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SALLY MODULES AND REDUCTION NUMBERS OF IDEALS

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Abstract. We study the relationship between the reduction number of a primary ideal of a local ring relative to one of its minimal reductions and the multiplicity of the corresponding Sally module. This paper is focused on three goals: (i) to develop a change of rings technique for the Sally module of an ideal to allow extension of results from Cohen–Macaulay rings to more general rings; (ii) to use the fiber of the Sally modules of almost complete intersection ideals to connect its structure to the Cohen–Macaulayness of the special fiber ring; (iii) to extend some of the results of (i) to two-dimensional Buchsbaum rings. Along the way, we provide an explicit realization of the S_2-fication of arbitrary Buchsbaum rings.

§1. Introduction

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 1$. We say that $\mathbf{R}$ is almost Cohen–Macaulay, or, briefly, aCM, if depth $\mathbf{R} \geqslant d-1$. Numerous classes of such rings occur among Rees algebras of ideals of Cohen–Macaulay rings (addressed in several classical papers of Sally, Valla and others; see the surveys [23, 24]). They also arise in the normalization of integral domains of low dimension and among rings of invariants. Obviously, information about the aCM condition rests in the cohomology of the algebra, which often is not fully available. As a corrective, we look for chunks of those data that may be available in the reduction number of the ideal, Hilbert coefficients, structure of the Sally module and in the relationships among these invariants.

The simplest to describe of these notions is that of reductions of ideals, but we only make use of a special case. For simplicity, we assume that the residue field $\mathbf{R}/\mathfrak{m}$ is infinite. If I is an $\mathfrak{m}$-primary ideal, a minimal reduction is an ideal $Q = (a_1, \ldots, a_d) \subset I$ such that $I^{n+1} = QI^n$

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for some integer $n \geqslant 0$. The smallest such integer is denoted $\mathrm{r}_Q(I)$ and called the reduction number of I relative to Q. We set $\mathrm{r}(I) = \inf\{\mathrm{r}_Q(I) \mid Q \text{ is a minimal reduction of } I\}$. A set of questions revolves around the relationship between the Hilbert coefficients of I and its reduction number. In turn, the mentioned coefficients are those extracted from the Hilbert function of I; that is, of the associated graded ring $\mathrm{gr}_I(\mathbf{R}) = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$, or, more precisely, from some of the coefficients of its Hilbert polynomial

$$\lambda(I^n/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d-1}{d-1} - \mathrm{e}_1(I)\binom{n+d-2}{d-2} + \text{lower terms}, \quad n \gg 0.$$

An instance of the relationships between these invariants of I is the following. If $\mathbf{R}$ is Cohen–Macaulay, we have from [29, Theorem 2.45]

$$\mathrm{r}(I) \leqslant \frac{d \cdot \mathrm{e}_0(I)}{o(I)} - 2d + 1,$$

where $\mathrm{e}_0(I)$ is the multiplicity of I and $o(I)$ is the order of I; that is, the largest positive integer n such that $I \subset \mathfrak{m}^n$. A non-Cohen–Macaulay version of similar character is given in [8, Theorem 3.3], with $\mathrm{e}_0(I)$ replaced by $\lambda(\mathbf{R}/Q)$ for any reduction Q of I. Since $\lambda(\mathbf{R}/Q) = \mathrm{e}_0(I) + \chi_1(Q) \leqslant \mathrm{hdeg}_I(\mathbf{R})$, by Serre's theorem [2, Theorem 4.7.10] and [9, Theorem 7.2] respectively, the reduction number $\mathrm{r}(I)$ can be bounded in terms of I alone. It is worth asking whether a sharper bound holds on replacing $\lambda(\mathbf{R}/Q)$ by $\mathrm{e}_0(I) - \mathrm{e}_1(Q)$ (see [10, Theorem 4.2]).

A distinct kind of bound was introduced by Rossi [22].

THEOREM 1.1. [22, Corollary 1.5] *If $(\mathbf{R}, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension at most 2, then, for any $\mathfrak{m}$-primary ideal I with a minimal reduction Q,*

$$\mathrm{r}_Q(I) \leqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 1.$$

We refer to the right-hand side as the Rossi index of I. The presence of the offsetting terms at times leads to smaller bounds. Before we explain how such bounds arise, it is worthwhile to point out that this has been observed in numerous cases in higher dimension. For instance, if the Rossi index is equal to 2, then equality holds in all dimensions (see [11, Theorem 3.1]). Another case is that of all the ideals of $k[x, y, z]$ of codimension 3 generated by five quadrics (see [16, Theorem 2.1 and Proposition 2.4]).

We prove some specialized higher-dimension versions of this inequality and also propose a version of it for non-Cohen–Macaulay rings. These require a natural explanation for these sums as the multiplicities of certain modules that occur in the theory of Rees algebras. At the center of our discussion is the Sally module [27, Definition 2.1]

$$S_Q(I) = \bigoplus_{n \geqslant 1} I^{n+1}/IQ^n$$

of I relative to Q. Its properties depend heavily on the value of $\mathrm{r}_Q(I)$ and of its multiplicity, which in the Cohen–Macaulay case is given by $\mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I)$ (see [27, Corollary 3.3]).

In Theorem 2.1, we give a quick presentation of some of the main properties of $S_Q(I)$. Making use of a theorem of Serre [2, Theorem 4.7.10], we provide proofs that extend the assertions in [27] from the Cohen–Macaulay case to general rings with some provisos. Its use requires the knowledge of the dimension of $S_Q(I)$ (which is readily accessible if $\mathbf{R}$ is Cohen–Macaulay), for which we appeal to the assertion of Theorem 2.1(3), that if $\dim S_Q(I) = d$, then $\mathrm{s}_Q(I) \leqslant \sum_{n=1}^{r-1} \lambda(I^{n+1}/QI^n)$, where $r = \mathrm{r}_Q(I)$ and $\mathrm{s}_Q(I)$ is an appropriate multiplicity of $S_Q(I)$. Furthermore, $S_Q(I)$ is Cohen–Macaulay if and only if equality holds. This is a direct extension of [17, Theorem 3.1] (see also [18, Theorem 4.7]) if $\mathbf{R}$ is Cohen–Macaulay. In these cases, the Rossi index of I is $\mathrm{s}_Q(I) + 1$ and $\mathrm{r}_Q(I) \leqslant \mathrm{s}_Q(I) + 1$.

We want to take as our starting point the value of $\mathrm{s}_Q(I)$. We first give a change of rings result (Theorem 2.6) to detect when $\dim S_Q(I) < d$ whenever we have a finite integral extension of $\mathbf{R}$ into a Cohen–Macaulay ring $\mathbb{S}$. This is used in Section 4 when we prove a version of Theorem 1.1 for Buchsbaum rings (Theorem 4.3).

Section 3 deals exclusively with almost complete intersections and derives reduction number bounds from the behavior of one component of the fiber of $S_Q(I)$. For instance, if $(\mathbf{R}, \mathfrak{m})$ is a Noetherian local ring, I is an $\mathfrak{m}$-primary ideal and Q is one of its minimal reductions, then if $\lambda(I^n/QI^{n-1}) = 1$ for some integer $n > 0$, we have that $\mathrm{r}(I) \leqslant n\nu(\mathfrak{m}) - 1$ (Proposition 3.1).

One of our main results (Theorem 3.7) gives a description of the properties of the special fiber ring, including a formula for its multiplicity $f_0(I)$.

THEOREM 3.7. *Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geqslant 1$ with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal which is an almost complete intersection. If for some minimal reduction Q of I,*

$\lambda(I^2/QI)=1$, *then*

(1) *the special fiber ring* $\mathcal{F}(I)$ *is Cohen–Macaulay;*
(2) $f_0(I)=\mathrm{r}(I)+1=\mathrm{e}_1(I)-\mathrm{e}_0(I)+\lambda(\mathbf{R}/I)+2$.

