A CHARACTERIZATION OF THE HUGHES PLANES

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1. Introduction. Baer **(1)** introduced the term "(p, L)-collineation" to denote a central collineation with centre p and axis L. We shall find it convenient to use a modification of the related notion of "(p, L)-transitivity."

DEFINITION. *Let π_0 be a subplane of the projective plane π. Let L be a fixed line of π_0, and let p be a fixed point of π_0. Let r and s be any two points of π_0 that are collinear with p, distinct from p, and not on L. If, for each such choice of r and s, there is a (p, L)-collineation of π that* (1) *carries π_0 into itself and* (2) *carries r into s, we shall say that π is (p, L, π_0)-transitive.*

In effect, the requirement is that the (p, L)-collineations of π induce a (p, L)-transitive group on π_0.

We shall be particularly interested in the case where every point of π not in π_0 belongs to (exactly one) line of π_0. If π is finite, this occurs precisely when π is of order q^2, where q is the order of π_0.

If (with the above restriction on order) π is (p, L, π_0)-transitive for each point p belonging to $L \cap \pi_0$ and some fixed line L, then π is a semi-translation plane with respect to L **(5)**. The "strict" semi-translation planes do not admit any (p, L)-transitivities. Just as planes in general may be classified in terms of their (p, L)-transitivities, we may hope to get some sort of classification of semi-translation planes in terms of (p, L, π_0)-transitivities.

The Hughes planes **(4)** play a special role in this situation. If π is a Hughes plane of order q^2, then π contains a subplane π_0 of order q such that π is (p, L, π_0)-transitive for every p and L in π_0 **(7; 8)**.

Our main result is that the Hughes planes are unique in this respect. Indeed we are able to prove the stronger result: Let π be a plane of order q^2 containing a subplane π_0 of order q. Let L_∞ be a line of π_0. Then π is a Hughes plane if π is (p, L, π_0)-transitive for each choice of p in $L_\infty \cap \pi_0$ and each choice of L (including L_∞) in π_0.

In the infinite case, the restriction on the orders of π and π_0 may be replaced by the condition that each point of π belongs to a line of π_0 and each line of π contains a point of π_0. While Hughes gave his definition for finite planes **(4)**, his chief tool is a left near-field of dimension two over its centre. We may thus speak of "infinite Hughes planes" and our result still holds for the infinite case. We shall find it convenient to represent the affine version of Hughes

Received June 19, 1964. This work was supported (in part) by grant number GP 1623 from the National Science Foundation.

https://doi.org/10.4153/CJM-1965-087-4 Published online by Cambridge University Press

planes in a form different from that originally given by Hughes. In a previous paper, we have shown that Hughes' representation can be reduced to ours **(6)**. The argument is reversible and we shall not give it in detail. For our purposes, then, a plane is considered to be a Hughes plane if it can be co-ordinatized in the following fashion:

Let $\mathfrak{T}$ be a left near-field, i.e. $[a(b + c) = ab + ac]$, which contains a subfield $\mathfrak{F}$ such that $\mathfrak{T}$ is a right vector space of dimension two over $\mathfrak{F}$. The affine points are ordered pairs (x, y) of elements of $\mathfrak{T}$. The lines are sets of points whose co-ordinates satisfy equations of any one of the following types:

$$(1)\ y = (x - \alpha)m + \beta, \qquad \alpha, \beta \in \mathfrak{F}, m \notin \mathfrak{F},$$

$$(2)\ y = x\delta + b, \qquad \delta \in \mathfrak{F},$$

$$(3)\ x = c.$$

To facilitate shifting back and forth between projective and affine planes, we shall consider an affine plane to be a projective plane with one line designated as L_∞.

We shall be using Hall's co-ordinate system **(3)**, although we shall make only limited use of the ternary as such. Where we do use it, $y = T(x, m, b)$ will denote the equation of the line of slope m with intercept b. We shall use (m) to denote the point at infinity on $y = xm$, (∞) to denote the point at infinity on $x = 0$.

