'$$



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is a $C^\infty$ fibre bundle, the Milnor fibration. The fibres $X_t = f^{-1}(t) \cap B, t \in T'$, have the homotopy type of a bouquet of $\mu$ $n$-spheres [Mi]. If $u\colon T_\infty \to T'$ is the universal covering and $X_\infty = X' \times_{T'} T_\infty$, then for any $\tau \in T_\infty$ the natural inclusion $X_{u(\tau)} \hookrightarrow X_\infty$ is a homotopy equivalence. $H_n(X_\infty, \mathbb{Z}) \cong \mathbb{Z}^\mu$ is the Milnor lattice.

We have the monodromy $M$, the intersection form $q$ and the Seifert form $l$ on $H_n(X_\infty, \mathbb{Z})$ [AGV] (Ch. 2). The variation Var: $H^n(X_\infty, \mathbb{Z}) \to H_n(X_\infty, \mathbb{Z})$ is an isomorphism. Seifert form $l$ and Variation Var determine one another by

$$l(a,b) = \langle \mathrm{Var}^{-1}(a), b\rangle \quad \text{for } a,b \in H_n(X_\infty, \mathbb{Z}).$$

The Seifert form $l$ determines monodromy $M$ and intersection form $q$ by

$$l(Ma,b) = (-1)^{n+1} l(b,a) \quad \text{and} \quad q(a,b) = -l(a,b) + (-1)^{n+1} l(b,a).$$

Thus any automorphism of the Milnor lattice $H_n(X_\infty, \mathbb{Z})$, which respects the Seifert form, also respects monodromy and intersection form. The group of all automorphisms of the Milnor lattice, which respect the Seifert form, will be denoted by $G_\mathbb{Z}$.

The intersection form $q$ is $(-1)^n$-symmetric, i.e. symmetric for even $n$, skewsymmetric for odd $n$. The radical of the intersection form is $\ker(M - \mathrm{id})$.

We have the monodromy $M = M_u \cdot M_s = M_s \cdot M_u$ with unipotent part $M_u$ and semisimple part $M_s$ on the cohomology $H^n(X_\infty, \mathbb{Q})$. The nilpotent part $N = \log M_u$ satisfies $N^{n+1} = 0$. We set

$$\begin{aligned} H^n(X_\infty, \mathbb{C})_\lambda &= \ker(M_s - \lambda \cdot \mathrm{id}) \subset H^n(X_\infty, \mathbb{C}), \\ H^n(X_\infty, \mathbb{C})_{\neq 1} &= \bigoplus_{\lambda \neq 1} H^n(X_\infty, \mathbb{C})_\lambda, \\ H^n(X_\infty, \mathbb{Z})_{\neq 1} &= H^n(X_\infty, \mathbb{Z}) \cap H^n(X_\infty, \mathbb{C})_{\neq 1}, \end{aligned}$$

and similarly $H^n(X_\infty, \mathbb{Z})_1,\ H^n(X_\infty, \mathbb{Q})_{\neq 1},\ H^n(X_\infty, \mathbb{Q})_1$.

The mixed Hodge structure on $H^n(X_\infty, \mathbb{C})$ was defined by Steenbrink [St], using resolutions of singularities. Varchenko [Va1] found a construction, which uses the Gauß–Manin connection, of a slightly different Hodge filtration. His construction was modified [Ph2] [SchSt] [SM1] to obtain Steenbrink's Hodge filtration. This is reproduced in Section 3. Hodge filtration and weight filtration are invariant with respect to $M_s$. The mixed Hodge structure splits into a PMHS of weight $n$ on $H^n(X_\infty, \mathbb{C})_{\neq 1}$ and a PMHS of weight $n+1$ on $H^n(X_\infty, \mathbb{C})_1$. This determines the weight filtration. To explain this shift of the index and the polarizing form $S$ for the PMHS, we will combine a result of Scherk [Sche] with the results of Schmid [Schm] and Steenbrink [St] (cf. [SchSt] for the following).

With a suitable coordinate change one can obtain [Br]



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(i) $f$ is a polynomial of arbitrary high degree,
(ii) 0 is the only singular point of the closure $Y_0$ of $f^{-1}(0)$ in $\mathbb{P}^{n+1}\mathbb{C}$.
(iii) the closure $Y_t$ of $f^{-1}(t)$ in $\mathbb{P}^{n+1}\mathbb{C}$ is smooth for $t \in T' = T'_\delta$ if $\delta$ is small enough.

Analogously to the Milnor fibration, we get a locally trivial $C^\infty$-fibration $\pi_f\colon Y' \to T'$ with

$$d = \deg f, \quad F(z_0, \ldots, z_{n+1}) = z_{n+1}^d \cdot f\left(\frac{z_0}{z_{n+1}}, \ldots, \frac{z_n}{z_{n+1}}\right),$$
$$Y = \{(z,t) \in \mathbb{P}^{n+1}\mathbb{C} \times T \mid F(z) - t \cdot z_{n+1}^d = 0\} = \{(z,t) \mid z \in Y_t, t \in T\},$$
$$\pi_f\colon Y \to T, \ (z,t) \mapsto t, \quad Y' = \pi_f^{-1}(T').$$

The monodromy $M_Y$ on the primitive part $P^n(Y_t, \mathbb{Q})$, $t \in T'$, of the middle cohomology of a regular fibre is quasiunipotent. $M_{Y,s}$ and $M_{Y,u}$ are the semisimple and unipotent part of $M_Y$. The nilpotent part $N_Y = \log M_{Y,u}$ satisfies $N_Y^{n+1} = 0$. There is a $(-1)^n$-symmetric nondegenerate intersection form $q_Y^*$ on $P^n(Y_t, \mathbb{Q})$; we set $S_Y := (-1)^{n(n-1)/2} \cdot q_Y^*.Y_\infty = Y' \times_{T'} T_\infty$ is defined analogously to $X_\infty$. For any $t \in T'$ the embedding $Y_{u(\tau)} \hookrightarrow Y_\infty$ is a homotopy equivalence.

$N_Y$ determines a weight filtration $W_\bullet$ on $P^n(Y_\infty, \mathbb{Q})$ with index $m = n$ as in Lemma 2.1. The pure Hodge structures on the primitive cohomology groups $P^n(Y_t, \mathbb{Q})$, $t \in T'$, are polarized by $S_Y$ and give a variation of Hodge structures in the sense of [Schm]. This induces a holomorphic mapping $\tau \mapsto F^\bullet_{u(\tau)}$ from the universal cover $T_\infty$ of $T'$ to a classifying space for Hodge filtrations on $P^n(Y_\infty)$, which satisfies $F^\bullet_{u(\tau+1)} = M_Y^{-1} F^\bullet_{u(\tau)}$. Following Schmid, the limit filtration

$$F^\bullet_\infty = \lim_{Im\tau \to \infty} \exp(N_Y \cdot \tau) F^\bullet_{u(\tau)}$$

on $P^n(Y_t, \mathbb{C})$ is well-defined.

THEOREM 3.1 ([Schm], (6.16)). *$S_Y$, $N_Y$, $W_\bullet$ and $F^\bullet_\infty$ give a* PMHS *of weight $n$ on $P^n(Y_\infty)$. It is invariant with respect to $M_{Y,s}$.*

Following Steenbrink, there is an exact sequence

$$0 \to H^n(Y_0) \to H^n(Y_\infty) \to H^n(X_\infty) \to H^{n+1}(Y_0) \to H^{n+1}(Y_\infty) \to 0.$$

The result of Scherk simplifies the situation.

THEOREM 3.2 ([Sche]). *If $f$ is a polynomial of sufficiently high degree with the properties* (ii) *and* (iii) *from above, then the mapping $i^*\colon P^n(Y_\infty) \to H^n(X_\infty)$ is surjective and the kernel is* $\ker i^* = \ker(M_Y - \mathrm{id})$.

THEOREM 3.3 ([St]). *Let $f$ be as in Theorem 3.2. The sequence*



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$$0 \to \ker(M_Y - \mathrm{id}) \to P^n(Y_\infty) \to H^n(X_\infty) \to 0$$

*is an exact sequence of mixed Hodge structures. Here $P^n(Y_\infty)$ carries Schmid's mixed Hodge structure, $H^n(X_\infty)$ carries Steenbrink's mixed Hodge structure. The mixed Hodge structures are invariant with respect to the semisimple parts of the monodromies.*

Theorem 3.3 explains the shift of the index of the weight filtration on $H^n(X_\infty)_1$. It also shows, how one has to define a nondegenerate bilinear form $S$ on $H^n(X_\infty, \mathbb{Q})$, which is invariant with respect to $M_s$, and which leads to a PMHS of weight $n$ on $H^n(X_\infty, \mathbb{C})_{\neq 1}$ and a PMHS of weight $n+1$ on $H^n(X_\infty, \mathbb{C})_1$.

The restriction $i^*\colon P^n(Y_\infty)_{\neq 1} \to H^n(X_\infty)_{\neq 1}$ is an isomorphism. $S_Y$ induces $S$ on $H^n(X_\infty)_{\neq 1}$. One can express this part of $S$ in terms of the intersection form $q$. The intersection form $q$ on $H_n(X_\infty, \mathbb{Q})_{\neq 1}$ is nondegenerate and induces an isomorphism $H^n(X_\infty, \mathbb{C})_{\neq 1} \cong H_n(X_\infty, \mathbb{C})_{\neq 1}$ and a nondegenerate bilinear form $q^*$ on $H^n(X_\infty, \mathbb{C})_{\neq 1}$. Then $S = (-1)^{n(n-1)/2} \cdot q^*$.

The restriction of $S$ to $H^n(X_\infty)_1$ is defined by $S(a,b) := S_Y(\tilde{a}, N_Y \tilde{b})$ for $a, b \in H^n(X_\infty)_1$, $\tilde{a}, \tilde{b} \in P^n(Y_\infty)_1$ such that $i^*\tilde{a} = a, i^*\tilde{b} = b$. This $S$ is well-defined and nondegenerate because of $\ker i^* = \ker N_Y \cap P^n(Y_\infty)_1$ and Lemma 2.1 for $N_Y$ and $S_Y$ on $P^n(Y_\infty)_1$. If one compares the pairings $S_l$ in Definition 2.2 with Theorem 2.5, one sees, that this $S$ is the right one for a PMHS of weight $n+1$ on $H^n(X_\infty)_1$.

The following lemma shows that $S$ is determined by the variation and the monodromy and is independent of the choice of the projective fibration $Y' \to T'$. We define a monodromy invariant isomorphism $\nu\colon H^n(X_\infty, \mathbb{Q}) \to H^n(X_\infty, \mathbb{Q})$ by

$$\nu = (M - \mathrm{id})^{-1} \quad \text{on } H^n(X_\infty, \mathbb{Q})_{\neq 1},$$

$$\nu = \sum_{l \geqslant 1} \frac{1}{l}(-1)^{l-1}(M - \mathrm{id})^{l-1} \quad \left(`= \frac{N}{M - \mathrm{id}}'\right) \quad \text{on } H^n(X_\infty, \mathbb{Q})_1.$$

LEMMA 3.4. *The bilinear form $S$ on $H^n(X_\infty, \mathbb{Q})$ is nondegenerate and invariant with respect to the monodromy. It is given by $S(a,b) = (-1)^{n(n-1)/2}\langle a, \mathrm{Var} \circ \nu(b)\rangle$ for $a, b \in H^n(X_\infty, \mathbb{Q})$. The restriction of $S$ to $H^n(X_\infty, \mathbb{Q})_{\neq 1}$ is equal to $(-1)^{n(n-1)/2} \cdot q^*$ and $(-1)^n$-symmetric. The restriction of $S$ to $H^n(X_\infty, \mathbb{Q})_1$ is $(-1)^{n+1}$-symmetric.*

*Proof.* The formula for the restriction to $H^n(X_\infty, \mathbb{Q})_{\neq 1}$ follows from the definition of Var. The formula for the restriction to $H^n(X_\infty, \mathbb{Q})_1$ follows from

$$N_Y = (M_{Y,u} - \mathrm{id}) \left( \sum_{l \geqslant 1} \frac{1}{l}(-1)^{l-1}(M_{Y,u} - \mathrm{id})^{l-1} \right)$$



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and

$$i_* \circ \text{Var} \circ i^* = \text{PD} \circ (M_Y - \text{id}),$$

where

$$i_*\colon H_n(X_\infty) \hookrightarrow P_n(Y_\infty),$$

and

$$\text{PD: } P^n(Y_\infty) \xrightarrow{\cong} P_n(Y_\infty) \text{ is Poincaré duality.}$$ □

*Remark.* $S$ induces an isomorphism $H^n(X_\infty, \mathbb{Q}) \cong H_n(X_\infty, \mathbb{Q})$ and a nondegenerate bilinear form $S_*$ on $H_n(X_\infty, \mathbb{Q})$. The restriction of $S_*$ to $H_n(X_\infty, \mathbb{Q})_{\neq 1}$ is $S_* = (-1)^{n(n-1)/2} \cdot q$. It is not difficult to compute the restriction of $S_*$ to $H_n(X_\infty, \mathbb{Q})_1$ in terms of the Seifert form:

$$S_*(a, b) = (-1)^{n(n-1)/2} l \left( \left( \sum_{l \geqslant 1} \frac{1}{l!} N^{l-1} \right) a, b \right)$$

for $a, b \in H_n(X_\infty, \mathbb{Q})_1$. Thus

$$S_*(N\, a, b) = (-1)^{n(n-1)/2} l((M_u - \text{id})a, b) = (-1)^{n(n-1)/2} q(a, b).$$

From the definition of $S$ and the previous theorems follows:

THEOREM 3.5. *Steenbrink's mixed Hodge structure and $S$ yield a* PMHS *of weight $n$ on $H^n(X_\infty, \mathbb{Z})_{\neq 1}$ and a* PMHS *of weight $n + 1$ on $H^n(X_\infty, \mathbb{Z})_1$.*

This sum of two PMHS's will also be called a PMHS. It is invariant with respect to the semisimple part $M_s$ of the monodromy. To obtain a classifying space for such PMHS's, one has to include $M_s$ and modify all definitions, statements and proofs in Section 2 from Lemma 2.3 to Proposition 2.6. This is not difficult, so only a few comments are necessary.