This is the case if $\mathbf{R}$ is Gorenstein and $\lambda(I/Q)=2$.

An issue is whether there are extensions of these formulas for more general classes of ideals of Cohen–Macaulay rings, or even non-Cohen–Macaulay rings of dimension 2. With regard to the second question, in Section 4 we consider extensions of Theorem 1.1 to Buchsbaum rings. We begin by proving that any Buchsbaum local ring $\mathbf{R}$ admits an S_2-fication $\mathbf{f}:\mathbf{R}\to\mathbb{S}$. A surprising fact is the intrinsic construction of $\mathbb{S}$, independent of the existence of a canonical module for $\mathbf{R}$ (Theorem 4.2). We make use of this morphism to derive in Theorem 4.3 a bound for the reduction numbers of the ideals of $\mathbf{R}$, in dimension 2, in terms of the multiplicity of the Sally module $S_Q(I)$ and the Buchsbaum invariant of $\mathbf{R}$.

§2. Sally modules and Rossi index

Let $(\mathbf{R},\mathfrak{m})$ be a Noetherian local ring of dimension $d\geqslant 1$, let I be an $\mathfrak{m}$-primary ideal, and let Q be a minimal reduction of I. Let $r=\mathrm{r}_Q(I)$ be the reduction number of I relative to Q. Let us recall the construction of the Sally module associated to these data and show how some of its properties, originally stated for Cohen–Macaulay rings, hold in greater generality. We denote the Rees algebras of Q and I by $\mathbf{R}[Q\mathbf{T}]$ and $\mathbf{R}[I\mathbf{T}]$ respectively and consider the exact sequence of finitely generated $\mathbf{R}[Q\mathbf{T}]$-modules

$$0\to I\mathbf{R}[Q\mathbf{T}]\longrightarrow I\mathbf{R}[I\mathbf{T}]\longrightarrow S_Q(I)\to 0.$$

Then, $S=S_Q(I)=\bigoplus_{n\geqslant 1} I^{n+1}/IQ^n$ is the Sally module of I relative to Q (see [27, Definition 2.1]). Note that

$$S/(Q\mathbf{T})S=\bigoplus_{n=1}^{r-1} I^{n+1}/QI^n$$

is a finitely generated $\mathbf{R}$-module. We refer to the component I^{n+1}/QI^n as the degree-n fiber of $S_Q(I)$. Note that for $1\leqslant n\leqslant r-1$, these components do not vanish. If $\dim S_Q(I)=d$, and $Q=(a_1,\ldots,a_d)$, then the elements $\{a_1\mathbf{T},\ldots,a_d\mathbf{T}\}$ give a system of parameters for $S_Q(I)$. We denote by $\mathrm{s}_Q(I)=\mathrm{e}_0((Q\mathbf{T}),S)$ the corresponding multiplicity. Its value is independent of Q if $\mathbf{R}$ is Cohen–Macaulay. Let us summarize some results of [2, 12, 17, 27] in the following, which we often need in this paper.

THEOREM 2.1. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d > 0$. *Let* I *be an* $\mathfrak{m}$*-primary ideal, and let* Q *be a minimal reduction of* I *with reduction number* $r = \mathrm{r}_Q(I)$.

(1) *The Sally module* $S_Q(I)$ *of* I *relative to* Q *is a finitely generated module over* $\mathbf{R}[Q\mathbf{T}]$ *of dimension at most* d.
(2) *If* $\mathbf{R}$ *is Cohen–Macaulay, then* $S_Q(I)$ *is either* 0 *or has dimension* d. *In the latter case,* $\mathfrak{m}\mathbf{R}[Q\mathbf{T}]$ *is its only associated prime and its multiplicity (or rank) is*
$$\mathrm{s}_Q(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I).$$
(3) *If* $\dim S_Q(I) = d$, *then*
   (a) $\mathrm{s}_Q(I) \leqslant \sum_{n=1}^{r-1} \lambda(I^{n+1}/QI^n)$;
   (b) *the equality in (a) holds if and only if* $S_Q(I)$ *is Cohen–Macaulay, in which case* $\mathrm{r}_Q(I) \leqslant s_Q(I) + 1$;
   (c) *if* $\mathbf{R}$ *is Cohen–Macaulay, the equality in (a) holds if and only if* $\operatorname{depth} \operatorname{gr}_I(\mathbf{R}) \geqslant d - 1$.

Proof. Let $S = S_Q(I)$. Since $\mathfrak{m}^l S = 0$ for some integer $l > 0$ [12, Lemma 2.1], $\dim_{\mathbf{R}[Q\mathbf{T}]}(S) \leqslant d$. The assertion (2) is proved in [27, Proposition 2.2, Corollary 3.3]. Now, suppose that $\dim S_Q(I) = d$. Then, by Serre's theorem [2, Theorem 4.7.10],
$$\lambda(S/(Q\mathbf{T})S) = \chi_0(Q\mathbf{T}, S) + \chi_1(Q\mathbf{T}, S) = \mathrm{s}_Q(I) + \chi_1(Q\mathbf{T}, S),$$
and since $\chi_1(Q\mathbf{T}, S) \geqslant 0$, the equality in (a) holds if and only if $Q\mathbf{T}$ yields a regular sequence on S. This proves the assertions (3)(a) and (3)(b). Assertion (3)(c) follows from [17, Theorem 3.1]. $\square$

REMARK 2.2. (see [6, Lemma 3.8]) We make several observations about the Sally fiber $S/(Q\mathbf{T})S$. A crude estimation for the length of the fiber of S is $\lambda(S/(Q\mathbf{T})S) \geqslant r - 1$. Refinements require information about the lengths of the modules I^{n+1}/QI^n. Let us consider a very extreme case. Suppose that I^{n+1}/QI^n is cyclic for some $n \geqslant 1$. Write $I^{n+1}/QI^n \simeq \mathbf{R}/L_n$, where L_n is an ideal of $\mathbf{R}$. We claim that $I^{n+2}/QI^{n+1} \simeq \mathbf{R}/L_{n+1}$, $L_n \subset L_{n+1}$. For notational simplicity, we pick $n = 1$, but the argument applies to all cases. We have $I^2 = (ab) + QI$, $a, b \in I$. We claim that $I^3 = (a^2b) + QI^2$. If $c, d, e \in I$, we have $cd \in (ab) + QI$, and thus $cde \in a(be) + QI^2 \subset (a^2b) + QI^2$. We also have $L_1(a^2b) = aL_1(ab) \subset QI^2$. This gives the following exact value for the length of the fiber of S: if $\lambda(I^2/QI) = 1$, then $\lambda(S/(Q\mathbf{T})S) = r - 1$.

We recall from the literature several properties of ideals I such that $\lambda(I^2/QI) = 1$.

PROPOSITION 2.3. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d > 0$. *Let* I *be an* $\mathfrak{m}$*-primary ideal, and let* Q *be a minimal reduction of* I *such that* $\lambda(I^2/QI) = 1$. *Then, the following hold:*

(1) depth $\mathrm{gr}_I(\mathbf{R}) \geqslant d - 1$;
(2) depth $\mathbf{R}[I\mathbf{T}] \geqslant d$;
(3) $S_Q(I)$ *is Cohen–Macaulay of dimension* d;
(4) $\mathrm{r}_Q(I)$ *is independent of the choice of a minimal reduction* Q;
(5) $\mathrm{r}(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 1 > 1$;
(6) $\mathrm{r}(I) + 1$ *is a bound on the degrees of the defining equations of* $\mathrm{gr}_I(\mathbf{R})$.