It will be understood throughout that the projective plane π contains a proper subplane π_0 such that each point of π is on at least one line of π_0 and each line of π contains at least one point of π_0. The line L_∞ will always be a line of π_0. The co-ordinate system for π will be chosen so that it induces a co-ordinate system for π_0. Thus the affine points of π_0 are those whose co-ordinates are chosen from a subsystem $\mathfrak{F}$. Small Greek letters (except in a few cases where they clearly denote operators) will always denote elements of $\mathfrak{F}$.

2. Development of the main theorem.

LEMMA 1. *Suppose that π is (p, L_∞, π_0)-transitive for each choice of $p \in L_\infty \cap \pi_0$. Then π can be co-ordinatized by a system $\mathfrak{T}$ containing a subsystem $\mathfrak{F}$ which is a right quasi-field (Veblen–Wedderburn system). Furthermore:*

(1) *Points of π_0 have co-ordinates in $\mathfrak{F}$.*
(2) *Lines of π_0 have equations of the form $y = x\alpha + \beta$ or $x = \gamma$.*
(3) *Lines of π whose slopes m are not in $\mathfrak{F}$ have equations of the form $y = (x - \alpha)m + \beta$.*
(4) *$(x - \alpha)\beta = x\beta - \alpha\beta$ for all $x \in \mathfrak{T}$ and all $\alpha, \beta \in \mathfrak{F}$.*
(5) *$c + (b + \beta) = (c + b) + \beta$ for all $c, b \in \mathfrak{T}$ and all $\beta \in \mathfrak{F}$.*

Proof. (Note that, as remarked in the Introduction, small Greek letters denote elements of $\mathfrak{F}$.) Lemma 1 is essentially the same as Theorem 8 of **(5)** so we shall not give the details. The collineations induced on π_0 by the assumed collineations of π require that π_0 be a translation plane. Hence π_0 can be co-ordinatized by a quasi-field $\mathfrak{F}$. The expression $x - \alpha$ denotes x plus the additive inverse of the element α which belongs to F. Part (5) of the conclusion is not mentioned in **(5)**, but follows from the fact that π admits translations that can be represented as mappings $(x, y) \to (x, y + \beta)$.

In the following lemmas, it is to be understood that π is co-ordinatized in the manner described in Lemma 1.

LEMMA 2. *If the hypotheses of Lemma* 1 *are satisfied and* π *admits an involutory collineation that fixes all points on the line* $y = x$ *and interchanges* (0) *with* (∞), *then*

(1) $(cm)m^{-1} = c$ *for all* $m \neq 0$ *and all* c *in* $\mathfrak{T}$.
(2) $(c + b) - b = c$ *for all* c, b *in* $\mathfrak{T}$.

Proof. The collineation must be represented by the mapping $(x, y) \to (y, x)$. Since $(0, 0)$ is fixed and $(1, m) \to (m, 1)$, we must have $y = xm \to y = xm^{-1}$, where $mm^{-1} = 1$. Since $(c, cm) \to (cm, c)$, we have $c = (cm)m^{-1}$, which establishes conclusion (1).

Since $(0, b) \to (b, 0)$ and (1) is fixed, $y = x + b \to y = x - b$, where $b - b = 0$. Conclusion (2) follows from the fact that $(c, c + b) \to (c + b, c)$.

LEMMA 3. *If the hypotheses of Lemma* 1 *are satisfied and* π *admits a homology with centre* (∞), *axis* $y = 0$, *which carries* (1) *into* (α), *then*

(1) $(cm)\alpha = c(m\alpha)$ *for all* c, m *in* $\mathfrak{T}$.
(2) $T(c, \alpha, b) = (c + b\alpha^{-1})\alpha$ *for all* c, b in $\mathfrak{T}$.

Proof. It is easily established that the homology must map (x, y) into $(x, y\alpha)$, (m) into $(m\alpha)$. Part (1) comes easily by looking at the point (c, cm) on $y = xm$. Note that $(0, b\alpha^{-1}) \to (0, b)$. Thus $y = x + b\alpha^{-1} \to y = T(x, \alpha, b)$. Since $(c, c + b\alpha^{-1}) \to (c, (c + b\alpha^{-1})\alpha)$ on $y = T(x, \alpha, b)$ we get part (2) of Lemma 3.