The spaces $I^{p,q}$ and $I_0^{p,q}$ of the Hodge decomposition in Lemma 2.3 are invariant under $M_s$. One has to show that the subspaces $B_l$ in Lemma 2.4 can be chosen as invariant spaces with respect to $M_s$. Let $P_l = \bigoplus_\lambda P_{\lambda,l}$ be the decomposition of $P_l$ into generalized eigenspaces with respect to $M_s$. The conditions $\dim F^p P_l = f_l^p$ have to be replaced by $\dim F^p P_{\lambda,l} = f_{\lambda,l}^p$. Now $D_{\text{prim}}$ is the product of classifying spaces for those pure polarized Hodge structures on the primitive subspaces $P_{\lambda,k} + P_{\bar{\lambda},k}$, which respect the decomposition into the eigenspaces $P_{\lambda,k}$ and $P_{\bar{\lambda},k}$. In the same manner, we can define the other classifying spaces. From now on the fibration $D_{\text{PMHS}} \to D_{\text{prim}}$ is denoted by $\pi_{\text{PMHS}}$.



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All the groups have to respect $M_s$. The group $G_{\mathbb{Z}}$ is the group of automorphisms of the Milnor lattice, which respect the Seifert form. It is canonically isomorphic to the group of automorphisms of $H^n(X_\infty, \mathbb{Z})$, which respect $M_s$, $N$ and $S$. The quotient $D_{\text{PMHS}}/G_{\mathbb{Z}}$ parametrizes the isomorphism classes of PMHS's on $H^n(X_\infty, \mathbb{Z})$. All the statements of Propositions 2.5 and 2.6 carry over to the situation which is considered here. The next proposition summarizes some of them.

PROPOSITION 3.6. *Let $f$ be an isolated hypersurface singularity with Hodge filtration $F_0^\bullet$ on $H^n(X_\infty, \mathbb{C})$. The space $D_{\text{PMHS}}$ is the classifying space of all Hodge filtrations on $H^n(X_\infty, \mathbb{C})$ which have the same Hodge numbers $f^p_{\lambda,l}$ as $F_0^\bullet$, and which give a* PMHS *with the same properties with respect to $N$, $S$ and $M_s$.*

(a) *$D_{\text{PMHS}}$ is a complex manifold and a homogeneous space with respect to a real Lie group. $\pi_{\text{PMHS}}\colon D_{\text{PMHS}} \to D_{\text{prim}}$ is a locally trivial fibre bundle with fibres isomorphic to $\mathbb{C}^{N_{\text{PMHS}}}$, $N_{\text{PMHS}} \in \mathbb{N}$.*
(b) *The group $G_{\mathbb{Z}}$ acts properly discontinuously on $D_{\text{PMHS}}$. The moduli space $D_{\text{PMHS}}/G_{\mathbb{Z}}$ for the isomorphism classes of polarized mixed Hodge structures on $H^n(X_\infty)$ is a normal complex space and has only quotient singularities.*

*Remark.* Varchenko [Va2] proved that the spectral numbers and spectral pairs (cf. Section 4) are constant within a $\mu$-constant family of singularities. Thus also the Hodge numbers $f^p_{\lambda,l}$ are constant. We have a period mapping from the parameter space of the $\mu$-constant family to the moduli space $D_{\text{PMHS}}/G_{\mathbb{Z}}$. This period mapping is locally liftable to $D_{\text{PMHS}}$.

## 4. Gauß–Manin Connection and Brieskorn Lattice

As in Section 3 let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a hypersurface singularity and $f\colon X' \to T'$ a Milnor fibration. The Brieskorn lattice of $f$ is $H_0'' = H_0''(f) = \Omega^{n+1}_{X,0}/\,\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,0}$ [Br]. Brieskorn lattice and Gauß–Manin connection determine Steenbrink's Hodge filtration on $H^n(X_\infty, \mathbb{C})$ [Va1] [Ph2] [SchSt]. The Brieskorn lattice induces an invariant of the right equivalence class of $f$, which is finer than the polarized mixed Hodge structure on $H^n(X_\infty)$. A classifying space for this invariant will be studied in Section 5.

Here, in Section 4, we will summarize the properties of the Gauß–Manin connection and the Brieskorn lattice. There are other presentations of these properties [He1] [He2] [SM1] [SM2] [SchSt]. The summary here is as elementary and short as possible. There is a new, explicit description of the relation between the pairing $S$ and K. Saito's higher residue pairing (4.1 and 4.4).

The cohomology bundle $H^n = \bigcup_{t \in T'} H^n(X_t, \mathbb{C})$ is a flat complex vector bundle. $H^n(X_\infty, \mathbb{C})$ can be identified with the space of the global flat many-valued sections in $H^n$. If $A \in H^n(X_\infty, \mathbb{C})_\lambda$ and $\alpha \in \mathbb{Q}$ such that $\mathrm{e}^{-2\pi i \alpha} = \lambda$, then



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$$s(A,\alpha)(t) = t^\alpha \exp\left(\log t \frac{-N}{2\pi i}\right) A(t)$$

is a unique holomorphic section in $H^n$. Let $\mathcal{H}^n$ be the sheaf of germs of holomorphic sections in $H^n$ and $i\colon T' \to T$ the inclusion. The germs $s(A,\alpha)_0 \in (i_*\mathcal{H}^n)_0$ in 0 of the sections $s(A,\alpha)(t)$ span a $\mathbb{C}\{t\}[t^{-1}]$-vector space $\mathcal{G}_0$ of dimension $\mu$. This space $\mathcal{G}_0$ is invariant with respect to the differential operator $\partial_t\colon (i_*\mathcal{H}^n)_0 \to (i_*\mathcal{H}^n)_0$, which is induced by the covariant derivative. $\mathcal{G}_0$ is a regular singular $\mathbb{C}\{t\}[t^{-1}]$-module; it is called the Gauß–Manin connection [Ph1]. The mapping

$$\psi_\alpha\colon H^n(X_\infty,\mathbb{C})_\lambda \to \mathcal{G}_0, \psi_\alpha(A) = s(A,\alpha)_0$$

is injective, the image $\psi_\alpha(H^n(X_\infty,\mathbb{C})_\lambda)$ is

$$C_\alpha = \ker(t\partial_t - \alpha)^{n+1} \subset \mathcal{G}_0.$$

These subspaces $C_\alpha$ are the key to understand the structure of $\mathcal{G}_0$. The mapping $\psi_\alpha$ satisfies

$$(t\partial_t - \alpha)\circ\psi_\alpha = \psi_\alpha \circ \left(\frac{-N}{2\pi i}\right) \quad \text{and} \quad t\circ\psi_\alpha = \psi_{\alpha+1}.$$

$t\colon C_\alpha \to C_{\alpha+1}$ is bijective, and $\partial_t\colon C_\alpha \to C_{\alpha-1}$ is bijective if $\alpha \neq 0$. The eigenspaces $C_\alpha$ induce the decreasing $V^\bullet$-filtration on $\mathcal{G}_0$,

$$V^\alpha = \sum_{\beta\geqslant\alpha} \mathbb{C}\{t\}C_\beta = \bigoplus_{\alpha\leqslant\beta<\alpha+1} \mathbb{C}\{t\}C_\beta,$$

$$V^{>\alpha} = \sum_{\beta>\alpha} \mathbb{C}\{t\}C_\beta = \bigoplus_{\alpha<\beta\leqslant\alpha+1} \mathbb{C}\{t\}C_\beta.$$

The ring

$$R = \mathbb{C}\{\{\partial_t^{-1}\}\} = \left\{ \sum_{i\geqslant 0} a_i\partial_t^{-i} \,\middle|\, \sum_{i\geqslant 0} a_i t^i/i! \in \mathbb{C}\{t\} \right\}$$

is the ring of microdifferential operators with constant coefficients [Ph1]. It is easy to see that the subspace $\mathbb{C}\{t\}\cdot C_\alpha$ is a free $R$-module of rank $\dim_{\mathbb{C}} C_\alpha$ if $\alpha \notin \mathbb{Z}_{<0}$. The subspace $V^{>-1}$ is a free $R$-module of rank $\mu$,

$$V^{>-1} = \bigoplus_{-1<\alpha\leqslant 0} R\cdot C_\alpha = \bigoplus_{-1<\alpha\leqslant 0} \mathbb{C}\{t\}\cdot C_\alpha.$$

The mappings $\psi_\alpha,\ -1<\alpha\leqslant 0$, are put together to give an isomorphism

$$\psi = \bigoplus_{-1<\alpha\leqslant 0} \psi_\alpha\colon H^n(X_\infty,\mathbb{C}) \to \bigoplus_{-1<\alpha\leqslant 0} C_\alpha$$

of vector spaces. We use the structure of $V^{>-1}$ as $R$-module, the isomorphism $\psi$, and the bilinear form $S$ on $H^n(X_\infty,\mathbb{C})$ to define a pairing on $V^{>-1}$.

DEFINITION 4.1. The pairing $P_S\colon V^{>-1}\times V^{>-1}\to R\cdot\partial_t^{-1}$ is defined by the following properties. Let $\alpha,\beta\in(-1,0]$, $a\in C_\alpha,\ b\in C_\beta,\ g_1(\partial_t^{-1}), g_2(\partial_t^{-1})\in R$, and

$$\begin{aligned}
&P_S(a,b)=0 \quad \text{if } \alpha+\beta\notin\mathbb{Z},\\
&P_S(a,b)=\frac{1}{(2\pi i)^n}S(\psi^{-1}(a),\psi^{-1}(b))\cdot\partial_t^{-1} \quad \text{if } \alpha+\beta=-1,\\
&P_S(a,b)=\frac{1}{(2\pi i)^{n+1}}S(\psi^{-1}(a),\psi^{-1}(b))\cdot\partial_t^{-2} \quad \text{if } \alpha=\beta=0,\\
&P_S(g_1(\partial_t^{-1})a, g_2(\partial_t^{-1})b)=g_1(\partial_t^{-1})g_2(-\partial_t^{-1})P_S(a,b).
\end{aligned}$$

$P_S^{(-l)}$ is the part of $P_S$ in $\mathbb{C}\cdot\partial_t^{-l}$, i.e. $P_S(a,b)=\sum_{l\geqslant 1}P_S^{(-l)}(a,b)$ and $P_S^{(-l)}(a,b)\in\mathbb{C}\cdot\partial_t^{-l}$.