Proof. Assertion (1) is proved in [6, Corollary 3.11]. Assertions (1) and (2) are equivalent over a Cohen–Macaulay ring of dimension d. Assertion (3) follows from (2). Assertion (4) is proved in [26, Theorem 1.2(i)]. Assertion (5) follows from (4), Theorem 2.1 and Remark 2.2. Assertion (6) is proved in [26, Theorem 1.2(ii)]. $\square$

The following construction leads to other classes of almost Cohen–Macaulay algebras. Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension d and depth r. Let Q be an ideal generated by a system of parameters that forms a proper sequence. Let us now explore the depth of the Rees algebra of Q. For this we make use of the approximation complex $\mathcal{Z}(Q)$ (see [14] for details):

$$\begin{aligned} 0 \to Z_d \otimes_{\mathbf{R}} \mathbf{B}[-d] \to \cdots \to Z_{d-r+1} \otimes_{\mathbf{R}} \mathbf{B}[-d+r-1] \\ \to \cdots \to Z_1 \otimes_{\mathbf{R}} \mathbf{B}[-1] \to \mathbf{B} \to 0, \end{aligned}$$

where $\mathbf{B} = \mathbf{R}[T_1, \ldots, T_d]$ and the module Z_i is that of the i-cycles of the Koszul complex $K(Q)$.

To determine the depth of the coefficient modules of $\mathcal{Z}(Q)$, consider the Koszul complex $K(Q)$:

$$0 \to K_d \longrightarrow K_{d-1} \longrightarrow \cdots \longrightarrow K_{d-r} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \to 0.$$

The subcomplex up to K_{d-r} is acyclic since depth $\mathbf{R} = r$. Thus, for $i \geqslant d - r + 1$, the module Z_i of i-cycles has a minimal free resolution

$$0 \to K_d \longrightarrow K_{d-1} \longrightarrow \cdots \longrightarrow K_{i+1} \longrightarrow Z_i \to 0,$$

in particular, projdim $Z_i = d - 1 - i$. By the Auslander–Buchsbaum equality, we have depth $Z_i = r - d + 1 + i$. Thus, depth $H_0(\mathcal{Z}) \geqslant r + 1$. In particular, if I is generated by a d-sequence, depth $\mathbf{R}[Q\mathbf{T}] = r + 1$.

PROPOSITION 2.4. *Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension d and depth $d - 1$, let I be an $\mathfrak{m}$-primary ideal, and let Q be one of its minimal reductions. Denote by φ the matrix of syzygies of Q. Suppose that Q is generated by a d-sequence and that the Sally module $S_Q(I)$ is Cohen–Macaulay. If the first Fitting ideal $I_1(\varphi)$ of Q is contained in I, then* depth $\mathrm{gr}_I(\mathbf{R}) \geqslant d - 1$.

Proof. We start with the exact sequence defining $S_Q(I)$:

$$0 \to I\mathbf{R}[Q\mathbf{T}] \longrightarrow I\mathbf{R}[I\mathbf{T}] \longrightarrow S_Q(I) \to 0.$$

From the theory of the approximation complex (see [14]), we have depth $\mathrm{gr}_Q(\mathbf{R}) = d - 1$, while the calculation above shows that depth $\mathbf{R}[Q\mathbf{T}] = d$. Since $I_1(\varphi) \subset I$, we have that $\mathbf{R}/I \otimes_{\mathbf{R}} \mathrm{gr}_Q(\mathbf{R}) = \mathbf{R}/I[\mathbf{T}_1, \ldots, \mathbf{T}_d]$. Consider the exact sequence

$$0 \to I\mathbf{R}[Q\mathbf{T}] \longrightarrow \mathbf{R}[Q\mathbf{T}] \longrightarrow \mathbf{R}/I[\mathbf{T}_1, \ldots, \mathbf{T}_d] \to 0.$$

It yields depth $I\mathbf{R}[QT] \geqslant d$. Since $S_Q(I)$ is Cohen–Macaulay, we have depth $I\mathbf{R}[I\mathbf{T}] \geqslant d$. Now, we make use of the tautological exact sequence

$$0 \to I\mathbf{T}\mathbf{R}[I\mathbf{T}] \longrightarrow \mathbf{R}[I\mathbf{T}] \longrightarrow \mathbf{R} \to 0$$

to get depth $\mathbf{R}[I\mathbf{T}] \geqslant d - 1$. Finally, from $I\mathbf{T}\mathbf{R}[I\mathbf{T}] \simeq I\mathbf{R}[I\mathbf{T}][-1]$ and the exact sequence

$$0 \to I\mathbf{R}[I\mathbf{T}] \longrightarrow \mathbf{R}[I\mathbf{T}] \longrightarrow \mathrm{gr}_I(\mathbf{R}) \to 0,$$

we get depth $\mathrm{gr}_I(\mathbf{R}) \geqslant d - 1$. □

COROLLARY 2.5. *Let $(\mathbf{R}, \mathfrak{m})$ be a Buchsbaum local ring of dimension d and depth $d - 1$. Let I be an $\mathfrak{m}$-primary ideal, and let Q be one of its minimal reductions. Denote by φ the matrix of syzygies of Q. Suppose that the Sally module $S_Q(I)$ is Cohen–Macaulay. If $I_1(\varphi) \subset I$, then* $\mathrm{r}(I) + 1$ *is a bound on the degrees of the defining equations of* $\mathrm{gr}_I(\mathbf{R})$.

Proof. The assertion follows from Proposition 2.4 and [26, Theorem 1.2(ii)]. □

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Change of rings for Sally modules

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 1$, and let $\varphi : \mathbf{R} \to \mathbb{S}$ be a finite homomorphism. Suppose that $\mathbb{S}$ is a Cohen–Macaulay ring. Let I be an $\mathfrak{m}$-primary $\mathbf{R}$-ideal with a minimal reduction Q. For each maximal ideal $\mathfrak{M}$ of $\mathbb{S}$, the ideal $I\mathbb{S}_{\mathfrak{M}}$ is $\mathfrak{M}\mathbb{S}_{\mathfrak{M}}$-primary and $Q\mathbb{S}_{\mathfrak{M}}$ is a minimal reduction of $I\mathbb{S}_{\mathfrak{M}}$. We define the Sally module of $I\mathbb{S}$ relative to $Q\mathbb{S}$ in the same manner:

$$S_{Q\mathbb{S}}(I\mathbb{S}) = \bigoplus_{n \geqslant 1} I^{n+1}\mathbb{S}/IQ^n\mathbb{S}.$$

Note that $S_{Q\mathbb{S}}(I\mathbb{S})$ is a finitely generated graded $\mathrm{gr}_Q(\mathbf{R})$-module. Its localization at $\mathfrak{M}$ gives $S_{Q\mathbb{S}_{\mathfrak{M}}}(I\mathbb{S}_{\mathfrak{M}})$.

Consider the natural mapping $\mathbf{f}_\varphi : \mathbb{S} \otimes_{\mathbf{R}} S_Q(I) \to S_{Q\mathbb{S}}(I\mathbb{S})$, which is a graded surjection. Let $\nu = \nu_{\mathbf{R}}(\mathbb{S})$ denote the minimal number of generators of $\mathbb{S}$ as an $\mathbf{R}$-module. Combining $\mathbf{f}_\varphi$ with a presentation $\mathbf{R}^\nu \to \mathbb{S} \to 0$ gives rise to a homogeneous surjection of graded modules

$$[S_Q(I)]^\nu \to S_{Q\mathbb{S}}(I\mathbb{S}).$$

The following takes information from this construction into Theorem 2.1. It is a factor in our estimation of several reduction number calculations.

THEOREM 2.6. (Change of Rings Theorem) *Let $\mathbf{R}, \mathbb{S}, I, Q$ be as above. Then,*

(1) *if* $\dim S_Q(I) < d$, *then* $S_{Q\mathbb{S}}(I\mathbb{S}) = 0$;
(2) *if* $\mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) \neq 0$, *then* $\dim S_Q(I) = d$;
(3) $\mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) \leqslant \nu_{\mathbf{R}}(\mathbb{S}) \cdot \mathrm{s}_Q(I)$.