LEMMA 4. *If* (1) *there is an affine line* $L' \in \pi_0$ *and a point* $r \in \pi_0 \cap L_\infty$ *such that* π *admits a central collineation of order two with axis* L' *and centre* r *which carries* π_0 *into itself and* (2) π *is* (p, L, π_0) *transitive whenever* p *is any point on* $L_\infty \cap \pi_0$ *and* L *is any line of* π_0, *then*

$$\left.\begin{array}{ll} (1) & c\alpha + b\alpha = (c + b)\alpha \\ (2) & T(c, \alpha, b) = c\alpha + b \\ (3) & c(\alpha + b) = c\alpha + cb \end{array}\right\} \text{ for all } c, b \in T \text{ and all } \alpha \in F;$$

(4) *addition is associative; the additive group is a right vector space over* $\mathfrak{F}$.

Proof. Note the hypothesis (1) is included in hypothesis (2) if π is finite.

Hypothesis (2) includes the hypotheses of Lemma 1 (with $L = L_\infty$) and of Lemma 3, for each α in $\mathfrak{F}$. The permutation group induced on $L_\infty \cap \pi_0$ by

https://doi.org/10.4153/CJM-1965-087-4 Published online by Cambridge University Press

the assumed collineations is triply transitive; the affine lines are all in a single transitive class under the group generated by these collineations. Hence the hypotheses of Lemma 2 are also satisfied.

By part (2) of Lemma 3, $T(c, \alpha, b\alpha) = (c + b)\alpha$. Thus $y = (x + b)\alpha$ is the equation of a line. Let us reconsider the involution given by the mapping $(x, y) \to (y, x)$, $(m) \to (m^{-1})$. Under this mapping, the line $y = (x + b)\alpha$ is mapped onto $y = (x - b\alpha)\alpha^{-1}$, which contains $(b\alpha, 0)$ and (α^{-1}). Since $(c, (c + b)\alpha)$ is on $y = (x + b)\alpha$, $((c + b)\alpha, c)$ must be on $y = (x - b\alpha)\alpha^{-1}$. Hence

$$[(c + b)\alpha - b\alpha]\alpha^{-1} = c.$$

We then obtain (1) by using the right inverse laws for multiplication and addition. Part (2) now follows from part (2) of Lemma 3.

Consider the elation with axis $x = 0$, centre (∞), which carries (0) into (δ). The line $y = b$ must map into $y = T(x, \delta, b)$, i.e., $y = x\delta + b$. Since the lines $x =$ constant are fixed, the mapping takes the form $(x, y) \to (x, x\delta + y)$. Since $(1, m) \to (1, \delta + m)$, the line $y = xm$ maps into $y = x(\delta + m)$. We then obtain (3) easily.

Since $(m) \to (\delta + m)$ and $(\alpha, 0) \to (\alpha, \alpha\delta)$,

$$y = (x - \alpha)m \to y = (x - \alpha)(\delta + m) + \alpha\delta.$$

But
$$(c, (c - \alpha)m) \to (c, c\delta + (c - \alpha)m).$$

Hence
$$c\delta + (c - \alpha)m = (c - \alpha)(\delta + m) + \alpha\delta.$$

Using (3) and (2),

$$c\delta + (c - \alpha)m = [(c\delta - \alpha\delta) + (c - \alpha)m] + \alpha\delta.$$

Let $\delta = 1$ and $(c - \alpha)m = b$; then

$$c + b = [(c - \alpha) + b] + \alpha$$

or
$$(c + b) - \alpha = (c - \alpha) + b.$$

Now if we use this last identity together with (3) and part 5 of Lemma 1,

$$\begin{aligned}(t\xi_1 + \eta_1) + (t\xi_2 + \eta_2) &= [(t\xi_1 + \eta_1) + t\xi_2] + \eta_2 \\ &= [(t\xi_1 + \eta_1) + \eta_2] + t\xi_2 \\ &= [t\xi_1 + (\eta_1 + \eta_2)] + t\xi_2 \\ &= (t\xi_1 + t\xi_2) + (\eta_1 + \eta_2) \\ &= t(\xi_1 + \xi_2) + (\eta_1 + \eta_2).\end{aligned}$$

