ON THE SPECTRUM OF $C_1$ AS AN OPERATOR ON $bv_0$

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Abstract

In 1985 John Reade determined the spectrum of $C_1$ regarded as an operator on the space $c_0$ of all null sequences normed by $\|x\| = \sup_{n \geq 0} |x_n|$. It is the purpose of this paper to determine the spectrum of $C_1$ regarded as an operator on the space $bv_0$ of all sequences $x$ such that $x_k \to 0$ as $k \to \infty$ and $\|x\| = \sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$.


Notation. $s$; $c_0$; $l_1$; $bv_0$; $bs$; $T^*$; $X^*B(X)$; $A^*$; $\sigma(T)$; $O(1)$; $o(1)$; $\approx$; $\Re(z)$; will denote the set of all sequences; convergent to zero sequences, that is, null sequences; sequences such that $\sum_{k=0}^{\infty} |x_k| < \infty$; sequences such that $\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$; bounded series, that is, sequences $x$ such that $\sup_{n \geq 0} |\sum_{k=0}^{n} x_k| < \infty$; the adjoint operator of $T$; the space of all continuous linear functionals on $X$, that is, the continuous dual of $X$; the linear space of all bounded linear operators, say, $T$ on $X$ into itself; the transposed matrix of $A$; the spectrum of $T$; capital order, that is, $x_n = O(1)$ if there exists $M \in \mathbb{R}^+$ such that $|x_n| \leq M$ for all $n$; small order, that is, $x_n = o(1)$ as $n \to \infty$, that is, $\lim_{n \to \infty} x_n = 0$; lies between two positive constant multiples, for example $a_n \asymp b_n$ means that there exist $m, M \in \mathbb{R}^+$ such that $mb_n \leq a_n \leq Mb_n$; the real part of the complex number $z$, respectively.

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1. Introduction

In his 1985 paper Reade considers the operator which converts a sequence \((x_n)_{n=0}^\infty\) into its sequence of averages
\[
\left(\frac{x_0 + x_1 + \cdots + x_n}{n+1}\right)_{n=0}^\infty.
\]
He shows it is a bounded operator on \(c_0\). We shall denote this operator by \(C_1 = (C, 1)\) and call it the Cesàro operator. It can be represented by the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\
& & & & \\
& & & & 
\end{pmatrix}.
\]
The key to determining the spectrum \(\sigma(C_1)\) of a bounded linear operator \(C_1: bv_0 \to bv_0\) on a Banach space \(bv_0\) is the determination of all eigenvalues of \(C_1^* \in B(bv_0^*)\), that is, the determination of all \(\lambda \in \mathbb{C}\) such that \((C_1 - \lambda I)^{-1} \in B(bv_0)\).

1.1 Theorem. Let \(T \in B(X)\), where \(X\) is any Banach space. Then the spectrum of \(T^*\) is identical with the spectrum of \(T\). Furthermore, \(R_\lambda(T^*) = (R_\lambda(T))^*\) for \(\lambda \in \rho(T) = \rho(T^*)\), where \(R_\lambda(T) = (T - \lambda I)^{-1}\) and \(\rho(T) = \{\lambda \in \mathbb{C}: (T - \lambda I)^{-1}\ exists\}\).

Proof. See [2, page 568] and [3, page 71].

1.2. Lemma. \(C_1: bv_0 \to bv_0\) and \(C_1 \in B(bv_0)\) with \(\|C_1\|_{bv_0} = 1\).

Proof. Since \(C_1: bv_0 \to bv_0\), write \(y_n = C_1x\) and define \(x_n = a_0 + a_1 + \cdots + a_n\). Then \(y_n \to 0\) as \(n \to \infty\) and
\[
\sum_{n=0}^{\infty} |y_n - y_{n+1}| = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \left| \sum_{\nu=1}^{n+1} \nu a_\nu \right|
\leq \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \sum_{\nu=1}^{n} \nu |a_\nu|
= \sum_{\nu=1}^{\infty} \nu |a_\nu| \sum_{n=\nu-1}^{\infty} \frac{1}{(n+1)(n+2)}
\leq \sum_{\nu=1}^{\infty} |a_\nu| = \sum_{\nu=0}^{\infty} |x_\nu - x_{\nu-1}|.
\]
Direct manipulation gives
\[ \| C_1 \|_{(bv_0, bv_0)} = \sup \{ 1, 0, 0, \ldots \} = 1. \]

Clearly \( \lim_{n \to \infty} \frac{1}{1 + n} = 0 \) and hence \( C_1 \in B(bv_0) \).