Proof. All of the assertions follow from the surjection $[S_Q(I)]^\nu \to S_{Q\mathbb{S}}(I\mathbb{S})$, where $\nu = \nu_{\mathbf{R}}(\mathbb{S})$, and the vanishing property of Sally modules over Cohen–Macaulay rings (Theorem 2.1(2)). ∎

REMARK 2.7. It would be good to prove that the condition 2.6(2) is actually an equivalence.

To allow a discussion of some results in Section 3, we give several observations on a technique to construct Rees algebras that are almost Cohen–Macaulay; that is, depth $\mathrm{gr}_I(\mathbf{R}) \geqslant d - 1$.

We assume that $(\mathbf{R}, \mathfrak{m})$ is a Gorenstein local ring and take a parameter ideal Q. Let I be an almost complete intersection, $Q \subset I$. We look only at cases when $\lambda(I/Q)$ is small. Recall that if $\lambda(I/Q) = 1$, then $I^2 = QI$. Let

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us consider Northcott ideals (see [29, Theorem 1.113]). Let $\mathbf{R}=k[x,y]$, and let $\mathfrak{m}=(x,y)$. Let $Q=(a,b)$ be an $\mathfrak{m}$-primary ideal of $\mathbf{R}$. Suppose that $Q\subset\mathfrak{m}^{n+1}$, $n\geqslant 2$, and let $L=(x,y^n)$. Then, $I=Q:L=(a,b,c)$ is a proper ideal of $\mathbf{R}$. Recall how c arises. Write

$$a = a_1x + a_2y^n,$$
$$b = b_1x + b_2y^n,$$

for some a_1,a_2,b_1,b_2 in $\mathbf{R}$. Then, we have $c=a_1b_2-a_2b_1$. Note that $Q:c=(x,y^n)$. Thus, $\lambda(I/Q)=\lambda(\mathbf{R}/(x,y^n))=n$.

For these ideals Q, we further have the following.

Proposition 2.8. *The ideal $Q:(x,y^n)$ is integral over Q.*

Proof. To control the integrality of direct links, one can make use of [4, Theorem 2.3], [30, Theorem 3.1] or [31, Theorem 7.4]:

$$(Q:\mathfrak{m}^n)^2=Q(Q:\mathfrak{m}^n);$$

in particular, $Q:\mathfrak{m}^n$ is integral over Q. Since $\mathfrak{m}^n\subset(x,y^n)$, $Q:(x,y^n)\subset Q:\mathfrak{m}^n$. Thus, $Q:(x,y^n)$ is also integral over Q. ∎

In this study, it is important to find the lengths of the modules I^{n+1}/QI^n. One case that is important for us is given by the following.

Proposition 2.9. [15, Proposition 3.12] *Let $\mathbf{R}$ be a Gorenstein local ring. Let I be an $\mathfrak{m}$-primary ideal of $\mathbf{R}$, and let Q be a minimal reduction of I. Suppose that $\nu(I/Q)=1$. Then,*

$$\lambda(I^2/QI)=\lambda(I/Q)-\lambda(\mathbf{R}/I_1(\varphi)),$$

where φ is the matrix of syzygies of I.

Example 2.10. Let us consider some examples. In the case of the Northcott ideals above, we have

$$\varphi=\begin{bmatrix}-b_1 & -b_2\\ a_1 & a_2\\ -y^n & x\end{bmatrix},$$

$I_1(\varphi)=(x,y^n,a_1,a_2,b_1,b_2)=(x,y^m)$, where $1\leqslant m\leqslant n$. Appropriate choices for a_2 or b_2 give rise to all values for m in the range. For example, let $Q=(x^4+y^5,x^2y^2+xy^3)$, let $L=(x,y^3)$, and let $I=Q:L$. Then,

$\lambda(I/Q)=3$ and $I_1(\varphi)=(x,y^2)$. Hence, by Proposition 2.9, $\lambda(I^2/QI)=1$. In fact, the reduction number $r_Q(I)=2$. More generally, choosing $m=n-1$ above gives $\lambda(I^2/QI)=1$, by Proposition 2.9. Moreover, if $(a_1,a_2,b_1,b_2)\subset(x,y^n)$, then $I^2=QI$, by Proposition 2.9.

§3. Almost complete intersections

This section makes use of the behavior of the components of Sally modules of almost complete intersections in order to derive uniform bounds for reduction numbers. We begin with the following result.

Proposition 3.1. *Let* $(\mathbf{R},\mathfrak{m})$ *be a Noetherian local ring of dimension* d *with infinite residue field, and let* I *be an* $\mathfrak{m}$*-primary almost complete intersection ideal with a minimal reduction* Q*. If for some integer* n*,* $\lambda(I^n/QI^{n-1})=1$*, then* $\mathrm{r}(I)\leqslant n\nu(\mathfrak{m})-1$*.*

Proof. Since Q is a minimal reduction of I, it is generated by a subset of the minimal set of generators of I; that is, $I=(Q,a)$. The expected number of generators of I^n is $\binom{d+n}{n}$. A lesser value for $\nu(I^n)$ would imply by [7, Theorem 1] that $I^n=JI^{n-1}$ for some minimal reduction J of I, and therefore $\mathrm{r}(I)\leqslant n-1$.

Suppose that $\nu(I^n)=\binom{d+n}{n}$. Since $\lambda(I^n/QI^{n-1})=1$, $\mathfrak{m}I^n\subset QI^{n-1}$. Moreover, we have

$$\mathfrak{m}QI^{n-1}\subset\mathfrak{m}I^n\subset QI^{n-1}\subset I^n.$$

Note that

$$\begin{aligned}\lambda(\mathfrak{m}I^n/\mathfrak{m}QI^{n-1}) &= \lambda(I^n/\mathfrak{m}QI^{n-1})-\lambda(I^n/\mathfrak{m}I^n)\\ &= \lambda(I^n/QI^{n-1})+\nu(QI^{n-1})-\nu(I^n)=0.\end{aligned}$$

It follows that $\mathfrak{m}I^n=\mathfrak{m}QI^{n-1}$. From the Cayley–Hamilton theorem, we have $(I^n)^s=(QI^{n-1})(I^n)^{s-1}$, where $s=\nu(\mathfrak{m})$. In particular, $I^{ns}=QI^{ns-1}$, as desired. ∎

Next, we give examples of ideals with $\lambda(I^n/QI^{n-1})=1$ for some integer n.

Example 3.2. Let $\mathbf{R}=k[x,y]$, where k is an infinite field, $Q=(x^6+y^5,x^7+x^5y)$, and consider the ideals $Q:L$, where $L=(x,y^n)$ for some positive integer n.

(1) If $I=Q:(x,y^2)$, then $I^2=QI$.
(2) If $I=Q:(x,y^3)$, then $\lambda(I^2/QI)=1$, $\mathrm{r}(I)=2$.

(3) If $I = Q:(x, y^4)$, then $\lambda(I^2/QI) = 3$, $\lambda(I^3/QI^2) = 2$, $\lambda(I^4/QI^3) = 1$, $\mathrm{r}(I) = 5$.
(4) $I = Q:(x, y^5) = (x^6 + x^4y, y^5 - x^4y)$ is a complete intersection. Notice that I is not integral over Q (see Proposition 2.8).