Here we may take t to be an arbitrary fixed element of $\mathfrak{T}$ that is not in $\mathfrak{F}$, while ξ_1, η_1, ξ_2, η_2 are arbitrary elements of $\mathfrak{F}$.

If c is any element of $\mathfrak{T}$, the point (t, c) is on some line $y = x\xi + \eta$ of π_0 by the standard assumption mentioned at the end of the Introduction. Thus

https://doi.org/10.4153/CJM-1965-087-4 Published online by Cambridge University Press

each element c of $\mathfrak{T}$ may be written in the form $c = t\xi + \eta$. The fact that addition is associative in the quasi-field $\mathfrak{F}$ together with the identity

$$(t\xi_1 + \eta_1) + (t\xi_2 + \eta_2) = t(\xi_1 + \xi_2) + (\eta_1 + \eta_2)$$

then implies associativity of addition in $\mathfrak{T}$.

The postulated collineations are enough to ensure that π_0 must be Desarguesian and $\mathfrak{F}$ must be a field. Using (1) of Lemma 3 and (1) of the present lemma, we get that the additive group in $\mathfrak{T}$ is a right vector space of dimension two over $\mathfrak{F}$.

LEMMA 5. *Under the hypotheses of Lemma* 4, *multiplication is associative.*

Proof. Suppose that $b \notin \mathfrak{F}$. Then if c is an arbitrary element of $\mathfrak{T}$, not in $\mathfrak{F}$, c may be written in the form $b^{-1}\alpha + \beta\alpha = (b^{-1} + \beta)\alpha$ for some choice of β and α. Then

$$(ab)c = (ab)[(b^{-1} + \beta)\alpha] = [(ab)(b^{-1} + \beta)]\alpha$$
$$= [a + (ab)\beta]\alpha = [a(1 + b\beta)]\alpha = a[(1 + b\beta)\alpha] = a(bc).$$

If $c \in \mathfrak{F}$, we have already established that $(ab)c = a(bc)$. Now we wish to show that $(a\alpha)c = a(\alpha c)$ for all α in $\mathfrak{F}$, a, c in $\mathfrak{T}$. Let $(a\alpha)c = d$. Then $a = (dc^{-1})\alpha^{-1} = d(c^{-1}\alpha^{-1})$. But $(\alpha c)(c^{-1}\,{}^{-}\alpha^{1}) = 1$; hence $ac = (c^{-1}\alpha^{-1})^{-1}$ from the right inverse property. Thus $a(\alpha c) = d(c^{-1}\alpha^{-1})(\alpha c) = d$; that is, $(a\alpha)c = a(\alpha c)$. The associative law is now established in all cases.

THEOREM 1. *Let π be a finite projective plane of order q^2 containing a subplane π_0 of order q. Let L_∞ be a fixed line of π_0. Suppose that π is (p, L, π_0)-transitive for every choice of p and L such that $p \in L_\infty \cap \pi_0$ and $L \in \pi_0$. Then π is a Hughes plane.*

Proof. In view of the previous lemmas, the co-ordinate system we have chosen satisfies all of the conditions of a left near-field that is a right vector space over $\mathfrak{F}$, with the possible exception of the left distributive law. However,

$$a(b + c) = a[b(1 + b^{-1}c)] = ab(1 + b^{-1}) = ab + ac.$$

Hence π is a Hughes plane as represented at the beginning of this paper.