1.3. **Lemma.** Each bounded linear operator \( T : X \to Y \), where \( X = c_0, l_p \) and \( Y = c_0, l_p \) \((1 \leq p < \infty), \ l_\infty \) (where \( l_p \) denotes sequences \( x \) such that \( \sum_{k=0}^{\infty} |x_k|^p < \infty \) and \( l_\infty \) denotes bounded sequences) determines and is determined by an infinite matrix of complex numbers.

**Proof.** See Taylor [13, pages 221–223].

1.4. **Lemma.** Let \( C_1 : bv_0 \to bv_0 \). Then \( C_1^* : bv_0^* \to bv_0^* \) is given by \( C_1^* \) and \( C_1^* \in B(bs) \).

**Proof.** Since \( bv_0 \) has \( AK \) and \( bv_0^* \) is isomorphic to \( bs \) under the map \( h : bv_0^* \to bs, h(f) = (t_0, t_1, t_2, \ldots) \), where \( t_n = f(\delta^n), n \geq 0, f \in bv_0^* \), we have (see Lemma 1.3)
\[
C_1^* = C_1^* = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.
\]

But for any operator \( T \) on a normed linear space \( X \), \( \|T\|_X = \|T^*\|_{X^*} \) (see [2, page 478], [3, page 54] and [7, page 232]), so
\[
\|C_1\|_{bv_0} = \|C_1^*\|_{bv_0^*} = \|C_1^*\|_{bs} = 1.
\]

Thus \( C_1^* \in B(bv_0^*) \), that is, \( C_1^* \in B(bs) \) since it is also clear that each column of \( C_1^* \) is null (\( C_1^* \) being a normal matrix).

1.5. **Corollary.** \( C_1 \in B(bv_0) \) has not eigenvalues.

**Proof.** The proof follows from the fact that \( C_1 \in B(c_0) \) has no eigenvalues (see [10]) since \( bv_0 \subset c_0 \).

1.6. **Lemma.** Let
\[
Z_n = \prod_{\nu=0}^{n} \left( 1 - \frac{1}{\lambda(\nu + 1)} \right), \quad \lambda \neq 0, \lambda \in \mathbb{C}.
\]

Then the partial sums of \( \sum_{n=1}^{\infty} Z_n \) are bounded if and only if \( \text{Re}(1/\lambda) \geq 1 \).

**Proof.** When \( \lambda = 1 \), \( Z_n = 0 \) for all \( n \) and so the partial sums of \( \sum_{n=0}^{\infty} Z_n \) are certainly bounded.
Let $C$ be a constant depending only on $\lambda$ which may be different at each occurrence and $A$ a non-zero constant. We have that

$$\log_e (1 - u) = \sum_{n=1}^{\infty} \frac{1}{n} u^n = -u + O(u^2)$$

uniformly in $|u| \leq \frac{1}{2}$, $u \in C$. Now given $\lambda \neq 0$ there is $\nu_0$ such that $|\lambda|(\nu + 1) > 2$ for $\nu \geq \nu_0$.

$$\sum_{\nu=0}^{n} \log \left(1 - \frac{1}{\lambda(\nu + 1)}\right) = C - \frac{1}{\lambda} \sum_{\nu=\nu_0}^{n} \frac{1}{1 + \nu} + \sum_{\nu=\nu_0}^{n} t_\nu$$

where $t_\nu = O(1/\nu^2)$, and

$$\sum_{\nu=\nu_0}^{n} t_\nu = \sum_{\nu=\nu_0}^{\infty} t_\nu - \sum_{\nu=\nu+1}^{\infty} t_\nu = C + O\left(\frac{1}{n}\right)$$

Also

$$\sum_{\nu=\nu_0}^{n} \frac{1}{\nu + 1} = C + \log n + O\left(\frac{1}{n}\right)$$

since if $C = \sum_{\nu=0}^{n} \frac{1}{\nu + 1} - \log n$, then

$$C_{n+1} - C_n = \frac{1}{2 + n} - \log \left(\frac{n + 1}{n}\right) = O\left(\frac{1}{n^2}\right)$$

Therefore

$$C_{n+1} = C + \sum_{\nu=0}^{n} (C_{\nu+1} - C_\nu)$$

$$= C + \sum_{\nu=0}^{\infty} (C_{\nu+1} - C_\nu) - \sum_{\nu=n+1}^{\infty} (C_{\nu+1} - C_\nu), \quad C_0 = 0,$$

that is,

$$C_{n+1} = C - \sum_{\nu=n+1}^{\infty} (C_{\nu+1} - C_\nu) = C + O\left(\frac{1}{n}\right)$$