LEMMA 4.2.
(i) $P_S\colon C_\alpha\times C_\beta\to 0$ *if* $\alpha+\beta\notin\mathbb{Z},\ \alpha,\beta>-1,$ $P_S\colon C_\alpha\times C_\beta\to\mathbb{C}\cdot\partial_t^{-\alpha-\beta-2}$ *is a perfect pairing if* $\alpha+\beta\in\mathbb{Z},\ \alpha,\beta>-1.$
(ii) $P_S^{(-l)}$ is $(-1)^{n+1+l}$*-symmetric.*
(iii) $[t,P_S(a,b)]=P_S(ta,b)-P_S(a,tb)$, *i.e.*

$$(l-1)P_S^{(-l+1)}(a,b)\cdot\partial_t^{-1}=P_S^{(-l)}(ta,b)-P_S^{(-l)}(a,tb).$$

*Proof.* (i) and (ii) follow from Definition 4.1; (iii) follows with an easy calculation from $(t\partial_t-\alpha)\circ\psi_\alpha=\psi_\alpha\circ(-N/2\pi i)$. □

*Remark.* $P_S$ is the restriction of K. Saito's higher residue pairing [SK1] [SK2] to $V^{>-1}$. This follows, because the residue pairing satisfies analogous properties to 4.1, 4.2, and 4.4 and also induces a polarization of the mixed Hodge structure [SM1] Section 2. This residue pairing is defined on the Gauß–Manin system $(\int_f^{n+1}\mathcal{O}_X)_0$ [SM1] Section 2. The space $V^{>-1}$ is canonically embedded in the Gauß–Manin system. But we prefer the more elementary approach with Gauß–Manin connection $\mathcal{G}_0$, Definition 4.1, Lemma 4.2, and Proposition 4.4.

If $\omega\in\Omega_X^{n+1}$ is a holomorphic $(n+1)$-form, then the Gelfand–Leray form $\omega/\mathrm{d}f|_{X_t}$ gives a holomorphic section $s[\omega](t)$ in the cohomology bundle $H^n$,

$$s[\omega](t)=\left[\frac{\omega}{\mathrm{d}f}\Big|X_t\right]\in H^n(X_t,\mathbb{C}),\quad t\in T'.$$

For any $\omega\in\Omega_{X,0}^{n+1}$ the germ $s[\omega]_0\in(i_*\mathcal{H}^n)_0$ of the section $s[\omega](t)$ is in $\mathcal{G}_0$ [Br] and even in $V^{>-1}$ [Ma]. The kernel of the mapping $\Omega_{X,0}^{n+1}\to V^{>-1},\ \omega\mapsto s[\omega]_0$ is $\mathrm{d}f\wedge\mathrm{d}\Omega_{X,0}^{n-1}$ [Ma]. The Brieskorn lattice $H_0''=\Omega_{X,0}^{n+1}/\,\mathrm{d}f\wedge\mathrm{d}\Omega_{X,0}^{n-1}$ will be identified with its image in $V^{>-1}$.



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PROPOSITION 4.3.

(i) $\mathbb{C}\{t\}[t^{-1}] \cdot H_0'' = \mathcal{G}_0, \quad H_0'' \subset V^{>-1}$.
(ii) $tH_0'' \subset H_0''$, $H_0''$ *is a free* $\mathbb{C}\{t\}$*-module of rank* $\mu$.
(iii) $\partial_t^{-1} H_0'' \subset H_0''$, $H_0''$ *is a free* $R$*-module of rank* $\mu$.

*Proof.*

(i) [Br] and [Ma].
(ii) follows from (i) and $t \cdot s[\omega]_0 = s[f\omega]_0$.
(iii) follows from (i) and $\partial_t^{-1} s[\,\mathrm{d}\eta]_0 = s[\,\mathrm{d}f \wedge \eta]_0$ for $\eta \in \Omega^n_{X,0}$ [Br]. □

The Grothendieck residue on the Jacobi algebra induces a nondegenerate pairing $\mathrm{Res}_f$ on $\Omega^{n+1}_{X,0}/\,\mathrm{d}f \wedge \Omega^n_{X,0} = H_0''/\partial_t^{-1} H_0''$ [SK1] [SK2] [Va4].

PROPOSITION 4.4.

(i) $P_S(H_0'', H_0'') \subset R \cdot \partial_t^{-n-1}$, *i.e.* $P_S^{(-l)}(H_0'', H_0'') = 0$ *if* $1 \leqslant l \leqslant n$.
(ii) $P_S^{(-n-1)}(s[\omega_1]_0, s[\omega_2]_0) = \mathrm{Res}_f(\omega_1, \omega_2) \cdot \partial_t^{-n-1}$ *if* $\omega_1, \omega_2 \in \Omega^{n+1}_{X,0}$.

*Proof.* Statements of this type can be found in [SM1] 2.7. But they are not specific about the constants in the Definition 4.1 of $P_S$. Explicit calculations which take into account all the constants can be found in [Va4] Section 3.3. Varchenko uses a projective fibration $Y' \to T'$ like the one which we used to define $S$ in Section 3. He gives results on the sections in the bundle $\bigcup_{t \in T'} P^n(Y_t)$ and on the pairing $q_Y^*$ in $P^n(Y_t)$. One has to translate these results into statements on $H_0''$ and $P_S$ and calculate all the constants in [Va4] Section 3.3. That gives (i) and (ii). □

COROLLARY 4.5.

(i) $H_0''$ *is isotropic of maximal size with respect to the antisymmetric bilinear form* $P_S^{(-n)}$, *i.e.* $P_S^{(-n)}(h, H_0'') = 0 \iff h \in H_0''$.
(ii) $H_0'' \supset V^{n-1}$, $\dim H_0''/V^{n-1} = \frac{1}{2} \dim V^{>-1}/V^{n-1}$.

*Proof.*

(i) This follows easily from $P_S^{(-n)}(H_0'', H_0'') = 0$ (4.4(i)) and from the fact that $P_S^{(-n-1)} = \mathrm{Res}_f \cdot \partial_t^{-n-1}$ is well-defined and nondegenerate on $H_0''/\partial_t^{-1} H_0''$.
(ii) $P_S^{(-n)}(V^{>-1}, V^{n-1}) = 0$ (4.2(i)), $H_0'' \subset V^{>-1}$ (4.3(i)), and (i) imply $H_0'' \supset V^{n-1}$ and

$$H_0'' = \left( H_0'' \cap \bigoplus_{-1<\alpha<n-1} C_\alpha \right) \oplus V^{n-1}.$$



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Now $P_S^{(-n)}$ is nondegenerate on $\bigoplus_{-1<\alpha<n-1} C_\alpha$ (4.2(i)), and $H_0'' \cap \bigoplus_{-1<\alpha<n-1} C_\alpha$ is a maximal isotropic subspace. □

Varchenko [Va1] used the Gauß–Manin connection and the Brieskorn lattice to construct a mixed Hodge structure on $H^n(X_\infty, \mathbb{C})$. His construction was modified later [Ph2] [SchSt] (cf. [SM1]) to obtain Steenbrink's [St] mixed Hodge structure. The modified version can be given as follows.

If a subspace $K \subset V^{>-1}$ satisfies $\mathbb{C}\{t\}[t^{-1}] \cdot K = \mathcal{G}_0$ and $\partial_t^{-1} K \subset K$, then $K$ induces a decreasing filtration $F_K^\bullet$ on $H^n(X_\infty, \mathbb{C})$, which is invariant with respect to $M_S$, by

$$F_K^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}(V^\alpha \cap \partial_t^{n-p} K + V^{>\alpha}/V^{>\alpha}),$$
$$\alpha \in (-1, 0],\ \mathrm{e}^{-2\pi i\alpha} = \lambda.$$

PROPOSITION 4.6. *$F_{H_0''}^\bullet$ is Steenbrink's Hodge filtration $F^\bullet$.*

*Remark.* Now $N(F^p) \subset F^{p-1}$ follows from $tH_0'' \subset H_0''$, and $S(F^p, F^{n+1-p}) = 0$ on $H^n(X_\infty)_{\neq 1}$ and $S(F^p, F^{n+2-p}) = 0$ on $H^n(X_\infty)_1$ follow from $P_S(H_0'', H_0'') \subset R \cdot \partial_t^{-n-1}$.

Proposition 4.6 motivates the definition of the spectral pairs [St] [AGV]. These are equivalent to the Hodge numbers $f_{\lambda,l}^p$ of the PMHS on $H^n(X_\infty)$, but they reflect better the embedding $H_0'' \subset \mathcal{G}_0$. They are $\mu$ pairs $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$ with multiplicities $d(\alpha, l)$, so

$$Spp(f) = \Sigma d(\alpha, l)(\alpha, l) \in \mathbb{Z}[\mathbb{Q} \times \mathbb{Z}],$$

$$d(\alpha, l) = \dim \mathrm{Gr}_F^{[n-\alpha]} \mathrm{Gr}_l^W H^n(X_\infty, \mathbb{C})_\lambda \quad \text{for } \mathrm{e}^{-2\pi i\alpha} = \lambda,\ \lambda \neq 1,$$

$$d(\alpha, l) = \dim \mathrm{Gr}_F^{n-\alpha} \mathrm{Gr}_{l+1}^W H^n(X_\infty, \mathbb{C})_1 \quad \text{for } \mathrm{e}^{-2\pi i\alpha} = 1.$$

They satisfy the symmetries (any two of the symmetries determine the third)

$$d(\alpha, l) = d(n-1-\alpha, 2n-l),$$

$$d(\alpha, l) = d(2n-1-l-\alpha, l),$$

$$d(\alpha, l) = d(\alpha-n+l, 2n-l).$$

This follows from the PMHS. But, in fact, the first symmetry follows already from Proposition 4.4, in the spirit and as an extension of Corollary 4.5. Together with $V^{>-1} \supset H_0''$, it implies $d(\alpha, l) = 0$ if $\alpha \notin (-1, n)$, so (again) $H_0'' \supset V^{n-1}$, and $F^{n+1} = 0$, $F^n H^n(X_\infty, \mathbb{C})_1 = 0$. The fact $N^{n+1} = 0$ implies $d(\alpha, l) = 0$ if $l \notin [0, 2n]$. If one forgets the second entries and the weight filtration, one obtains



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the spectral numbers, $\mu$ rational numbers $\alpha$ with multiplicities $d(\alpha)$,

$$Sp(f) = \Sigma d(\alpha)(\alpha) \in \mathbb{Z}[\mathbb{Q}],$$

$$d(\alpha) = \dim \mathrm{Gr}_F^{[n-\alpha]} H^n(X_\infty)_\lambda \quad \text{if } \lambda = \mathrm{e}^{-2\pi i\alpha}$$

$$= \dim \mathrm{Gr}_V^\alpha H_0'' - \dim \mathrm{Gr}_V^\alpha \partial_t^{-1} H_0''.$$

They satisfy the symmetry $d(\alpha) = d(n-1-\alpha)$.

PROPOSITION 4.7 ([Va2]). *The spectral pairs are constant within a $\mu$-constant family of singularities.*

Thus the PMHS's of the singularities in a $\mu$-constant family are contained in the same classifying space $D_{\text{PMHS}}$. We have a period mapping from the parameter space of the $\mu$-constant family to the quotient $D_{\text{PMHS}}/G_{\mathbb{Z}}$, which is locally liftable to $D_{\text{PMHS}}$. But the discussion in Section 6 will show that this period mapping often is not good enough for Torelli type questions. One loses information if one considers only $F^\bullet_{H_0''}$ and $\mathrm{Gr}^V_\bullet H_0''$ instead of $H_0''$.

## 5. Classifying Spaces for Brieskorn Lattices

In this section, a classifying space $D_{\text{BL}}$ for Brieskorn lattices with fixed spectral pairs will be constructed. This is the technical center piece of the paper.

The canonical projection $D_{\text{BL}} \to D_{\text{PMHS}}$ will turn out to be a locally trivial bundle with fibres $\mathbb{C}^{N_{\text{BL}}}$ just as the bundle $D_{\text{PMHS}} \to D_{\text{prim}}$. The similarity with $D_{\text{PMHS}}$ is quite strong: $D_{\text{PMHS}}$ is the classifying space for the Hodge filtrations, $D_{\text{prim}}$ can be seen as the classifying space for the Hodge filtrations $F^\bullet \mathrm{Gr}^W_\bullet$ on the quotients of the weight filtration; $D_{\text{BL}}$ will be the classifying space for the Brieskorn lattices in $\mathcal{G}_0$, $D_{\text{PMHS}}$ can be seen as the classifying space for the quotients $\mathrm{Gr}^\bullet_V H_0''$ with respect to the $V^\bullet$-filtration.

