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FINITE COVERINGS OF RINGS BY IDEALS

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ABSTRACT. Necessary and sufficient conditions are given for a ring to be a union of finitely many proper two-sided ideals.

Several authors have investigated the problem of when an algebraic structure can be written as a set theoretic union of finitely many proper substructures.

B. H. Neumann [4] characterized those groups which can be written as a union of finitely many proper subgroups, while Baer [4] did the same thing in the case where the subgroups are abelian. More recently, Brodie, Chamberlain and Kappe [1] characterized those groups which can be written as a union of finitely many proper normal (or normal abelian or verbal) subgroups (see also [5]).

Motivated by a question of J. Bergen, Lanski [2] characterized those rings which can be written as a union of finitely many proper right annihilators. In the first page of his work, Lanski also makes a few general observations about rings which are a union of ideals.

In this very brief note, we obtain necessary and sufficient conditions for a ring to be a union of finitely many proper two-sided ideals.

Crucial to all of this work is the following fundamental result of B. H. Neumann [3].

LEMMA 1. *Let a group G be a finite union of cosets a_iG_i of subgroups of G. If we omit any coset a_iG_i for which $[G : G_i]$ is infinite, then G is still the union of the remaining cosets.*

Here is the main result.

THEOREM 2. *For an associative ring R, the following are equivalent.*

(i) R is a union of finitely many proper two-sided ideals.

(ii) R has a finite homomorphic image S such that $S^2 = 0$ and $(S, +)$ is a nontrivial noncyclic group.

PROOF (ii) $\Rightarrow$ (i). This is clear because any such $(S, +)$ is a finite union of proper subgroups, each of which is an ideal of S, and the preimages of these ideals gives the result for R.

(i) $\Rightarrow$ (ii). Lemma 1 tells us that we may assume R is a union of finitely many proper two-sided ideals, each of which is of finite index in R. Factoring out the intersection of those ideals, we see that we may assume R is finite.

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For this finite ring R, condition (i) is clearly equivalent to the property that for any $x \in R$, the two-sided ideal generated by x, which we will denote $\langle x \rangle$, is not equal to R.

Two cases will be considered.

CASE I. Assume first that $R = R^2$.

Let $J(R)$ denote the Jacobson radical of R. Since $J(R)$ is nilpotent, we know that $J(R) \neq R$. Thus, $R/J(R)$, being semisimple artinian, contains a multiplicative identity which can be lifted to an idempotent e in R. Hence, for any $r \in R$, we have $r - er \in J(R)$.

Choose n such that $[J(R)]^n = 0$, and let $a \in R$. Since $R = R^n$, the previous remarks tell us that $a \in \langle e \rangle$.

Thus $\langle e \rangle = R$, contradicting our earlier observation. We conclude that this case is impossible.

CASE II. Assume now that $R \neq R^2$. Choose n such that $R^n = R^{n+1}$.

Let $S = R/R^2$. Certainly, S is nontrivial and $S^2 = 0$. We will be finished if we can show that $(S, +)$ is not cyclic. Assume to the contrary that $(S, +)$ is cyclic, generated by $r + R^2$.

Let y belong to R. Then $y = zr + b$ for some $z \in Z$, $b \in R^2$. Continuing this procedure on each component of each term in a representation of b as an element of R^2, we get that $y \in \langle r \rangle \bmod R^4$. Eventually, we have $y = p + q$, where $p \in \langle r \rangle$ and $q \in R^n = R^{n+1}$.

If $R^n = 0$, we have the contradiction that $R = \langle r \rangle$. So assume $R^n \neq 0$.

In that case, $R \neq J(R)$, and the multiplicative identity of $R/J(R)$ can be lifted to an idempotent e in R. We claim that the ideal $\langle r - e - er \rangle$ is equal to R, thus giving our contradiction.

Note first that since $e = -e(r - e - er)$, e is in $\langle r - e - er \rangle$. Next observe that the argument given in Case I, using the nilpotence of $J(R)$ and the fact that $q \in R^n = R^{n+1}$, tells us that $q \in \langle e \rangle \subseteq \langle r - e - er \rangle$. Finally, we note that since $e \in \langle r - e - er \rangle$, it must also be true that $r \in \langle r - e - er \rangle$, so $p \in \langle r \rangle \subseteq \langle r - e - er \rangle$. Thus, $y = p + q \in \langle r - e - er \rangle$, and we are done. ∎

It's not clear what happens if "two-sided ideals" is weakened to "one-sided ideals" or "subrings" in the statement of the theorem. The group theoretic results of [1] and [4] tell us to expect a similar condition on $(S, +)$ for some homomorphic image S of R, but the exact multiplicative condition which is forced on S is unclear.

At any rate, we will close with an example showing that the "one-sided" and "two-sided" results will be different.

EXAMPLE. Let A denote a 4-element semigroup with zero (θ), *i.e.* $A = \{\theta, x, y, z\}$, having binary operation as shown.

	θ	x	y	z
θ	θ	θ	θ	θ
x	θ	θ	x	θ
y	θ	θ	y	z
z	θ	θ	θ	θ

Let F denote the field of 2 elements, and F_0A the associated contracted semigroup algebra (*i.e.* elements of F_0A are of the form $\alpha_1 x + \alpha_2 y + \alpha_3 z$ where $\alpha_1, \alpha_2, \alpha_3 \in F$).

We note that F_0A is not a union of finitely many proper two-sided ideals since $F_0A = \langle y \rangle$. However, F_0A is the union of the proper right ideals generated by $y, x + y$ and $x + z$.

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