Hence as $n \to \infty$, $\log Z_n = C - 1/\lambda \log n + O(1/n)$ so $Z_n = A n^{-1/\lambda}(1 + O(1/n)) = A n^{-1/\lambda} + O(n^{-\text{Re}(1/\lambda)-1})$. If $\text{Re}(1/\lambda) \geq 1$, $\lambda = 1$, the partial sums of $\sum_{n=1}^{\infty} n^{-1/\lambda}$ are bounded and $\sum_{n=1}^{\infty} n^{-\text{Re}(1/\lambda)-1} < \infty$, so the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded. If $0 < \text{Re}(1/\lambda) < 1$ or $\lambda = 1$ then the partial sums
of $\sum_{n=1}^{\infty} n^{-1/\lambda}$ are unbounded, but we still have $\sum_{n=1}^{\infty} n^{-\text{Re}(1/\lambda)-1} < \infty$. If $\text{Re}(1/\lambda) < 0$ then

$$\sum_{n=1}^{N} n^{-1/\lambda} \asymp N^{1-1/\lambda} \cdot \left(1 - \frac{1}{\lambda}\right).$$

Now

$$\sum_{n=1}^{N} n^{-\text{Re}(1/\lambda)-1} = \begin{cases} O(N^{-\text{Re}(1/\lambda)}), & \text{if } \text{Re}(1/\lambda) < 0, \\ O(\log N), & \text{if } \text{Re}(1/\lambda) = 0. \end{cases}$$

Using (1.6) we see that the partial sums of $\sum_{n=1}^{\infty} n^{-1/\lambda}$ are unbounded although $\sum_{n=1}^{\infty} n^{-\text{Re}(1/\lambda)-1} < \infty$, and hence we conclude that the partial sums of $\sum_{n=1}^{\infty} z_n$ are bounded if and only if $\text{Re}(1/\lambda) \geq 1$.

### 2. Determination of the spectrum of $C_1$ on $bv_0$

#### 2.1. Theorem. The eigenvalues of $C_1^* \in B(bv_0^*)$, that is, $C_1 \in B(bs)$, are all $\lambda \in \mathbb{C}$ satisfying $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$.

**Proof.** Suppose $C_1^* x = \lambda x$, $x \in bs$, $x \neq \theta$ where $\theta$ is the zero sequence. Then as in Lemma 1.4,

$$C_1^* = C_1^t = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and solving the system of equations

\[
\begin{align*}
x_0 + \frac{1}{2}x_1 + \frac{1}{3}x_2 + \cdots &= x_0 \\
\frac{1}{2}x_1 + \frac{1}{3}x_2 + \cdots &= x_1 \\
\frac{1}{3}x_2 + \cdots &= x_2 \\
\cdots \\
\frac{1}{n}x_{n-1} + \frac{1}{n+1}x_n + \cdots &= x_{n-1} \\
\frac{1}{n+1}x_n + \cdots &= x_n \\
\cdots
\end{align*}
\]
we obtain

\[
x_1 = \left(1 - \frac{1}{\lambda}\right)x_0
\]

\[
x_2 = \left(1 - \frac{1}{\lambda}\right) \left(1 - \frac{1}{2\lambda}\right)x_0
\]

\[
x_3 = \left(1 - \frac{1}{\lambda}\right) \left(1 - \frac{1}{2\lambda}\right) \left(1 - \frac{1}{3\lambda}\right)x_0.
\]

\[
\ldots
\]

\[
x_N = \prod_{n=1}^{N} \left(1 - \frac{1}{n\lambda}\right)x_0.
\]

By Lemma 1.6, \((x_N)^\infty\in bs\) if and only if \(\text{Re}(1/\lambda) \geq 1\), that is, \(|\lambda - \frac{1}{2}| \leq \frac{1}{2}\). Hence the result follows.

2.2. **Theorem.** Let \(C : bv_0 \rightarrow bv_0\). Then the spectrum of \(C\) is

\[
\sigma(C) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}.
\]

2.3. **Definition (Weighted mean method).** The weighted mean method is a matrix \(A = (a_{nk})\) with

\[
a_{nk} = p_k|P_n|, \quad P_n = \sum_{k=0}^{n} p_k \neq 0.
\]

2.4. **Lemma.** If \((M, p) = (N, p)\) is a regular (conservative) weighted mean method then \((M, p) = (N, p)\) is absolutely regular (conservative).

(See [1], [14] for further details.)