But there is no transitive group action on $D_{\text{BL}}$ present. The conditions for the Brieskorn lattices, which have to be controlled, are more involved than those for the PMHS's.

The following results can be seen as a continuation of the discussion of the structure of the Brieskorn lattice in [SM1] Section 3. The existence of bases of $H_0''$ with very special properties is one of the main results of that paper. A more explicit and refined version of the construction of such bases is given in Proposition 5.1 and Lemma 5.2, the analysis in 5.3–5.6 goes beyond [SM1].

We fix several data of a singularity, which were considered in Sections 3 and 4 and which are locally constant or canonically isomorphic in $\mu$-constant families: the cohomology $H^n(X_\infty, \mathbb{Z})$ with $M_s, N, S$ on $H^n(X_\infty, \mathbb{Q})$, the Gauß–Manin connection $\mathcal{G}_0 \supset V^{>-1}$ with $t, \partial_t, V^\alpha, V^{>\alpha}, C_\alpha, P_S$, and the spectral pairs. These data determine classifying spaces $D_{\text{PMHS}}$ and $\check{D}_{\text{PMHS}}$. The $\mu$ spectral numbers



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will be indexed such that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_\mu$. Because of the symmetry, they satisfy $\alpha_i + \alpha_{\mu+1-i} = n-1$. Recall from Section 4 that any subspace $K \subset V^{>-1}$ satisfying $\partial_t^{-1} K \subset K$ induces a decreasing filtration $F_K^\bullet$ on $H^n(X_\infty, \mathbb{C})$, which is invariant with respect to $M_s$, by

$$F_K^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}(\mathrm{Gr}_V^\alpha \partial_t^{n-p} K), \quad \alpha \in (-1, 0],\ \mathrm{e}^{-2\pi i\alpha} = \lambda.$$

A classifying space for Brieskorn lattices should consist of subspaces $K \subset V^{>-1}$ having the following properties:

(i) $\partial_t^{-1} K \subset K$,
(ii) $F_K^\bullet \in D_{\mathrm{PMHS}}$,
(ii)$'$ $F_K^\bullet \in \check{D}_{\mathrm{PMHS}}$,
(iii) $t\,K \subset K$,
(iv) $P_S^{(-l)}$: $K \times K \to 0$ for $1 \leqslant l \leqslant n$.

(i) and (ii)$'$ (resp. (iii) and (ii)$'$) imply that $K$ is a free $R$-module (resp. $\mathbb{C}\{t\}$-module) of rank $\mu$. We define classifying spaces for Brieskorn lattices,

$$\begin{aligned} D_{\mathrm{BL}} &= \{K \subset V^{>-1} \mid K \text{ satisfies (i), (ii), (iii), (iv)}\}, \\ \check{D}_{\mathrm{BL}} &= \{K \subset V^{>-1} \mid K \text{ satisfies (i), (ii)}', \text{(iii), (iv)}\}, \end{aligned}$$

the canonical projection $\pi_{\mathrm{BL}}: D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$ is the restriction of the projection $\check{\pi}_{\mathrm{BL}}: \check{D}_{\mathrm{BL}} \to \check{D}_{\mathrm{PMHS}}$ to $D_{\mathrm{BL}} = \check{\pi}_{\mathrm{BL}}^{-1}(D_{\mathrm{PMHS}})$.

*Remarks.* (a) The following remark is clear from the definition, but useful: If $K_1, K_2 \subset V^{>-1}$ are subspaces such that $\partial_t^{-1} K_i \subset K_i$, $i = 1, 2$, then $\mathrm{Gr}_V^\alpha K_1 = \mathrm{Gr}_V^\alpha K_2$ for any $\alpha \iff F_{K_1}^\bullet = F_{K_2}^\bullet$.

(b) M. Saito [SM2] (2.9) considers two larger classifying spaces

$$\mathbf{L}(\mathcal{G})' = \{K \subset V^{>-1} \mid K \supset V^{n-1}\} \text{ (a union of Grassmann manifolds)},$$

$$\mathbf{L}(\mathcal{G}) = \{K \subset V^{>-1} \mid \mathrm{Gr}_V^\alpha \text{ has the right dimension for all } \alpha,$$
$$\text{and } K \text{ satisfies (i) and (iii)}\},$$

so $\check{D}_{\mathrm{BL}} \subset \mathbf{L}(\mathcal{G}) \subset \mathbf{L}(\mathcal{G})'$. Here $\mathbf{L}(\mathcal{G})$ is not only a locally closed analytic subspace of the manifold $\mathbf{L}(\mathcal{G})'$ [SM2] (2.9), but satisfies properties similar to those for $\check{D}_{\mathrm{BL}}$ in Theorem 5.6. It is a holomorphic locally trivial fibre bundle with affine fibres and smooth base $\{F_K^\bullet \mid K \in \mathbf{L}(\mathcal{G})\}$. But, as condition (iv) is not used, this base is larger than $\check{D}_{\mathrm{PMHS}}$ and the dimension of the fibres is larger than $N_{\mathrm{BL}}$.

The choice of elements $s_1, \ldots, s_\mu$ as in Proposition 5.1 is essential for the whole Section 5.

PROPOSITION 5.1. *Let $F^\bullet \in \check{D}_{\mathrm{PMHS}}$. There exist elements $s_i \in C_{\alpha_i}$, $i = 1, \ldots, \mu$, with the properties*



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($\alpha$) $s_1, \ldots, s_\mu$ *project onto a* $\mathbb{C}$*-basis of* $\bigoplus_{-1<\alpha<n} \mathrm{Gr}_V^\alpha K/\mathrm{Gr}_V^\alpha \partial_t^{-1} K$ *for (one or equivalently) any* $K$ *such that* $\partial_t^{-1}K \subset K$ *and* $F_K^\bullet = F^\bullet$.

($\beta$) $s_{\mu+1} := 0$*; there exists a map* $\nu\colon \{1, \ldots, \mu\} \to \{1, \ldots, \mu, \mu+1\}$ *with*

$$(t - (\alpha_i + 1)\partial_t^{-1})s_i = s_{\nu(i)}.$$

($\gamma$) *There exists an involution* $\kappa\colon \{1, \ldots, \mu\} \to \{1, \ldots, \mu\}$ *with* $\kappa(i) = \mu + 1 - i$ *if* $\alpha_i \neq \frac{1}{2}(n-1)$ *and* $\kappa(i) = \mu + 1 - i$ *or* $\kappa(i) = i$ *if* $\alpha_i = \frac{1}{2}(n-1)$ *and* $P_S(s_i, s_j) = \delta_{\kappa(i)j} \cdot \partial_t^{-n-1}$.

*Remarks.* (i) Condition ($\alpha$) is the most simple to obtain and the most important. It implies

$$\mathrm{Gr}_V^\alpha \partial_t^q K = \bigoplus_{\substack{i,p \\ \alpha_i - p = \alpha, p \leqslant q}} \mathbb{C} \cdot \partial_t^p s_i.$$

Condition ($\alpha$) corresponds to the notion of an opposite filtration in [SM1] Section 3.

(ii) Condition ($\beta$) is the next important. But without loosing too much, one could replace it by the weaker condition

$$(t - (\alpha_i + 1)\partial_t^{-1})s_i \in \bigoplus_{\alpha_j = \alpha_i + 1} \mathbb{C} \cdot s_j.$$

Together with ($\alpha$), that corresponds to the notion of an opposite (B)-filtration in [SM1] Section 3. With the weaker condition instead of ($\beta$), the involution $\kappa$ in ($\gamma$) can be chosen as $\kappa(i) = \mu + 1 - i$ for any $i$.

*Proof of Proposition 5.1.* It will suffice to prove the existence of $s_1, \ldots, s_\mu$ for one filtration $F^\bullet \in D_{\mathrm{PMHS}}$: any $g \in G_\mathbb{C}$ induces an automorphism of $\mathcal{G}_0$, which maps $s_1, \ldots, s_\mu$ to elements with the same properties ($\beta$) and ($\gamma$) and the analogous property ($\alpha$) for $g(F^\bullet) \in \check{D}_{\mathrm{PMHS}}$. The group $G_\mathbb{C}$ acts transitively on $\check{D}_{\mathrm{PMHS}}$.

So, let $F^\bullet \in D_{\mathrm{PMHS}}$. The proof uses Deligne's Hodge Decomposition $I^{p,q}$ and a version of Lemma 2.3, which takes into account the semisimple part $M_s$ of the monodromy.

*Some notation.* If $\lambda$ is an eigenvalue of $M_s$ then $\alpha$ denotes the number such that $\mathrm{e}^{-2\pi i\alpha} = \lambda$, $\alpha \in (-1, 0]$, and $m := n$ if $\lambda \neq 1$, $m := n+1$ if $\lambda = 1$.

Let $I_0^{p,q} = \bigoplus_\lambda (I_0^{p,q})_\lambda$ be the decomposition into eigenspaces of $M_s$. The cohomology $H^n(X_\infty, \mathbb{C})$ decomposes into $H^n(X_\infty, \mathbb{C}) = \bigoplus_{i,p,q,\lambda} N^i(I_0^{p,q})_\lambda$. We define a mapping $\Xi\colon H^n(X_\infty, \mathbb{C}) \to \bigoplus_{-1<\beta<n} \mathrm{Gr}_V^\beta K$ for $K$ as in ($\alpha$) by

$$\Xi \mid N^i(I_0^{p,q})_\lambda\colon = \partial_t^{(p-i)-n} \circ \psi_\alpha \mid N^i(I_0^{p,q})_\lambda.$$



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The composition of $\Xi$ with the canonical projection

$$\bigoplus_{-1<\beta<n} \mathrm{Gr}_V^\beta K \to \bigoplus_{-1<\beta<n} \mathrm{Gr}_V^\beta K/\mathrm{Gr}_V^\beta \partial_t^{-1} K$$

is an isomorphism. Any union of bases of the subspaces $N^i(I_0^{p,q})_\lambda$ maps under $\Xi$ to a set of elements which satisfy condition $(\alpha)$. Now observe

$$(t-(\alpha+k+1)\partial_t^{-1})\circ\partial_t^{-k}\circ\psi_\alpha = \partial_t^{-k-1}\circ\psi_\alpha\circ\left(\frac{-N}{2\pi i}\right)$$

to see that any union of bases of the primitive subspaces $(I_0^{p,q})_\lambda$ together with all nonvanishing images under $(-N/2\pi i)$ maps under $\Xi$ to a set of elements which satisfy $(\alpha)$ and $(\beta)$.

To also obtain $(\gamma)$, we will choose bases of the primitive subspaces $(I_0^{p,q})_\lambda$ with good properties with respect to $S$ and apply $\Xi$ to them and to all nonvanishing images under $(-N/2\pi i)$.

In the decomposition of $H^n(X_\infty,\mathbb{C})$, all subspaces except one are orthogonal to $N^i(I_0^{p,q})_\lambda$ with respect to $S$; the subspaces $N^i(I_0^{p,q})_\lambda$ and $N^{p+q-m-i}(I_0^{q,p})_{\overline{\lambda}}$ are dual (Lemma 2.3). The pairing

$$\frac{1}{(2\pi i)^m}(-1)^{p-(m-n)}S\left(\bullet,\left(\frac{-N}{2\pi i}\right)^{p+q-m}\bullet\right):(I_0^{p,q})_\lambda\times(I_0^{q,p})_{\overline{\lambda}}\to\mathbb{C}$$

is a perfect pairing. Because of

$$\begin{aligned}&(-1)^{p-(m-n)}S\left(v_1,\left(\frac{-N}{2\pi i}\right)^{p+q-m}v_2\right)\\ &\quad=(-1)^{q-(m-n)}S\left(v_2,\left(\frac{-N}{2\pi i}\right)^{p+q-m}v_1\right)\end{aligned}$$

for $v_1\in(I_0^{p,q})_\lambda,\ v_2\in(I_0^{q,p})_{\overline{\lambda}}$, we obtain the same pairing if we exchange $(I_0^{p,q})_\lambda$ and $(I_0^{q,p})_{\overline{\lambda}}$. If $(I_0^{p,q})_\lambda\neq(I_0^{q,p})_{\overline{\lambda}}$ we choose any two bases of $(I_0^{p,q})_\lambda$ and $(I_0^{q,p})_{\overline{\lambda}}$ which are dual with respect to this pairing. If $p=q$ and $\lambda\in\{\pm1\}$ we can choose either an orthonormal basis of $(I_0^{p,p})_{\pm1}$ with respect to this (symmetric) pairing, or a basis of $(I_0^{p,p})_{\pm1}$ such that each element of the basis is dual to another element of the basis – except for one, which is selfdual, if $\dim(I_0^{p,p})_{\pm1}$ is odd.