COROLLARY 3.3. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Gorenstein local ring of dimension* d *with infinite residue field, and let* I *be an* $\mathfrak{m}$*-primary ideal with a minimal reduction* Q*. If* $\lambda(I/Q) = 2$*, then* $\mathrm{r}(I) \leqslant 2\nu(\mathfrak{m}) - 1$*.*

Proof. According to Proposition 2.9, since $\lambda(I/Q) = 2$, we have that $\lambda(I^2/QI) = 1$. □

COROLLARY 3.4. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* d *with infinite residue field, and let* I *be an* $\mathfrak{m}$*-primary almost complete intersection ideal with a minimal reduction* Q*. If* $\lambda(I^2/QI) = 1$*, then* $\mathfrak{m}I^n = \mathfrak{m}IQ^{n-1}$ *for* $n \geqslant 2$*. In particular,* $\mathfrak{m}S_Q(I) = 0$*.*

Proof. Recall that $\mathrm{r}(I) > 1$, by Proposition 2.3(5). Then, from the proof of Proposition 3.1, we have that $\mathfrak{m}I^2 = \mathfrak{m}QI$. If $n > 2$, it follows that

$$\mathfrak{m}I^n = \mathfrak{m}I^2 I^{n-2} = \mathfrak{m}QI^{n-1} = \cdots = \mathfrak{m}Q^{n-1}I,$$

as desired. □

Following [5], we study relationships between $\mathrm{r}(I)$ and the multiplicity $f_0(I)$ of the special fiber $\mathcal{F}(I) = \mathbf{R}[I\mathbf{T}] \otimes \mathbf{R}/\mathfrak{m}$ of the Rees algebra of I.

PROPOSITION 3.5. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geqslant 1$*, and let* I *be an* $\mathfrak{m}$*-primary ideal that is an almost complete intersection. Let* Q *be a minimal reduction of* I*, and let* $S_Q(I)$ *denote the Sally module of* I *relative to* Q*. Setting* $\mathbf{A} = \mathcal{F}(Q)$*, there is an exact sequence*

$$\mathbf{A} \oplus \mathbf{A}[-1] \xrightarrow{\overline{\varphi}} \mathcal{F}(I) \longrightarrow S_Q(I)[-1] \otimes \mathbf{R}/\mathfrak{m} \to 0. \tag{1}$$

Proof. Let $I = (Q, a)$. Consider the exact sequence introduced in [5, Proof of 2.1]:

$$\mathbf{R}[Q\mathbf{T}] \oplus \mathbf{R}[Q\mathbf{T}][-1] \xrightarrow{\varphi} \mathbf{R}[I\mathbf{T}] \longrightarrow S_Q(I)[-1] \to 0,$$

where φ is the map defined by $\varphi(f, g) = f + ag\mathbf{T}$. Tensoring this sequence with $\mathbf{R}/\mathfrak{m}$ yields the desired sequence and the induced map $\overline{\varphi}$. □

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COROLLARY 3.6. *In the set up of Proposition 3.5, suppose that* $\mathbf{R}$ *has infinite residue field and that* $\lambda(I^2/QI) = 1$. *Then,*

(1) $\mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 1 \leqslant f_0(I) \leqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 2 = \mathrm{r}(I) + 1$;
(2) $\mathcal{F}(I)$ *is Cohen–Macaulay if and only if* $f_0(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 2$.

Proof. First, we notice that $\mathfrak{m}S_Q(I) = 0$ by Corollary 3.4, so that $S_Q(I) \otimes \mathbf{R}/\mathfrak{m} \simeq S_Q(I)$. Then, $\mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 1 \leqslant f_0(I)$ follows because $\mathbf{A}$ must embed in $\mathcal{F}(I)$ (see, for example, [19, Corollary 8.3.5]), and the cokernel of this embedding has multiplicity greater than or equal to the multiplicity of $S_Q(I)$, which is $\mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I)$. The inequality $f_0(I) \leqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 2$ follows by reading multiplicities in the exact sequence (1), as in [5, Proof of 2.1]. The last equality follows from Proposition 2.3(5).

Now, assume that $\mathcal{F}(I)$ is Cohen–Macaulay. Since I is an almost complete intersection, we have that $\mathcal{F}(I)$ is a finite extension of $\mathbf{A}$ with a presentation $\mathbf{A}[z]/L$, where L has codimension 1. Since $\mathbf{A}[z]$ is factorial and L is unmixed, it follows that L is principal. Write $L = (h(z))$. Note that $h(z)$ must be monic and its degree is $f_0(I) = \mathrm{r}(I) + 1$. Now, the conclusion follows from part (1).

Conversely, the equality $f_0(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 2$ implies that $\overline{\varphi}$ is injective. Since $S_Q(I)$ is Cohen–Macaulay by Proposition 2.3(3), we have that $\mathcal{F}(I)$ is Cohen–Macaulay. □

We are now ready for one of our main results.

THEOREM 3.7. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geqslant 1$ *with infinite residue field, and let* I *be an* $\mathfrak{m}$*-primary ideal which is an almost complete intersection. If for some minimal reduction* Q *of* I, $\lambda(I^2/QI) = 1$, *then the special fiber ring* $\mathcal{F}(I)$ *is Cohen–Macaulay. In particular,* $\mathrm{r}(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 1 = f_0(I) - 1$.

Proof. The Cohen–Macaulayness of $\mathcal{F}(I)$ follows once we show that the map $\overline{\varphi}$ in sequence (1) is injective. To prove that $\overline{\varphi}$ is an embedding, we must show that its image C has multiplicity 2. Since

$$C = \mathbf{R}/\mathfrak{m} \oplus \left(\bigoplus_{n \geqslant 1} (IQ^{n-1} + \mathfrak{m}I^n)/\mathfrak{m}I^n \right),$$

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we have that the cokernel D of the embedding $\mathbf{A} \hookrightarrow C$ is

$$D = \bigoplus_{n \geqslant 1} (IQ^{n-1} + \mathfrak{m}I^n)/(Q^n + \mathfrak{m}I^n).$$

We examine its Hilbert function, $\lambda((IQ^{n-1} + \mathfrak{m}I^n)/(Q^n + \mathfrak{m}I^n))$, for $n \gg 0$. Since $\mathfrak{m}I^n = \mathfrak{m}Q^{n-1}I$ by Corollary 3.4, we have

$$\begin{aligned} &\lambda((IQ^{n-1} + \mathfrak{m}I^n)/(Q^n + \mathfrak{m}I^n)) \\ &\quad = \lambda(Q^{n-1}/(Q^n + \mathfrak{m}I^n)) - \lambda(Q^{n-1}/(IQ^{n-1} + \mathfrak{m}I^n)) \\ &\quad = \lambda((Q^{n-1}/Q^n) \otimes \mathbf{R}/(Q + \mathfrak{m}I)) - \lambda((Q^{n-1}/Q^n) \otimes \mathbf{R}/I) \\ &\quad = \lambda(I/(Q + \mathfrak{m}I)) \binom{n+d-2}{d-1}, \end{aligned}$$

which shows that D has dimension d and multiplicity $\lambda(I/(Q + \mathfrak{m}I)) = 1$. This concludes the proof. The last assertion follows from Corollary 3.6. $\square$

§4. Buchsbaum rings

The formula of Theorem 1.1, $\mathrm{s}_Q(I) + 1 = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + 1 \geqslant \mathrm{r}_Q(I)$, suggests several questions. Does it hold in higher dimensions? If $\mathbf{R}$ is not Cohen–Macaulay but still $\dim \mathbf{R} = 2$, what is a possible bound? We aim at exploring bounds of the form

$$\mathrm{s}_Q(I) + 1 + \text{Cohen–Macaulay deficiency of } \mathbf{R} \geqslant \mathrm{r}_Q(I).$$

We study this issue for Buchsbaum rings, where the natural Cohen–Macaulay deficiency is $I(\mathbf{R})$, the Buchsbaum invariant of $\mathbf{R}$ (see [25, Chapter 1, Theorem 1.12]). We recall that $I(\mathbf{R}) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(\mathrm{H}^i_{\mathfrak{m}}(\mathbf{R}))$, or $I(\mathbf{R}) = \sum_{i=1}^{d} (-1)^i \mathrm{e}_i(Q)$ (see [25, Chapter 1, Propositions 2.6, 2.7]).