**Proof.** Since \((N, p)\) is a regular (conservative) mean method we have by the Kojima-Schur conditions

\[
|P_n| \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty,
\]

where \(P_n = \sum_{\nu=0}^{n} \rho_{\nu}\) and

\[
P_n^* = \sum_{\nu=0}^{n} |\rho_{\nu}| = O(P_n).
\]

We need to prove that \((N, p)\) is absolutely regular (conservative), that is, that

\[
P_{k-1} \sum_{n=k}^{\infty} \left| \frac{1}{P_n} - \frac{1}{P_{n-1}} \right| \leq M.
\]
Let $P_n^* = \sum_{\nu=0}^{n} |p_{\nu}|$. Then (2.4) becomes $P_n^* \leq K |P_n|$ for all $n \geq 1$ ($K$ some constant). Thus

$$|P_{k-1}| \sum_{n=k}^{\infty} \frac{|P_n|}{|P_n| |P_{n-1}|} \leq |P_{k-1}| \sum_{n=k}^{\infty} \frac{|P_n| K^2}{P_n^* \cdot P_{n-1}^*}$$

$$\leq K^2 |P_{k-1}| \sum_{n=k}^{\infty} \left( \frac{1}{P_{n-1}^*} - \frac{1}{P_n^*} \right).$$

Since $|P_n| \to \infty$ as $n \to \infty$ by (2.3), we have that $P_n^* \to \infty$ (since $P_n^* \geq |P_n|$), therefore $\sum_{n=k}^{\infty} (1/P_{n-1}^* - 1/P_n^*) = 1/P_{k-1}^*$ and so (2.5) follows, provided that $|P_{k-1}|/P_{k-1}^* \leq M$ for some $M$. But $M = 1$ will do and the result follows.

We now prove Theorem 2.2.

**Proof.** By virtue of Theorem 2.1 and the fact that $\sigma(C_1) = \sigma(C_1^*)_a$ (see Theorem 1.1), it is enough to prove that $B = (C_1 - \lambda I)^{-1} \in B(bv_0)$ for all $|\lambda - \frac{1}{2}| > \frac{1}{2}$, that is, that $Q$ is absolutely regular where $B = -I/\lambda - Q/\lambda(\lambda - 1)$ except when $\lambda$ is the reciprocal of a positive integer, $B = (C - \lambda I)^{-1} = I/\lambda - Q/\lambda(\lambda - 1)$, where $Q = (q_{nk})$, $q_{nk} = A_{k-1}^{-1/\lambda} A_{n-1}^{1-1/\lambda}$,

$$A_n^\alpha = \binom{n + \alpha}{n} = \frac{(\alpha + n) \cdots (\alpha + 1)}{n!}$$

is a Hausdorff matrix $(\mu, \mu_n)$,

$$\mu_n = \frac{1}{\lambda} \left( -1 - \frac{\frac{1}{2}}{(n + 1) - \frac{1}{\lambda}} \right).$$

It is also clear that $Q$ is the Hausdorff matrix $(\mu, (1 - \frac{1}{\lambda})/((n + 1) - \frac{1}{\lambda}))$. The proof of this is trivial (see Rhoades [11]).

Now $Q$ is a regular Hausdorff transformation when Re$(1/\lambda) < 1$. To see this we simply check the regularity conditions, namely:

(i) \( \lim_{n \to \infty} q_{nk} = \lim_{n \to \infty} A_{n-1}^{-1/\lambda} A_{n-1}^{1-1/\lambda} = 0 \) since

$$|q_{nk}| = |A_{k-1}^{-1/\lambda} A_{n-1}^{1-1/\lambda}| = |A_{k-1}^{-1/\lambda}| \cdot O(n^{\alpha - 1})$$

and $\alpha = \text{Re}(1/\lambda) < 1$, whence $q_{nk} \to 0$ as $n \to \infty$;

(ii) $\sum_{k=1}^{n} A_{k-1}^{-1/\lambda} = A_{n-1}^{-1/\lambda}$, and therefore $\lim_{n \to \infty} \sum_{k=1}^{n} q_{nk} = 1$;

(iii) $\sum_{k=1}^{n} A_{k-1}^{-1/\lambda} = \sum_{k=1}^{n} O(k^{-\alpha}) = O(n^{1-\alpha}) \sim O(|A_{n-1}^{-1/\lambda}|)$ and therefore

$$\sum_{k=1}^{n} |A_{k-1}^{-1/\lambda}| = O(|A_{n-1}^{-1/\lambda}|).$$
It is clear that $Q = (q_{nk})$ is a weighted mean method (matrix) $(N, A_{k-1}^{-1/2})$ with $\sum_{k=1}^{n} A_{k-1}^{-1/2} = A_{n-1}^{-1/2}$. Since $Q$ is a weighted mean method and a regular Hausdorff method, theorem 2.2 follows.

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References