The union of such bases of the subspaces $(I_0^{p,q})_\lambda$ together with all nonvanishing images under $-N/2\pi i$ maps under $\Xi$ to a set of elements, which satisfy $(\alpha)$, $(\beta)$, and $(\gamma)$ if they are indexed properly. This follows from the definition of the pairing $P_S$ on $V^{>-1}$. Here it is useful to observe that

$$\begin{aligned}&P_S(a,(t-(\beta+1)\partial_t^{-1})b)\\ &\quad=P_S((t-(\alpha+1)\partial_t^{-1})a,b)\quad\text{if } a\in C_\alpha, b\in C_\beta,\end{aligned}$$

which follows from Lemma 4.2(iii) and which is a translation to $P_S$ of the fact that $N$ is an infinitesimal isometry with respect to $S$. □

The next lemma is essentially a translation of part of [SM1] 3.4 into the more explicit data which are used here.

LEMMA 5.2. *Suppose, $F^\bullet \in \check{D}_{\text{PMHS}}$ and elements $s_1, \ldots, s_\mu$ satisfying condition $(\alpha)$ of 5.1 are given. Then for any $K \subset V^{>-1}$ such that $\partial_t^{-1} K \subset K$ and $F_K^\bullet = F^\bullet$ and for any $i \in \{1, \ldots, \mu\}$, the intersection $\left(s_i + \sum_{\substack{j,p \\ p\geqslant 1, \alpha_j - p > \alpha_i}} \mathbb{C}\cdot \partial_t^p s_j\right) \cap K$ consists of a single element $h_i$,*

$$h_i = s_i + \sum_{\substack{j,p \\ p\geqslant 1, \alpha_j - p > \alpha_i}} c_{ij}^{(p)} \cdot \partial_t^p s_j.$$

*The elements $h_i, i = 1, ..., \mu$, form an $R$-basis of $K$.*

*Proof.* We start with elements $\tilde{h}_i \in (s_i + V^{>\alpha_i}) \cap K$. Any such elements $\tilde{h}_1, \ldots, \tilde{h}_\mu$ form an $R$-basis of $K$. Condition $(\alpha)$ implies

$$\mathrm{Gr}_V^\alpha K = \bigoplus_{\substack{i,p \\ p\leqslant 0, \alpha_i - p = \alpha}} \mathbb{C}\cdot \partial_t^p s_i \subset C_\alpha = \mathrm{Gr}_V^\alpha K \oplus \bigoplus_{\substack{i,p \\ p\geqslant 1, \alpha_i - p = \alpha}} \mathbb{C}\cdot \partial_t^p s_i.$$

For any $i$, we can add a suitable finite linear combination of $\{\partial_t^p \tilde{h}_j \mid \alpha_j - p > \alpha_i, p \leqslant 0\}$ to $\tilde{h}_i$ such that the sum $\tilde{\tilde{h}}_i$ is

$$\tilde{\tilde{h}}_i \in \left(s_i + \bigoplus_{\substack{j,p \\ p\geqslant 1, \alpha_j - p > \alpha_i}} \mathbb{C}\cdot \partial_t^p s_j + V^{n-1}\right) \cap K.$$

As $K \supset V^{n-1}$ (because of $F_K^\bullet = F^\bullet$), there exists an element $h_i$ as claimed. The uniqueness can be seen at once if one looks at the difference of two such elements. □

In the situation of Lemma 5.2 we set $c_{ij}^{(p)} := 0$ for $i, j \in \{1, \ldots, \mu\}$, $p \geqslant 1$, if $\alpha_j - p \leqslant \alpha_i$. We obtain an infinite sequence of $\mu \times \mu$-matrices $(c_{ij}^{(p)})_{ij}$ of which only the first $n$ can have entries $\neq 0$, because $\alpha_j - p > \alpha_i$ implies $p \leqslant n$.

There is a canonical $\mathbb{C}^*$-action on $\mathcal{G}_0$, given by

$$c * \sum \sigma_\alpha := \sum c^{\mathrm{ord}_M \cdot (-\alpha)} \cdot \sigma_\alpha \quad \text{if } \sigma_\alpha \in C_\alpha,\ c \in \mathbb{C}^*,$$

$$\mathrm{ord}_M := \min\,(l \mid M_s^l = \mathrm{id}) = \gcd(\text{ denominators of } \alpha_1, \ldots, \alpha_\mu).$$



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In general, $c*$ is not an automorphism of the Gauß–Manin connection $\mathcal{G}_0$, but it satisfies

$$(c*) \circ \partial_t = c^{\operatorname{ord}_M} \cdot \partial_t \circ (c*),$$
$$(c*) \circ t = c^{-\operatorname{ord}_M} \cdot t \circ (c*),$$
$$P_S^{(-l)}(c*a, c*b) = c^{\operatorname{ord}_M \cdot (-l+2)} \cdot P_S^{(-l)}(a,b), \qquad \text{if } a, b \in V^{>-1}.$$

The induced action

$$c*K = \{c*\sigma \mid \sigma \in K\}, \quad c \in \mathbb{C}^*,\ K \subset V^{>-1},$$

on the set of subsets of $V^{>-1}$ satisfies

$$F^\bullet_{c*K} = F^\bullet_K, \partial_t^{-1}(c*K) = c*(\partial_t^{-1}K), t \cdot (c*K) = c*(t\,K),$$
$$P_S^{(-l)}(c*K, c*K) = 0 \quad \text{if } P_S^{(-l)}(K,K) = 0.$$

The sets $\{K \subset V^{>-1} \mid \partial_t^{-1}K \subset K, F_K^\bullet = F^\bullet\}$ and $\check{\pi}_{\mathrm{BL}}^{-1}(F^\bullet)$ for fixed $F^\bullet \in \check{D}_{\mathrm{PMHS}}$ are invariant under this $\mathbb{C}^*$-action.

COROLLARY 5.3. *Suppose, $F^\bullet \in \check{D}_{\mathrm{PMHS}}$ and elements $s_1, \ldots, s_\mu$ satisfying condition $(\alpha)$ of* 5.1 *are given, and*

$$N_1 := \sharp\{(i,j,p) \mid 1 \leqslant i, j \leqslant \mu,\ p \geqslant 1,\ \alpha_j - p > \alpha_i\}.$$

*Lemma* 5.2 *yields a mapping*

$$\{K \subset V^{>-1} \mid \partial_t^{-1}K \subset K,\ F_K^\bullet = F^\bullet\} \to \mathbb{C}^{N_1}, K \mapsto (c_{ij}^{(p)} \mid \alpha_j - p > \alpha_i).$$

*This mapping is bijective. It induces a canonical affine algebraic structure on the set $\{K \subset V^{>-1} \mid \partial_t^{-1}K \subset K,\ F_K^\bullet = F^\bullet\}$. The $\mathbb{C}^*$-action on this affine algebraic space has negative weights* $\operatorname{ord}_M \cdot (\alpha_i - (\alpha_j - p))$.

*Proof.* Because of condition $(\alpha)$, the elements $s_i + \Sigma_{j,p} c_{ij}^{(p)} \cdot \partial_t^p s_j, i = 1, \ldots, \mu$, generate a free $R$-module of rank $\mu$ for any choice of $(c_{ij}^{(p)} | 1 \leqslant i, j \leqslant \mu,\ p \geqslant 1,\ \alpha_j - p > \alpha_i) \in \mathbb{C}^{N_1}$. With Lemma 5.2, this yields the isomorphism to $\mathbb{C}^{N_1}$. This isomorphism induces a $\mathbb{C}^*$-action on $\mathbb{C}^{N_1}$ with weight $\operatorname{ord}_M \cdot (\alpha_i - (\alpha_j - p)) < 0$ for the coordinate $c_{ij}^{(p)}$. □

By Corollary 5.3 the space $\{K \subset V^{>-1} | \partial_t^{-1}K \subset K, F_K^\bullet = F^\bullet\}$ for fixed $F^\bullet \in \check{D}_{\mathrm{PMHS}}$ is equipped with a system of coordinates. The equations for the conditions $t\,K \subset K$ and $P_S(K,K) \subset R \cdot \partial_t^{-n-1}$ in these coordinates are not independent of one another. The equations for $t\,K \subset K$ will be given in Proposition 5.4 and in Proposition 5.5 additional independent equations for $t\,K \subset K$ and $P_S(K,K) \subset R \cdot \partial_t^{-n-1}$ will be given.



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PROPOSITION 5.4. *Suppose, the following are given: A subspace* $K \subset V^{>-1}$ *with* $\partial_t^{-1}K \subset K$, $F_K^\bullet \in \check{D}_{\text{PMHS}}$; *elements* $s_1, \ldots, s_\mu$ *and a map* $\nu\colon \{1, \ldots, \mu\} \to \{1, \ldots, \mu+1\}$, *which satisfy conditions* $(\alpha)$ *and* $(\beta)$ *of 5.1; coefficients* $c_{ij}^{(p)}$ *and elements* $h_i$ *as in 5.2. Then* $t\,K \subset K$ *holds if and only if for* $p \geqslant 2$

$$c_{ik}^{(p)} \cdot (\alpha_k - p - \alpha_i) = -c_{i\nu^{-1}(k)}^{(p-1)} + c_{\nu(i)k}^{(p-1)} + \sum_j (\alpha_j - 1 - \alpha_i) \cdot c_{ij}^{(1)} \cdot c_{jk}^{(p-1)}.$$

*(Here the first summand* $-c_{i\nu^{-1}(k)}^{(p-1)}$ *is meaningful only if* $k \in \nu(\{1, \ldots, \mu\})$ *and has to be omitted otherwise.) Hence, if* $t\,K \subset K$, *then the coefficients* $c_{ij}^{(1)}$ *determine all the higher coefficients* $c_{ij}^{(p)}$, $p \geqslant 2$, *recursively. Furthermore, then*

$$t\,h_i = (\alpha_i + 1)\partial_t^{-1}h_i + h_{\nu(i)} + \sum_j c_{ij}^{(1)}(\alpha_j - 1 - \alpha_i)h_j.$$

*Proof.* $K = \bigoplus_{i=1}^{\mu} R \cdot h_i$. The condition $t\,K \subset K$ is equivalent to $t\,h_i \in K$ for all $i = 1, \ldots, \mu$.

$$\begin{aligned} t\,h_i &= (\alpha_i + 1)\partial_t^{-1}s_i + s_{\nu(i)} + \sum_{j,p} c_{ij}^{(p)}((\alpha_j - p + 1)\partial_t^{p-1}s_j + \partial_t^p s_{\nu(j)}) \\ &= (\alpha_i + 1)\partial_t^{-1}h_i + h_{\nu(i)} + \sum_j c_{ij}^{(1)}((\alpha_j - 1 - \alpha_i)h_j + \\ &\quad + \sum_{\substack{j,p\\ p\geqslant 2}} c_{ij}^{(p)}(\alpha_j - p - \alpha_i)\partial_t^{p-1}s_j + \sum_{j,p} c_{ij}^{(p)}\partial_t^p s_{\nu(j)} - \\ &\quad - \sum_{j,p} c_{\nu(i)j}^{(p)}\partial_t^p s_j - \sum_j c_{ij}^{(1)}(\alpha_j - 1 - \alpha_i) \cdot \sum_{k,p} c_{jk}^{(p)}\partial_t^p s_k. \end{aligned}$$

The sum of the last four terms vanishes if and only if $t\,h_i \in K$. This yields the recursive formulas for the coefficients $c_{ij}^{(p)}$, $p \geqslant 2$, and the formula for $t\,h_i$. □

*Remark.* The core of [SM1] Section 3 consists of the choice of elements $s_1, \ldots, s_\mu$, which satisfy conditions $(\alpha)$ and $(\beta)$ of 5.1 (or $(\alpha)$ and the weaker condition $(t - (\alpha_i + 1)\partial_t^{-1})s_i \in \bigoplus_{\alpha_j = \alpha_i + 1} \mathbb{C} \cdot s_j$, see the remark after 5.1), the existence and uniqueness of elements $h_i$ as in 5.2, and the identity

$$t\,h_i \in (\alpha_i + 1)\partial_t^{-1}h_i + \sum_{\alpha_j - 1 \geqslant \alpha_i} \mathbb{C} \cdot h_j$$

for $K$ such that $F_K^\bullet = F^\bullet$, $\partial_t^{-1}K \subset K$, $t\,K \subset K$.