S_2-fication of generalized Cohen–Macaulay rings

If $\mathbf{R}$ is a Noetherian ring with total ring of fractions K, the smallest finite extension $\varphi : \mathbf{R} \to \mathbb{S} \subset K$ with the S_2 property is called the S_2-fication of $\mathbf{R}$ (see [29, Definition 6.20]). We refer to either φ or $\mathbb{S}$ as the S_2-ficaton of $\mathbf{R}$. Let $\mathbf{R}$ be a generalized Cohen–Macaulay local ring of dimension $d \geqslant 2$ and positive depth. Suppose that $\mathbf{R}$ has a canonical module $\omega_{\mathbf{R}}$. Then, the natural morphism

$$\varphi : \mathbf{R} \to \mathbb{S} = \mathrm{Hom}_{\mathbf{R}}(\omega_{\mathbf{R}}, \omega_{\mathbf{R}})$$

is an S_2-fication of $\mathbf{R}$ (see [1, Theorem 3.2]). Note that $\ker(\varphi) = \mathrm{H}^0_{\mathfrak{m}}(\mathbf{R})$ (see [1, 1.11], [25, Chapter 1, Lemma 2.2]). Let us describe some elementary properties of $\mathbb{S}$.

PROPOSITION 4.1. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a generalized Cohen–Macaulay local ring of dimension at least* 2 *and of positive depth. Suppose that* $\mathbf{R}$ *has a canonical module* $\omega_{\mathbf{R}}$*, and let* $\mathbb{S} = \mathrm{Hom}_{\mathbf{R}}(\omega_{\mathbf{R}}, \omega_{\mathbf{R}})$.

(1) *If* $\mathfrak{a} = \mathrm{ann}(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$*, then* $\mathfrak{a}\mathbb{S} = \mathfrak{a}$.
(2) $\mathbb{S}$ *is a semi-local ring.*
(3) $\lambda(\mathbb{S}/\mathfrak{m}\mathbb{S}) = 1 + \nu(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})) \leqslant 1 + \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathbf{R} \xrightarrow{\varphi} \mathbb{S} \longrightarrow D \longrightarrow 0. \tag{2}$$

Note that $D \simeq \mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})$ is a module of finite length. From $\mathfrak{a}\mathbb{S} \subset \mathbf{R}$, we have $\mathfrak{a}\mathbb{S} \cdot \mathbb{S} \subset \mathbf{R}$, so that $\mathfrak{a}\mathbb{S} \subset \mathfrak{a}$. On the other hand, by Nakayama's lemma, $\mathfrak{m}\mathbb{S}$ is contained in the Jacobson radical of $\mathbb{S}$. Therefore, $\mathbb{S}$ is semi-local. The remaining assertion is clear. □

Now, we give an explicit description of the S_2-fication of these rings.

THEOREM 4.2. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a generalized Cohen–Macaulay local ring of dimension at least* 2 *and positive depth. Let* $\mathfrak{a} = \mathrm{ann}(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$*. Then,* $\mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathfrak{a})$ *is an* S_2*-fication of* $\mathbf{R}$*. Moreover, if* $\mathbf{R}$ *is a Buchsbaum ring, then* $\mathrm{Hom}_{\mathbf{R}}(\mathfrak{m}, \mathfrak{m})$ *is an* S_2*-fication of* $\mathbf{R}$.

Proof. We denote $\mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathfrak{a})$ by E. Then, there is a natural morphism $\varphi : \mathbf{R} \hookrightarrow E$, which is an isomorphism at each localization at $\mathfrak{p} \neq \mathfrak{m}$. We must show that E satisfies (S_2). It suffices to show that grade$(\mathfrak{m})$ relative to E is at least 2.

To see this, we pass to the completion $\widehat{\mathbf{R}}$ of $\mathbf{R}$ and show that grade$(\widehat{\mathfrak{m}})$ relative to $\widehat{E}$ is at least 2. Since $\widehat{\mathbf{R}}$ has a canonical module ω, let $\mathbb{S} = \mathrm{Hom}_{\widehat{\mathbf{R}}}(\omega, \omega)$ be the S_2-fication of $\widehat{\mathbf{R}}$. Now, we identify $\mathbb{S}$ to a subring of the total ring of fractions of $\widehat{\mathbf{R}}$. By flatness, $\mathrm{ann}(\mathbb{S}/\widehat{\mathbf{R}}) = \widehat{\mathfrak{a}}$, and by Proposition 4.1, we have $\widehat{\mathfrak{a}}\mathbb{S} = \widehat{\mathfrak{a}}$. Thus,

$$\mathbb{S} \subset \widehat{\mathfrak{a}} : \widehat{\mathfrak{a}} = \mathrm{Hom}_{\widehat{\mathbf{R}}}(\widehat{\mathfrak{a}}, \widehat{\mathfrak{a}}) = \widehat{E} \subset \mathrm{Hom}_{\mathbb{S}}(\widehat{\mathfrak{a}}, \widehat{\mathfrak{a}}) = \mathbb{S}.$$

The last equality is because $\widehat{\mathfrak{a}}$ is an ideal of grade at least 2 of the ring $\mathbb{S}$. □

The Rossi index for Buchsbaum rings

The next result is an extension of Theorem 1.1 to Buchsbaum rings. We denote by $\mathrm{s}_Q(I)$ the multiplicity of the Sally module of I relative to Q.

THEOREM 4.3. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a two-dimensional Buchsbaum local ring of positive depth. Let* $\mathbb{S}$ *be the* S_2*-fication of* $\mathbf{R}$*. Let* I *be an* $\mathfrak{m}$*-primary ideal with a minimal reduction* Q*.*

(1) *If* $I = I\mathbb{S}$*, then* $\mathrm{s}_Q(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) - \mathrm{e}_1(Q)$ *and* $\mathrm{r}_Q(I) \leqslant \mathrm{s}_Q(I) + 1$.
(2) *If* $I \subsetneq I\mathbb{S}$*, then* $\mathrm{r}_Q(I) \leqslant (1 - \mathrm{e}_1(Q))^2 \mathrm{s}_Q(I) - 2\mathrm{e}_1(Q) + 1$.

Proof. We may assume that $\mathbf{R}$ is complete (see [25, Chapter 1, Lemma 1.13]). Let $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_t\}$ be the maximal ideals of $\mathbb{S}$, and denote by $\mathbb{S}_1, \ldots, \mathbb{S}_t$ the corresponding localizations. For each $i = 1, \ldots, t$, let f_i be the relative degree $[\mathbb{S}/\mathfrak{M}_i : \mathbf{R}/\mathfrak{m}]$. If V is an $\mathbf{R}$-module of finite length which is also an $\mathbb{S}$-module, then we have that

$$\lambda_{\mathbf{R}}(V) = \sum_{i=1}^{t} \lambda_{\mathbb{S}_i}(V\mathbb{S}_i) f_i.$$

Applying this formula to the modules $I^n\mathbb{S}/I^{n+1}\mathbb{S}$, $I^{n+1}\mathbb{S}/IQ^n\mathbb{S}$, for $n \gg 0$, and $\mathbb{S}/I\mathbb{S}$, we get

$$\mathrm{e}_0^{\mathbf{R}}(I\mathbb{S}) = \sum_{i=1}^{t} \mathrm{e}_0^{\mathbb{S}_i}(I\mathbb{S}_i) f_i, \qquad \mathrm{e}_1^{\mathbf{R}}(I\mathbb{S}) = \sum_{i=1}^{t} \mathrm{e}_1^{\mathbb{S}_i}(I\mathbb{S}_i) f_i,$$
$$\mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) = \sum_{i=1}^{t} \mathrm{s}_{Q\mathbb{S}_i}(I\mathbb{S}_i) f_i, \qquad \lambda_{\mathbf{R}}(\mathbb{S}/I\mathbb{S}) = \sum_{i=1}^{t} \lambda_{\mathbb{S}_i}(\mathbb{S}_i/I\mathbb{S}_i) f_i.$$

Note that

$$\mathrm{r}_{Q\mathbb{S}}(I\mathbb{S}) = \max\{\mathrm{r}_{Q\mathbb{S}_1}(I\mathbb{S}_1), \ldots, \mathrm{r}_{Q\mathbb{S}_t}(I\mathbb{S}_t)\}.$$

Without loss of generality, we may assume that $\mathrm{r}_{Q\mathbb{S}}(I\mathbb{S}) = \mathrm{r}_{Q\mathbb{S}_1}(I\mathbb{S}_1)$.