THEOREM 2. *Let π be a projective plane containing a subplane π_0 such that every point of π is on a line of π_0 and every line of π contains a point of π_0. Let L_∞ be a line of π_0. Suppose that* (1) *π admits an involutory central collineation that carries π_0 into itself, has an affine line of π_0 as its axis, and has some point in $L_\infty \cap \pi_0$ for its centre;* (2) *π is (p, L, π_0)-transitive for each choice of p and L such that $p \in L_\infty \cap \pi_0$ and $L \in \pi_0$. Then π is an "infinite Hughes plane."*

Proof. The proof is essentially the same as the proof of Theorem 1.

3. Concluding remarks. It may be worth while to look at some examples of planes that are not Hughes planes but do admit a fair amount of (p, L, π_0)-transitivity. But first we remark that the whole idea is of interest chiefly in situations where there is a "natural" choice for π_0. We might also note that instances of (p, L)-transitivity will frequently induce (p, L, π_0)-transitivity for an appropriately chosen π_0. In such cases, the fact that we have (p, L, π_0)-transitivity would seem to be relatively unimportant—unless we can make further choices of p and L for the same subplane to get additional (p, π, L_0)-transitivity. Note Example 1 below.

We now give examples of finite planes of order q^2 for some q. In each case, π_0 is a suitably chosen subplane of order q. In each case, there is a natural choice of the co-ordinate system so that the affine points of π_0 have co-ordinates taken from a subfield $\mathfrak{F}$. We shall indicate choices of p and L (in terms of the co-ordinate system) for which the plane is (p, L, π_0)-transitive.

1. π is a dual translation plane of dimension two over its kernel. Here we may take $p = (\infty)$ together with L as any line of π_0.

2. Fryxell **(2)** has studied a class of dual semi-translation planes (which are not dual translation planes) in which we can make the same choices of p and L as in case 1.

3. π is a self-dual semi-translational plane; cf. **(5**, part V**)** for an example. Here we may take $L = L_\infty$, p as any point of $L_\infty \cap \pi_0$. Dually, we may also take $p = (\infty)$ and L as any line of π_0 that goes through (∞).

4. Fryxell **(2)** has investigated another class of self-dual semi-translation planes. Besides the (p, L, π_0)-transitivities of case 3, here we may also take p as any point of $\pi_0 \cap L_\infty$ together with L as any line of π_0 that goes through (∞).

In each of these classes there are planes that are not Hughes planes. The same remark applies to the dual of case 2.

The outstanding unknown case is the one where p and L may be taken as any incident point-line pair in π_0. This corresponds to the case where π is a strict semi-translation with respect to each line of π_0. The Hughes planes have this property. It is not known whether there are any other planes that have it. A property that can be shown to be identical with this one is the following: π admits a group of collineations (carrying π_0 into itself), which induces a doubly transitive group of collineations on π_0. In this case, π_0 is Desarguesian and the induced group includes the little projective group of π_0. If the little projective group of π_0 is the full projective group (which it will be if $q \not\equiv 1 \pmod 3$), we could apply our Theorem 1 if we could guarantee that the homologies of π_0 extend to homologies of π. The author has been unable to do this without making additional assumptions.

References

1. R. Baer, *Homogeneity of projective planes*, Amer. J. Math., *64* (1939), 110–141.

2. R. C. Fryxell, *Sequences of planes constructed from near-field planes of square order*, Ph.D. Thesis, Wash. State University (1964).

3. M. Hall, *Projective planes*, Trans. Amer. Math. Soc., *54* (1943), 229–277.
4. D. R. Hughes, *A class of non-Desarguesian projective planes*, Can. J. Math., *9* (1957), 378–388.
5. T. G. Ostrom, *Semi-translation planes*, Trans. Amer. Math. Soc., *111* (1964), 1–18.
6. ——— *Finite planes with a single* (p, L) *transitivity*, Arch. Math., *15* (1964), 378–384.
7. L. A. Rosati, *I gruppi di collineazioni dei piani di Hughes*, Boll. Un. Mat. Ital., *13* (III) (1958), 505–513.
8. G. Zappa, *Sui gruppi di collineazioni dei piani di Hughes*, Boll. Un. Mat. Ital., *12* (III) (1957), 507–516.

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