PROPOSITION 5.5. *Suppose, the following are given: A subspace* $K \subset V^{>-1}$ *such that* $\partial_t^{-1}K \subset K$, $F_K^\bullet \in \check{D}_{\text{PMHS}}$, $t\,K \subset K$; *elements* $s_1, \ldots, s_\mu$ *and maps* $\nu$, $\kappa$ *which satisfy* $(\alpha), (\beta)$, *and* $(\gamma)$ *of* 5.1; *coefficients* $c_{ij}^{(p)}$ *and elements* $h_i$ *as in* 5.2. *Then*



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(i) $P_S^{(-n)}(h_i,h_j)=0$ *for all* $i,j \iff P_S^{(-l)}(K,K)=0$ *for all* $1\leqslant l\leqslant n$,
(ii) $P_S^{(-n)}(h_i,h_j)=0 \iff c^{(1)}_{i\kappa(j)}=c^{(1)}_{j\kappa(i)}$.

*Proof.* (i)

CLAIM. *Let* $l\leqslant n$. *Then* $P_S^{(-l)}(h_i,h_j)=0$ *for all* $i,j$ *implies* $P_S^{(-l+1)}(h_i,h_j)=0$ *for all* $i,j$.

The claim and the assumption $P_S^{(-n)}(h_i,h_j)=0$ for all $i,j$ yield inductively

$$P_S^{(-l)}(h_i,h_j)=0 \ \text{ for all } \ i,j\in\{1,\dots,\mu\},\ l\in\{1,\dots,n\}.$$

This implies $P_S^{(-l)}(K,K)=0$ for $l\in\{1,\dots,n\}$ because of the $R$-sesquilinearity.

*Proof of the claim.* If $\alpha_i+\alpha_j>l-3$, then $P_S^{(-l+1)}(h_i,h_j)=0$ because of $P_S\colon C_\alpha\times C_\beta\to\mathbb{C}\cdot\partial_t^{-\alpha-\beta-2}$. If $\alpha_i+\alpha_j=l-3$, then $P_S^{(-l+1)}(h_i,h_j)=P_S^{(-l+1)}(s_i,s_j)=0$, because $P_S(s_i,s_j)=P_S^{(-n-1)}(s_i,s_j)\in\mathbb{C}\cdot\partial_t^{-n-1}$ (condition $\gamma$)). If $\alpha_i+\alpha_j<l-3$, then

$$\begin{aligned}
&(l-1)\partial_t^{-1}P_S^{(-l+1)}(h_i,h_j)\\
&=[t,P_S^{(-l+1)}(h_i,h_j)]\\
&=P_S^{(-l)}(t\,h_i,h_j)-P_S^{(-l)}(h_i,t\,h_j)\\
&=P_S^{(-l)}((\alpha_i+1)\partial_t^{-1}h_i,h_j)-P_S^{(-l)}(h_i,(\alpha_j+1)\partial_t^{-1}h_j)\\
&=(\alpha_i+\alpha_j+2)\partial_t^{-1}P_S^{(-l+1)}(h_i,h_j),
\end{aligned}$$

so, also in this case $P_S^{(-l+1)}(h_i,h_j)=0$. Here we have used the formula $t\,h_i\in(\alpha_i+1)\partial_t^{-1}h_i+\sum_{\alpha_j-1\geqslant\alpha_i}\mathbb{C}\cdot h_j$ from 5.4 and the hypothesis $P_S^{(-l)}(h_i,h_j)=0$.

*Proof.* (ii)

$$\begin{aligned}
&P_S^{(-n)}(h_i,h_k)\\
&=P_S^{(-n)}\left(s_i+\sum_{j,p}c_{ij}^{(p)}\partial_t^p s_j\,,\ s_k+\sum_{l,q}c_{kl}^{(q)}\partial_t^q s_l\right)\\
&=\partial_t\cdot P_S^{(-n-1)}\left(\sum_j c_{ij}^{(1)}s_j,s_k\right)+(-\partial_t)\cdot P_S^{(-n-1)}\left(s_i,\sum_l c_{kl}^{(1)}s_l\right)
\end{aligned}$$



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$$= \partial_t^{-n}(c^{(1)}_{i\kappa(k)} - c^{(1)}_{k\kappa(i)}). \qquad \square$$

THEOREM 5.6. $\check{\pi}_{\rm BL}\colon \check{D}_{\rm BL} \to \check{D}_{\rm PMHS}$ *is a locally trivial holomorphic bundle with fibres isomorphic to* $\mathbb{C}^{N_{\rm BL}}$,

$$\begin{aligned} N_{\rm BL} &= \sharp\{(i,j) \mid 1 \leqslant i \leqslant j \leqslant \mu,\ \alpha_i + \alpha_j < n-2\} \\ &= \sum_{\alpha+\beta<n-2,\alpha<\beta} d(\alpha)\cdot d(\beta) + \sum_{2\alpha<n-2} \tfrac{1}{2} d(\alpha)(d(\alpha)+1) < \tfrac{1}{4}\mu^2, \end{aligned}$$

*here* $d(\alpha)$ *is the multiplicity of* $\alpha$ *as spectral number. There is a canonical* $\mathbb{C}^*$*-action on the fibres with negative weights* $({\rm ord}_M \cdot (\alpha_i + \alpha_j - n + 2) \mid 1 \leqslant i \leqslant j \leqslant \mu, \alpha_i + \alpha_j < n-2)$. *Thus there is a canonical zero section. The* $\mathbb{C}^*$*-action commutes with the action of* $G_{\mathbb{C}}$ *on* $\check{D}_{\rm BL}$.

$\pi_{\rm BL}\colon D_{\rm BL} \to D_{\rm PMHS}$ *is the restriction of* $\check{\pi}_{\rm BL}$ *to* $D_{\rm BL} = \check{\pi}^{-1}_{\rm BL}(D_{\rm PMHS})$.

*Proof.* For fixed $F^\bullet \in \check{D}_{\rm PMHS}$, 5.4 and 5.5 show that a coordinate system of $\check{\pi}^{-1}_{\rm BL}(F^\bullet)$ is given by those $c^{(1)}_{i\kappa(j)}$ such that $i \leqslant j$ and $\alpha_{\kappa(j)} - 1 > \alpha_i$. Thus $\check{\pi}^{-1}_{\rm BL}(F^\bullet)$ is isomorphic to $\mathbb{C}^{N_{\rm BL}}$,

$$\begin{aligned} N_{\rm BL} &= \sharp\{(i,j) \mid 1 \leqslant i \leqslant j \leqslant \mu,\ \alpha_{\kappa(j)} - 1 > \alpha_i\} \\ &= \sharp\{(i,j) \mid 1 \leqslant i \leqslant j \leqslant \mu,\ \alpha_i + \alpha_j < n-2\} \\ &= \sum_{\alpha+\beta<n-2,\alpha<\beta} d(\alpha)\cdot d(\beta) + \sum_{2\alpha<n-2} \tfrac{1}{2} d(\alpha)(d(\alpha)+1) \\ &< \tfrac{1}{4}\mu^2, \end{aligned}$$

here we used $\alpha_{\kappa(j)} + \alpha_j = n-1$, $\sum_\alpha d(\alpha) = \mu$, $d(\alpha) = d(n-1-\alpha)$. With respect to these coordinates, the $\mathbb{C}^*$-action on $\check{\pi}^{-1}_{\rm BL}(F^\bullet)$ has the weights ${\rm ord}_M \cdot (\alpha_i - (\alpha_{\kappa(j)} - 1)) = {\rm ord}_M \cdot (\alpha_i + \alpha_j - n + 2) < 0$. The group $G_{\mathbb{C}}$ acts on $\mathcal{G}_0 \supset V^{>-1}$, on $\check{D}_{\rm BL}$, and on $\check{D}_{\rm PMHS}$. By definition, the $\mathbb{C}^*$-action on $\check{D}_{\rm BL}$ commutes with the action of $G_{\mathbb{C}}$ on $\check{D}_{\rm BL}$.

If $s_1, \ldots, s_\mu$ satisfy $(\alpha), (\beta), (\gamma)$ of 5.1 for $F^\bullet$, then for any $g \in G_{\mathbb{C}}$ the images $g(s_1), \ldots, g(s_\mu)$ satisfy $(\alpha), (\beta), (\gamma)$ of 5.1 for $g(F^\bullet)$. Thus the bijection $g\colon \check{\pi}^{-1}_{\rm BL}(F^\bullet) \to \check{\pi}^{-1}_{\rm BL}(g(F^\bullet))$ respects the affine algebraic and the holomorphic structure of these fibres of $\check{\pi}_{\rm BL}$. Any local section of the bundle $G_{\mathbb{C}} \to G_{\mathbb{C}} \cdot F^\bullet = \check{D}_{\rm PMHS}$ induces a local trivialisation of the bundle $\check{\pi}_{\rm BL}$ around $F^\bullet$. $\square$

COROLLARY 5.7. *The group* $G_{\mathbb{Z}}$ *acts properly discontinuously on* $D_{\rm BL}$. *The quotient* $D_{\rm BL}/G_{\mathbb{Z}}$ *is a normal complex space and has at most quotient singularities.*

*Proof.* Proposition 3.6(b) and Theorem 5.6. $\square$

*Remark.* If $F^\bullet \in D_{\rm PMHS}$ then the group $G_{\mathbb{R}} \cap$ Stab $(F^\bullet)$ is compact because of Lemma 2.3, and the group $G_{\mathbb{Z}} \cap$ Stab $(F^\bullet)$ is finite. The projection $D_{\rm BL}/G_{\mathbb{Z}} \to$



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$D_{\text{PMHS}}/G_{\mathbb{Z}}$ is a locally trivial bundle if and only if the images of $G_{\mathbb{Z}} \cap$ Stab $(F^\bullet) \to$ Aut$(\pi_{\text{BL}}^{-1}(F^\bullet))$ are isomorphic for all $F^\bullet \in D_{\text{PMHS}}$. For example, this is satisfied for $E_{3,0}, Z_{1,0}, W_{1,0}, U_{1,0}$, but not for $Q_{2,0}, S_{1,0}$ (Sect. 6).

## 6. Period Mappings, Examples

The elements of $D_{\text{BL}}/G_{\mathbb{Z}}$ are equivalence classes of subspaces of $\mathcal{G}_0$ with respect to the operation of $G_{\mathbb{Z}}$ on $\mathcal{G}_0$. The equivalence class in $D_{\text{BL}}/G_{\mathbb{Z}}$ of a Brieskorn lattice $H_0'' = H_0''(f)$ is an invariant of the right equivalence class of $f$. We call this invariant $\text{BL}(f)$. The space $D_{\text{BL}}/G_{\mathbb{Z}}$ is a moduli space for such invariants. There is another description of the elements of $D_{\text{BL}}/G_{\mathbb{Z}}$: We can consider $G_{\mathbb{Z}}$ as the automorphism group of the tuple $(H^n(X_\infty, \mathbb{Z}),\ M_s,\ N,\ S,\ \psi,\ \mathcal{G}_0)$ and $D_{\text{BL}}/G_{\mathbb{Z}}$ as the set of isomorphism classes of tuples $(H^n(X_\infty, \mathbb{Z}),\ M_s,\ N,\ S,\ \psi,\ \mathcal{G}_0,\ K \subset \mathcal{G}_0)$.

The invariant BL was defined and studied first in [He1] [He2], under the name LBL and together with two weaker invariants, one of which is the Picard–Fuchs singularity [AGV]. It contains very fine analytic information and is a good candidate for Torelli theorems for hypersurface singularities. In [He2] the following conjecture is formulated.

CONJECTURE. *The invariant* $\text{BL}(f)$ *of a hypersurface singularity* $f$ *determines the right equivalence class of* $f$.

In [He2] [He3] global Torelli theorems for several families of singularities are proved, which confirm this conjecture.

THEOREM 6.1 ([He2] [He3]). *The invariant $BL$ determines the right equivalence class for*

(i) *all unimodal singularities,*
(ii) *all bimodal singularities, possibly with the exception of the subseries* $Z_{1,14k}$, $S_{1,10k}, S^\sharp_{1,10k}$ $(k \geqslant 1)$ (*these cases are open*),
(iii) *all semiquasihomogeneous singularities with weights* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,
(iv) *all semiquasihomogeneous singularities with weights* $(1/a_0, \ldots, 1/a_n)$ *and pairwise coprime* $a_i$.