Applying Rossi's formula [22, Corollary 1.5] to $I\mathbb{S}_1$, we get

$$\begin{aligned}
\mathrm{r}_{Q\mathbb{S}}(I\mathbb{S}) = \mathrm{r}_{Q\mathbb{S}_1}(I\mathbb{S}_1) &\leqslant 1 + \mathrm{e}_1^{\mathbb{S}_1}(I\mathbb{S}_1) - \mathrm{e}_0^{\mathbb{S}_1}(I\mathbb{S}_1) + \lambda^{\mathbb{S}_1}(\mathbb{S}_1/I\mathbb{S}_1) \\
&\leqslant 1 + [\mathrm{e}_1^{\mathbb{S}_1}(I\mathbb{S}_1) - \mathrm{e}_0^{\mathbb{S}_1}(I\mathbb{S}_1) + \lambda^{\mathbb{S}_1}(\mathbb{S}_1/I\mathbb{S}_1)] f_1 \\
&\leqslant 1 + \sum_{i=1}^{t} [\mathrm{e}_1^{\mathbb{S}_i}(I\mathbb{S}_i) - \mathrm{e}_0^{\mathbb{S}_i}(I\mathbb{S}_i) + \lambda^{\mathbb{S}_i}(\mathbb{S}_i/I\mathbb{S}_i)] f_i \\
&\leqslant 1 + \sum_{i=1}^{t} \mathrm{s}_{Q\mathbb{S}_i}(I\mathbb{S}_i) f_i \\
&= 1 + \mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}).
\end{aligned}$$

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Suppose that $I = I\mathbb{S}$. Then, $S_Q(I) = S_{Q\mathbb{S}}(I\mathbb{S})$. Recall that by the exact sequence (2), we have that $\lambda_{\mathbf{R}}(\mathbb{S}/\mathbf{R}) = \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$. Furthermore, $\lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})) = -\mathrm{e}_1(Q)$ because $\mathbf{R}$ is Buchsbaum (see [25, Chapter 1, Propositions 2.6, 2.7]). Therefore,

$$\begin{aligned}
\mathrm{s}_Q(I) = \mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) &= \sum_{i=1}^{t} \mathrm{s}_{Q\mathbb{S}_i}(I\mathbb{S}_i) f_i \\
&= \sum_{i=1}^{t} [\mathrm{e}_1^{\mathbb{S}_i}(I\mathbb{S}_i) - \mathrm{e}_0^{\mathbb{S}_i}(I\mathbb{S}_i) + \lambda^{\mathbb{S}_i}(\mathbb{S}_i/I\mathbb{S}_i)] f_i \\
&= \mathrm{e}_1^{\mathbf{R}}(I\mathbb{S}) - \mathrm{e}_0^{\mathbf{R}}(I\mathbb{S}) + \lambda_{\mathbf{R}}(\mathbb{S}/I\mathbb{S}) \\
&= \mathrm{e}_1^{\mathbf{R}}(I\mathbb{S}) - \mathrm{e}_0^{\mathbf{R}}(I\mathbb{S}) + \lambda_{\mathbf{R}}(\mathbf{R}/I\mathbb{S}) + \lambda_{\mathbf{R}}(\mathbb{S}/\mathbf{R}) \\
&= \mathrm{e}_1^{\mathbf{R}}(I\mathbb{S}) - \mathrm{e}_0^{\mathbf{R}}(I\mathbb{S}) + \lambda_{\mathbf{R}}(\mathbf{R}/I\mathbb{S}) - \mathrm{e}_1(Q) \\
&= \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) - \mathrm{e}_1(Q).
\end{aligned}$$

Notice that if $I = Q\mathbb{S}$, then $\mathrm{r}_Q(I) = 1$, and the conclusion of the theorem holds. Therefore, we may assume that $I \neq Q\mathbb{S}$. In this case, we have that

$$\mathrm{r}_Q(I) = \mathrm{r}_{Q\mathbb{S}}(I) = \mathrm{r}_{Q\mathbb{S}}(I\mathbb{S}) \leqslant \mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) + 1 = \mathrm{s}_Q(I) + 1.$$

Now, assume that $I \subsetneq I\mathbb{S}$. Note that $\nu(\mathbb{S}) = 1 + \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})) = 1 - \mathrm{e}_1(Q)$, by Proposition 4.1(3). Let $r = \mathrm{r}_{Q\mathbb{S}}(I\mathbb{S})$. Then, $I^{r+1}\mathbb{S} = QI^r\mathbb{S}$. By [29, Proposition 1.118], we have

$$I^{(r+1)\nu(\mathbb{S})} = QI^{(r+1)\nu(\mathbb{S})-1}.$$

Using Theorem 2.6(3), we obtain

$$\begin{aligned}
\mathrm{r}_Q(I) \leqslant \nu(\mathbb{S})r + \nu(\mathbb{S}) - 1 &\leqslant \nu(\mathbb{S})\mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) + 2\nu(\mathbb{S}) - 1 \\
&\leqslant \nu(\mathbb{S})^2 \mathrm{s}_Q(I) + 2\nu(\mathbb{S}) - 1 \\
&= (1 - \mathrm{e}_1(Q))^2 \mathrm{s}_Q(I) - 2\mathrm{e}_1(Q) + 1. \qquad \square
\end{aligned}$$

REMARK 4.4. In the set-up of Theorem 4.3, suppose that $I \subsetneq I\mathbb{S}$, and let $r = \mathrm{r}_{Q\mathbb{S}}(I\mathbb{S})$. Since $\mathfrak{m}\mathbb{S} = \mathfrak{m}$, by Proposition 4.1, we have

$$I^{r+1}\mathfrak{m} = I^{r+1}\mathfrak{m}\mathbb{S} = QI^r\mathfrak{m}\mathbb{S} = QI^r\mathfrak{m}.$$

Thus, we obtain

$$\begin{aligned}\mathrm{r}_Q(I) \leqslant \nu(\mathfrak{m})r + \nu(\mathfrak{m}) - 1 &\leqslant \nu(\mathfrak{m})\mathrm{s}_{Q\mathbb{S}}(I\mathbb{S}) + 2\nu(\mathfrak{m}) - 1\\ &\leqslant \nu(\mathfrak{m})\nu(\mathbb{S})\mathrm{s}_Q(I) + 2\nu(\mathfrak{m}) - 1\\ &= \nu(\mathfrak{m})(1 - \mathrm{e}_1(Q))\mathrm{s}_Q(I) + 2\nu(\mathfrak{m}) - 1.\end{aligned}$$

COROLLARY 4.5. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a two-dimensional Buchsbaum local ring of depth zero. Let* $\mathbb{S}$ *be the* S_2*-fication of* $\mathbf{R}$*, and let* I *be an* $\mathfrak{m}$*-primary ideal with a minimal reduction* Q*. Set* $\overline{\mathbf{R}} = \mathbf{R}/\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R})$*,* $\overline{I} = I\overline{\mathbf{R}}$ *and* $\overline{Q} = Q\overline{\mathbf{R}}$*.*

(1) *If* $\overline{I} = \overline{I}\mathbb{S}$*, then* $\mathrm{r}_Q(I) \leqslant \mathrm{s}_Q(I) + 2$*.*
(2) *If* $\overline{I} \subsetneq \overline{I}\mathbb{S}$*, then* $\mathrm{r}_Q(I) \leqslant (1 - \mathrm{e}_1(Q))^2\mathrm{s}_Q(I) - 2\mathrm{e}_1(Q) + 2$*.*