After some general remarks about the period mapping, these families will be discussed in some detail.

Apart from these global Torelli theorems, the conjecture is confirmed by an infinitesimal Torelli theorem for all hypersurface singularities [SM2] (3.1 and 3.2). Let $f_0$ be a hypersurface singularity and $S$ a sufficiently small open subset of the $\mu$-constant stratum in some semiuniversal unfolding of $f_0$. Within this $\mu$-constant stratum, the topological data like Milnor lattice and Seifert form and also the spectral pairs are constant. We obtain a period mapping $\Phi\colon S \to D_{\text{BL}}, s \mapsto H_0''(f)$ for sufficiently small $S$. The period mapping $\Phi$ is holomorphic.



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THEOREM 6.2 ([SM2]).
(a) *If $S$ is smooth then $\Phi\colon S \to D_{\mathrm{BL}}$ is an immersion.*
(b) *Even if $S$ is not smooth any fibre of $\Phi$ is finite.*

In the case of semiquasihomogeneous singularities, we know more about $S$ and the period mapping $\Phi$. Let $f_0$ be a quasihomogeneous singularity with weights $(w_0, \ldots, w_n)$ and degree 1. Then $D_{\mathrm{PMHS}} = D_{\mathrm{prim}}$ and $N_{\mathrm{PMHS}} = 0$, because the monodromy is finite. The $\mu$-constant stratum $S$ in a suitable semiuniversal unfolding is a product $S = S^0 \times S^-$ with $S^- = \mathbb{C}^{\dim S^-}$. Here

$$f_s = f_{(s^0,0)} + \sum_i s_i^- d_i\ , \quad s = (s^0, s^-) = (s^0, (s_i^-)_i) \in S^0 \times S^-,$$

$f_{(s^0,0)}$ is quasihomogeneous, the $d_i$ are the monomials of degree $> 1$ in a monomial basis of the Jacobi algebra, and [Va2]

$$\dim S^0 = d(\alpha_1 + 1), \quad \dim S^- = \sum_{\alpha > \alpha_1 + 1} d(\alpha),$$

here $\alpha_1 = -1 + \sum w_i$ is the smallest spectral number. Setting

$$\deg s_i^- = \mathrm{ord}_M \cdot (1 - \deg d_i) \in \mathbb{Z}_{<0},$$

$$\mathrm{ord}_M = \min\,(l \mid M^l = \mathrm{id}) = \gcd\,(\text{denominators of } w_0, \ldots, w_n),$$

we obtain a $\mathbb{C}^*$-action with negative weights on the fibres $S^-$ of the trivial bundle $S^0 \times S^- \to S^0$. We also have a $\mathbb{C}^*$-action with negative weights on the fibres of the bundle $\pi_{\mathrm{BL}}\colon D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$ (Prop. 5.5). In Proposition 6.3, $S^0$ is supposed to be sufficiently small.

PROPOSITION 6.3. *In the case of semiquasihomogeneous singularities, the period mapping $\Phi\colon S \to D_{\mathrm{BL}}$ is a fibre preserving $\mathbb{C}^*$-equivariant embedding of the bundle $S \to S^0$ into the bundle $D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$.*

The proposition follows from [He2] (2.4). There a monomial differential form $\omega = (\text{ monomial in } x_0, \ldots, x_n) \cdot \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_n$ and its values

$$s[\omega]_0(s^0, s^-) \in H_0''(f_{(s^0,s^-)}) = \Omega^{n+1} / \mathrm{d}f_{(s^0,s^-)} \wedge \mathrm{d}\Omega^{n-1} \subset V^{>-1}$$

for fixed $s^0$ and varying $s^-$ are considered. Then (cf. [Br])

$$s[\omega]_0(s^0, 0) \in C_\alpha, \qquad \text{where } \alpha = \deg_w(\text{monomial}) - 1 + \Sigma w_i;$$

$s[\omega]_0(s^0, s^-)$ has only eigenvalue parts in $C_\beta$ for $\beta \geqslant \alpha$; the eigenvalue part in $C_\alpha$ is constant $= s[\omega]_0(s^0, 0)$; the coefficients of the higher eigenvalue parts are quasihomogeneous polynomials in $(s_i^-)_i$ such that

$$s[\omega]_0(s^0, c{*}s^-) = c^{\mathrm{ord}_M \cdot \alpha} \cdot c{*}\, s[\omega]_0(s^0, s^-).$$



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The proof uses formulas for the derivatives $\partial_t, \partial_{s_i^-}$ in the Gauß–Manin connection and a power series ansatz for the holomorphic coefficients of the eigenvalue parts. We obtain $H_0''(f_{(s^0,c*s^-)}) = c*\,H_0''(f_{(s^0,s^-)})$ and (together with 6.2) Proposition 6.3.

This shows that very often the PMHS of a singularity is not good enough for Torelli theorems. All the semiquasihomogeneous singularities with the same quasihomogeneous part have the same PMHS. Probably, the simple elliptic and the hyperbolic singularities are the only singularities where the PMHS determines the right equivalence class.

Now we come to the discussion of the families of singularities which are listed in Theorem 6.1. The following table gives the dimensions dim $S$ (=modality), dim $D_{\text{prim}}$, $N_{\text{PMHS}}$, and $N_{\text{BL}}$. In the case of semiquasihomogeneous singularities, we write dim $S =$ dim $S^0 +$ dim $S^-$.

Table I.

| Singularities | dim $S$ | dim $D_{\text{prim}}$ | $N_{\text{PMHS}}$ | $N_{\text{BL}}$ |
|---|---|---|---|---|
| $\tilde{E}_6,\ \tilde{E}_7,\ \tilde{E}_8$ | $1 = 1 + 0$ | 1 | 0 | 0 |
| $T_{pqr},\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ | 1 | 0 | 1 | 0 |
| 14 exceptional unimodal | $1 = 0 + 1$ | 0 | 0 | 1 |
| $E_{3,0}, Z_{1,0}, Q_{2,0}, W_{1,0}, S_{1,0}, U_{1,0}$ | $2 = 1 + 1$ | 1 | 0 | 1 |
| 14 exceptional bimodal | $2 = 0 + 2$ | 0 | 0 | 2 |
| 8 bimodal series | 2 | 0 or 1 | 0 | $\geqslant 2$ |
| semiqh. with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $5 = 4 + 1$ | 4 | 0 | 1 |
| semiqh. with weights $(1/a_0, \dots, 1/a_n)$ and pairwise coprime $a_i$ | dim $S = 0 +$ dim $S^-$ | 0 | 0 | $\geqslant$ dim $S^-$ |

Table I shows that any level of the double fibration $D_{\text{BL}} \to D_{\text{PMHS}} \to D_{\text{prim}}$ can contain geometric information. dim $S =$ dim $D_{\text{BL}}$ for 6 of the 8 listed classes. That is not typical. In general, one can expect that dim $D_{\text{BL}}$ is much bigger than the dimension of the $\mu$-constant stratum, and that dim $D_{\text{prim}}$, $N_{\text{PMHS}}$, and $N_{\text{BL}}$ are not 0.

With the exception of the semiquasihomogeneous singularities with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the proof of the global Torelli theorems proceeds in the following way. First, from Arnold's lists families are chosen which contain representatives of each right equivalence class in the $\mu$-homotopy class. The base spaces $S$ are not small. Then the (in most cases) many-valued period mapping $S \to D_{\text{BL}}$ is computed. Finally the action of $G_{\mathbb{Z}}$ on $D_{\text{BL}}$ is determined and compared with the right equivalence relation in $S$, with the result that the induced mapping $S/\textit{right equivalence} \to D_{\text{BL}}/G_{\mathbb{Z}}$ is injective. Often, controlling the action of $G_{\mathbb{Z}}$, is most difficult. But also the computation of the period mapping is easy only for



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semiquasihomogeneous parameters. In the following the most remarkable features of the single families are discussed.

### 6.1. THE SIMPLE SINGULARITIES $A_k,\ D_k,\ E_k$

Here the difference $\alpha_\mu - \alpha_1 = n + 1 - 2\sum_{i=0}^n w_i$ of the largest and the smallest spectral number is smaller than 1. Thus $H_0'' = V^{\alpha_1}$ and $D_{\mathrm{BL}} = D_{\mathrm{PMHS}} = D_{\mathrm{prim}} = \{pt\}$. In view of Theorem 6.2, this implies that these singularities are simple.

### 6.2. THE SIMPLE ELLIPTIC SINGULARITIES $\tilde{E}_6,\ \tilde{E}_7,\ \tilde{E}_8$

[He1] [He2], $D_{\mathrm{BL}} = D_{\mathrm{PMHS}} = D_{\mathrm{prim}}$ is isomorphic to the upper half plane $\mathbb{H}$. The group $G_{\mathbb{Z}}$ acts on $D_{\mathrm{prim}}$ as $PSL(2,\mathbb{Z})$ acts on $\mathbb{H}$. There exist Legendre normal forms with parameter space $S = \mathbb{C} - \{0; 1\}$ and $S/$*right equivalence* $= S/S_3$,

$$S_3 = \left\{\lambda \to \lambda, 1-\lambda, \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}\right\}.$$

The period mapping $S/$*right equivalence* $\to D_{\mathrm{BL}}/G_{\mathbb{Z}}$ is an isomorphism. Everything is as in the case of elliptic curves. In the case of surface singularities, the invariant BL can be identified with the pure Hodge structure of the elliptic curve in the minimal resolution of the singularity.

### 6.3. THE HYPERBOLIC SINGULARITIES $T_{pqr},\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$

[He1] [He2], $D_{\mathrm{prim}} = \{pt\},\ D_{\mathrm{PMHS}} = D_{\mathrm{BL}} \cong \mathbb{C},\ D_{\mathrm{BL}}/G_{\mathbb{Z}} \cong \mathbb{C}/\mathbb{Z} \cong \mathbb{C} - \{0\}$. There exist normal forms with $S = \mathbb{C} - \{0\}$ and $S/$*right equivalence* $= \mathbb{C} - \{0\}/\langle e^{2\pi i/kgV(p,q,r)}\rangle$. The period mapping $S/$*right equivalence* $\to D_{\mathrm{BL}}/G_{\mathbb{Z}}$ is an isomorphism.

### 6.4. THE 14 EXCEPTIONAL UNIMODAL SINGULARITIES

[He1] [He2], $D_{\mathrm{prim}} = D_{\mathrm{PMHS}} = \{pt\},\ D_{\mathrm{BL}} \cong \mathbb{C},\ D_{\mathrm{BL}}/G_{\mathbb{Z}} \cong \mathbb{C}/\langle e^{2\pi i/m}\rangle$ for some $m \in \mathbb{N}$. There exist normal forms with $S = \mathbb{C}$ and $S/$*right equivalence* $= \mathbb{C} - \{0\}/\langle e^{2\pi i/m}\rangle$. The period mappings $S \to D_{\mathrm{BL}}$ and $S/$*right equivalence* $\to D_{\mathrm{BL}}/G_{\mathbb{Z}}$ are isomorphisms.

### 6.5. THE 6 BIMODAL SINGULARITIES $E_{3,0}, Z_{1,0}, Q_{2,0}, W_{1,0}, S_{1,0}, U_{1,0}$

[He1] [He2], $D_{\mathrm{prim}} = D_{\mathrm{PMHS}} \cong \mathbb{H},\ D_{\mathrm{BL}} \cong \mathbb{H} \times \mathbb{C}$. There exist normal forms $f_{s^0,s^-} = f_{s^0,0} + s^- \cdot d$ with $S = S^0 \times S^- = (\mathbb{C} - \{0; 1\}) \times \mathbb{C}$. Here $f_{s^0,0}$ is quasihomogeneous and $d$ is a monomial of degree $> 1$. $D_{\mathrm{BL}}$ and $S$ are vector bundles.



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Let $m$ be the number 9,7,6,9,6,5 for $E, Z, Q, W, S, U$ respectively. There exist groups $G(S)$, $G(S^0)$, $G(D_{\text{BL}})$, $G(D_{\text{PMHS}})$ of automorphisms of $S$ (as vector bundle), $S^0$, $D_{\text{BL}}$ (as vector bundle), and $D_{\text{PMHS}}$ with the properties:

*$S/$right equivalence $= S/G(S)$,*
*$D_{\text{BL}}/G_{\mathbb{Z}} = D_{\text{BL}}/G(D_{\text{BL}})$,*
*$G(S)$ and $G(D_{\text{BL}})$ are central extensions of $G(S^0)$ and $G(D_{\text{PMHS}})$ by the cyclic group $\langle \mathrm{e}^{2\pi i/m}\rangle$, which acts on the fibres of the vector bundles by multiplication.*

The period mapping $S \to D_{\text{BL}}$ is many-valued, locally it is an isomorphism of vector bundles, the image is the whole of $D_{\text{BL}}$ with the exception of the fibres over a discrete set of points of $D_{\text{PMHS}}$.

The period mapping $S/$*right equivalence* $\to D_{\text{BL}}/G_{\mathbb{Z}}$ is injective. The moduli space $S/$*right equivalence* is isomorphic to

$$\mathbb{C}^2 \quad \text{for } E, Z, U$$
$$(\mathbb{C}-\{0\})\times\mathbb{C} \quad \text{for } W$$
$$\mathbb{C}^2/\{\pm\mathrm{id}\} \quad \text{for } Q$$
$$((\mathbb{C}-\{1,-1\})\times\mathbb{C})/\{\pm\mathrm{id}\} \quad \text{for } S.$$

It is smooth for $E, Z, U, W$. In the case of $Q$ and $S$, it has an $A_1$-singularity at the right equivalence class of $f_{\frac{1}{2},0}$. The projection $S/$*right equivalence* $\to$ $S^0/$*right equivalence* is a locally trivial bundle for $E, Z, U, W$, but not for $Q$ and $S$. In the case of $Q$ and $S$, the group $G_{\mathbb{Z}} \cap$ Stab $(F^\bullet)$ is isomorphic to $\langle \mathrm{e}^{2\pi i/m}\rangle$ for generic $F^\bullet \in D_{\text{PMHS}}$, but this group has the double size for the Hodge filtration of $f_{\frac{1}{2},0}$ (compare the remark at the end of Section 5). In the case of surface singularities, the four branches of the minimal resolution of $f_{\frac{1}{2},0}$ intersect the central curve in four points with double ratio $\frac{1}{2}$.

For the reader who wants to check the statements on the moduli space $S/$*right equivalence* or who wants to know the group $G(D_{\text{BL}})$, here are some more details on the groups $G(D_{\text{PMHS}})$, $G(S^0)$, $G(S)$:

$G(D_{\text{PMHS}})$ is a triangle group of type $(2,3,2m)$ for $E, Z, Q, U$ and a triangle group of type $(2,2m,2m)$ for $W, S$.

$$G(S^0) = S_3 = \{\lambda \to \lambda, 1-\lambda, 1/\lambda, \lambda/\lambda-1, 1/1-\lambda, \lambda-1/\lambda\} \quad \text{for } E, Z, Q, U,$$
$$G(S^0) = S_2 = \{\lambda \to \lambda, 1-\lambda\} \quad \text{for } W, S.$$

There exists a holomorphic function $\kappa$: $G(S^0)\times S^0 \to \mathbb{C}-\{0\}$ such that

$$\exists\, \gamma \in G(S) \text{ such that } (s_1^0, s_1^-) = \gamma(s_2^0, s_2^-)$$
$$\iff \quad \exists\, g \in G(S^0) \text{ such that } s_1^0 = g s_2^0 \text{ and } (s_1^-)^m = \kappa(g, s_2^0)\cdot(s_2^-)^m.$$



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The group $G(S)$ is uniquely determined by $G(S^0)$ and $\kappa$.

In [He2] for some of the classes $E, Z, Q, U, W, S$, the monomial $d$ was badly chosen, because it was an element of the Jacobi ideal of $f_{s^0,0}$ for special values of $s^0$. A better choice of $d$ is $x^2y^4,\ x^2y^4,\ xz^2,\ y^4z,\ x^2y^4,\ x^2y^3$ for $E, Z, Q, U, W, S$ respectively, if one uses the same $f_{s^0,0}$ as in [He2] (Table I). Then $\kappa$ is determined by

$$\kappa(\lambda \mapsto 1-\lambda, s) = \left(\frac{s}{s-1}\right)^{18}, \left(\frac{s}{s-1}\right)^{14}, -1, 1, 1, -1$$

for $E, Z, Q, U, W, S$, respectively, and

$$\kappa\left(\lambda \mapsto \frac{1}{\lambda}, s\right) = s^{-12}, s^{-10}, s^3, -s^{-21} \text{ for } E, Z, Q, U, \text{ respectively.}$$

### 6.6. THE 14 EXCEPTIONAL BIMODAL SINGULARITIES

[He1] [He2], $D_{\text{prim}} = D_{\text{PMHS}} = \{pt\},\ D_{\text{BL}} \cong \mathbb{C}^2$, the quotient $D_{\text{BL}}/G_{\mathbb{Z}}$ has a cyclic quotient singularity,

$$D_{\text{BL}}/G_{\mathbb{Z}} \cong \mathbb{C}^2 / \left\langle \begin{pmatrix} e^{2\pi i w_r} & 0 \\ 0 & e^{2\pi i w_s} \end{pmatrix} \right\rangle \text{ for some } w_r, w_s \in \mathbb{Q}.$$

The period mappings $S = S^- \to D_{\text{BL}}$ and $S/$*right equivalence* $\to D_{\text{BL}}/G_{\mathbb{Z}}$ are isomorphisms.

### 6.7. THE 8 BIMODAL SERIES

[He1] [He2], here let $m := 18, 14, 12, 12, 12, 10, 10, 9$ for $E_{3,p}$, $Z_{1,p}$, $Q_{2,p}$, $W_{1,p}$, $W^\sharp_{1,p}$, $S_{1,p}$, $S^\sharp_{1,p}$, $U_{1,p}$ $(p \geqslant 1)$, respectively.

$D_{\text{prim}} = D_{\text{PMHS}} = \{pt\},\ D_{\text{BL}} \cong \mathbb{C}^{[p/m]+2}$ for the singularities with $p \not\equiv 0(\text{mod}\, m)$,

$D_{\text{prim}} = D_{\text{PMHS}} \cong \mathbb{H},\ D_{\text{BL}} \cong \mathbb{H} \times \mathbb{C}^{[p/m]+2}$ for the subseries with $p \equiv 0(\text{mod}\, m)$.

There exist normal forms with $S = (\mathbb{C} - \{0\}) \times \mathbb{C}$.

For $p \not\equiv 0(\text{ mod } m)$ the following holds: The image of the (many-valued) period mapping $S \to D_{\text{BL}}$ is invariant under $G_{\mathbb{Z}}$. The period mapping $S/$*right equivalence* $\to D_{\text{BL}}/G_{\mathbb{Z}}$ is injective. $S/$*right equivalence* is the quotient of $S$ by some finite group, which acts on the factors $\mathbb{C} - \{0\}$ and $\mathbb{C}$ of $S = (\mathbb{C} - \{0\}) \times \mathbb{C}$ by multiplication with unit roots. This group is cyclic for all series with the exception of the subseries $U_{2q}$.



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For $p \equiv 0 (\mathrm{mod}\ m)$ the following holds: The image of the (many-valued) period mapping $S \to D_{\mathrm{BL}}$ is contained in one fibre $\pi_{\mathrm{BL}}^{-1}(F_0^\bullet)$ of $\pi_{\mathrm{BL}}$: $D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$ and is invariant under $G_{\mathbb{Z}} \cap$ Stab $(F_0^\bullet)$. $G_{\mathbb{Z}}$ acts on $D_{\mathrm{PMHS}} \cong \mathbb{H}$ like a triangle group and on $D_{\mathrm{BL}}$ like a central extension of this triangle group by some finite cyclic group, which acts on the fibres of $\pi_{\mathrm{BL}}$. For nearly all fibres $\pi_{\mathrm{BL}}^{-1}(F^\bullet)$, the stabilizer group $G_{\mathbb{Z}} \cap$ Stab $(F^\bullet)$ is isomorphic to this finite group, but for some exceptional fibres the group $G_{\mathbb{Z}} \cap$ Stab $(F^\bullet)$ might be larger. Unfortunately, it is not clear, in which fibre the image $\Phi(S)$ of the period mapping is contained. The period mapping $S/$*right equivalence* $\to D_{\mathrm{BL}}/G_{\mathbb{Z}}$ is injective only if this fibre is not one of the exceptional fibres with larger stabilizer group. This is the reason for the uncertainty, whether the invariant BL determines the right equivalence class for some of the subseries with $p \equiv 0 (\mathrm{mod}\ m)$.

## 6.8. THE SEMIQUASIHOMOGENEOUS SINGULARITIES WITH WEIGHTS $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

[He3], the space $D_{\mathrm{prim}} = D_{\mathrm{PMHS}} \cong \{z \in \mathbb{C}^4 \mid |z| < 1\}$ is isomorphic to the classifying space of polarized pure Hodge structures on $H^3(X_\infty, \mathbb{C})_{\neq 1}$, which are invariant under the monodromy.

Any homogeneous singularity with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the cone over a smooth cubic in $\mathbb{P}^3$. The coarse moduli space $\mathcal{M}_{\mathrm{cubics}}$ for smooth cubics in $\mathbb{P}^3$ is four dimensional, an affine variety, and it coincides with the coarse moduli space for the homogeneous singularities with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ up to right equivalence. The period mapping $\mathcal{M}_{\mathrm{cubics}} \to D_{\mathrm{PMHS}}/G_{\mathbb{Z}}$ is an open embedding. It yields a Torelli theorem for smooth cubics in $\mathbb{P}^3$ in terms of some pure Hodge structure. This is remarkable, because the Hodge structures on the cohomology groups of the smooth cubics are trivial.

The proof of this Torelli theorem for the homogeneous singularities does not use some global family and a many-valued period mapping as in all other cases. The ingredients are [He3] a projective closure in $\mathbb{P}^4$ of the Milnor fibres of a homogeneous singularity, an exact sequence of mixed Hodge structures of Steenbrink, the global Torelli theorem for cubics in $\mathbb{P}^4$ of Tjurin, Clemens and Griffiths, and the cancellation property of space germs of Hauser and Müller.

There also exist semihomogeneous singularities with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. As before Proposition 6.3, locally one can choose a $\mu$-constant stratum of the form $S = S^0 \times S^- = S^0 \times \mathbb{C}$, where $S^- = \mathbb{C}$ gives the one semihomogeneous parameter. The classifying space $D_{\mathrm{BL}} \cong D_{\mathrm{PMHS}} \times \mathbb{C}$ is a vector bundle with one dimensional fibre, too. For sufficiently small $S^0$ the period mapping $S \to D_{\mathrm{BL}}$ is an open embedding of vector bundles. The proof of the global Torelli theorem for semihomogeneous singularities uses that for homogeneous singularities, this period mapping, and some statement on the action of $G_{\mathbb{Z}}$ on the fibres of the bundle $D_{\mathrm{BL}} \to D_{\mathrm{PMHS}}$.



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[He3], any singularity which is $\mu$-homotopic to a Brieskorn–Pham singularity is right equivalent to a semiquasihomogeneous singularity with weights $(1/a_0, \ldots, 1/a_n)$. If the exponents $a_0, \ldots, a_n$ are pairwise coprime, there exists a normal form with parameter space $S = S^- \cong \mathbb{C}^{\dim S}$ and a $\mathbb{C}^*$-action with negative weights on $S$ such that $S/$*right equivalence* $= S/\langle e^{2\pi i/a}\rangle$, $a = a_0 \cdot \ldots a_n$. All eigenspaces of the monodromy are one-dimensional, so $D_{\text{prim}} = D_{\text{PMHS}} = \{pt\}$. Also the classifying space $D_{\text{BL}} \cong \mathbb{C}^{N_{\text{BL}}}$ is equipped with some $\mathbb{C}^*$-action with negative weights. The period mapping $S \to D_{\text{BL}}$ is a $\mathbb{C}^*$-equivariant embedding. The induced period mapping $S/$*right equivalence* $\to D_{\text{BL}}/G_{\mathbb{Z}}$ is injective, because $D_{\text{BL}}/G_{\mathbb{Z}} = D_{\text{BL}}/\langle e^{2\pi i/a}\rangle$. To prove this, one has to show $G_{\mathbb{Z}} = \langle \pm M\rangle$, which is a consequence of the very special properties of the integral monodromy $M$ for these singularities.

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