Proof. Note that $\overline{\mathbf{R}}$ is a two-dimensional Buchsbaum local ring of positive depth. Let $r = \mathrm{r}_{\overline{Q}}(\overline{I})$. Then, $I^{r+1} - QI^r \subset \mathrm{H}^0_{\mathfrak{m}}(\mathbf{R})$, which implies that $I^{r+2} - QI^{r+1} \subset I\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R}) = 0$. Thus, $\mathrm{r}_Q(I) \leqslant \mathrm{r}_{\overline{Q}}(\overline{I}) + 1$. Furthermore, the surjective map $S_Q(I) \to S_{\overline{Q}}(\overline{I})$ gives $\mathrm{s}_Q(I) \geqslant \mathrm{s}_{\overline{Q}}(\overline{I})$. Now, the conclusion follows from Theorem 4.3 applied to $\overline{\mathbf{R}}$. ∎

COROLLARY 4.6. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a two-dimensional generalized Cohen–Macaulay local ring. Let* $\mathbb{S}$ *be the* S_2*-fication of* $\mathbf{R}$*. Let* I *be an* $\mathfrak{m}$*-primary ideal with a minimal reduction* Q*. Suppose that* $\mathbf{R}$ *has positive depth.*

(1) *If* $I = I\mathbb{S}$*, then* $\mathrm{s}_Q(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \lambda(\mathbf{R}/I) + \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$ *and* $\mathrm{r}_Q(I) \leqslant \mathrm{s}_Q(I) + 1$*.*
(2) *If* $I \subsetneq I\mathbb{S}$*, then* $\mathrm{r}_Q(I) \leqslant (1 + \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})))^2\mathrm{s}_Q(I) + 2\lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})) + 1$*.*

Suppose that $\mathbf{R}$ *has depth zero. Set* $\overline{\mathbf{R}} = \mathbf{R}/\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R})$*,* $\overline{I} = I\overline{\mathbf{R}}$ *and* $\overline{Q} = Q\overline{\mathbf{R}}$*.*

(3) *If* $\overline{I} = \overline{I}\mathbb{S}$*, then* $\mathrm{r}_Q(I) \leqslant \mathrm{s}_Q(I) + 1 + \lambda(\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R}))$*.*
(4) *If* $\overline{I} \subsetneq \overline{I}\mathbb{S}$*, then* $\mathrm{r}_Q(I) \leqslant (1 + \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})))^2\mathrm{s}_Q(I) + 2\lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R})) + 1 + \lambda(\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R}))$*.*

Proof. The proof is almost identical to those of Theorem 4.3 and Corollary 4.5, except that $\lambda_{\mathbf{R}}(\mathbb{S}/\mathbf{R}) = \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$, $\nu(\mathbb{S}) \leqslant 1 + \lambda(\mathrm{H}^1_{\mathfrak{m}}(\mathbf{R}))$ and $I^g\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R}) = 0$, where $g = \lambda(\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R}))$. ∎

EXAMPLE 4.7. [13, Example 2.8] Let k be a field, and let $\mathbf{R} = k[[X, Y, Z, W]]/(X, Y) \cap (Z, W)$, where X, Y, Z, W are variables. Then, $\mathbf{R}$ is a Buchsbaum ring of dimension 2, depth 1, and $\mathrm{e}_1(Q) = -1$ for all parameter ideals Q. Let x, y, z, w denote the images of X, Y, Z, W in $\mathbf{R}$. Choose an integer $q>0$, and set $I=(x^4, x^3y, xy^3, y^4)+(z,w)^q$. Notice that the S_2-fication of $\mathbf{R}$ is $\mathbb{S}=k[[z,w]]\oplus k[[x,y]]$, so that $I=I\mathbb{S}$. For every $n \geqslant 0$, we have an exact sequence

$$0 \to \mathbf{R}/I^{n+1} \to k[[x,y,z,w]]/[(x,y)+I^{n+1}]$$
$$\oplus\ k[[x,y,z,w]]/[(z,w)+I^{n+1}] \to k \to 0,$$

which gives that $\lambda(\mathbf{R}/I^{n+1})=(q^2+16)\binom{n+2}{2}-\frac{q^2-q+12}{2}\binom{n+1}{1}-1$ for all $n \geqslant 1$ and $\lambda(\mathbf{R}/I)=\frac{q^2+q+20}{2}$. Hence, $\mathrm{e}_0(I)=q^2+16, \mathrm{e}_1(I)=\frac{q^2-q+12}{2}$, so that $\mathrm{e}_0(I)-\lambda(\mathbf{R}/I)=\mathrm{e}_1(I)$.

Let $Q=(x^4-z^q, y^4-w^q)$. It is easy to see that Q is a minimal reduction of I with $\mathrm{r}_Q(I)=2$. By Theorem 4.3(1), we have that $\mathrm{s}_Q(I)=1$. Hence, the bound of Theorem 4.3(1) is sharp: $\mathrm{r}_Q(I)=\mathrm{s}_Q(I)+1$.

Sally modules for Buchsbaum rings

The preceding section requires formulas for the multiplicity $\mathrm{s}_Q(I)$ of the Sally module of I relative to Q, at least in the case that $\mathbf{R}$ is Buchsbaum. We recall a result of Corso [3, Proposition 2.8].

PROPOSITION 4.8. *Let* $(\mathbf{R},\mathfrak{m})$ *be a Noetherian local ring of dimension* d*, let* I *be an* $\mathfrak{m}$*-primary ideal, and let* Q *be one of its minimal reductions. Suppose that* $\dim S_Q(I)=d$*. Then,*

$$\mathrm{s}_Q(I) \leqslant \mathrm{e}_1(I)-\mathrm{e}_1(Q)-\mathrm{e}_0(I)+\lambda(\mathbf{R}/I).$$

The following was proved in [20, Proposition 2.5(1)(i)], but we give a slightly different proof for the result, which uses the approximation complex.

COROLLARY 4.9. *Let* $(\mathbf{R},\mathfrak{m})$ *be a Noetherian local ring of dimension* d*, let* I *be an* $\mathfrak{m}$*-primary ideal, and let* Q *be one of its minimal reductions. Suppose that* $\dim S_Q(I)=d$*. Let* φ *be the matrix of syzygies of* Q*. If* Q *is generated by a* d*-sequence and* $I_1(\varphi)\subset I$*, then*

$$\mathrm{s}_Q(I)=\mathrm{e}_1(I)-\mathrm{e}_1(Q)-\mathrm{e}_0(I)+\lambda(\mathbf{R}/I).$$

Proof. Let $\overline{G}=\mathrm{gr}_Q(\mathbf{R})\otimes \mathbf{R}/I$. By the proof of [3, Proposition 2.8], we have

$$\mathrm{s}_Q(I)=\mathrm{e}_1(I)-\mathrm{e}_0(I)-\mathrm{e}_1(Q)+\mathrm{e}_0(\overline{G}).$$

Set $\mathbf{B}=\mathbf{R}[T_1,\ldots,T_d]$. Since Q is generated by a d-sequence, the approximation complex

$$0 \to \mathrm{H}_d(Q)\otimes \mathbf{B}[-d] \to \cdots \to \mathrm{H}_1(Q)\otimes \mathbf{B}[-1] \to \mathrm{H}_0(Q)\otimes \mathbf{B} \to \mathrm{gr}_Q(\mathbf{R}) \to 0$$

is acyclic (see [14, Theorem 5.6]). By tensoring this complex by $\mathbf{R}/I$, we get the exact sequence

$$\mathrm{H}_1(Q) \otimes \mathbf{R}/I \otimes \mathbf{B}[-1] \xrightarrow{\phi} \mathbf{R}/I \otimes \mathbf{B} \longrightarrow \overline{G} \to 0.$$

Since $I_1(\varphi) \subset I$, the mapping ϕ is trivial, so that $\mathrm{e}_0(\overline{G}) = \lambda(\mathbf{R}/I)$. □

REMARK 4.10. One instance when $I_1(\varphi) \subset I$ occurs if $\mathbf{R}$ is unmixed and I is integrally closed, according to [21, Proposition 3.1].

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