ON THE SPECTRA OF PISOT NUMBERS

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(Received 16 March 2010; revised 14 August 2010; accepted 11 June 2011)

Abstract. Let \( \theta \) be a real number greater than 1, and let \(((\ ))\) be the fractional part function. Then, \( \theta \) is said to be a \(Z\)-number if there is a non-zero real number \( \lambda \) such that \((((\lambda \theta^n))) < \frac{1}{2}\) for all \( n \in \mathbb{N} \). Dubickas in \([3]\) defined a subset \( G \) of \( \mathbb{Z} \) such that a real number \( \theta \) is a Pisot number if and only if there exists \( \lambda \neq 0 \) such that \((((\lambda \alpha))) < \frac{1}{2}\) for all \( \alpha \in \{\theta^n \mid n \in \mathbb{N}\} \cup \{\sum_{n=0}^{N} \theta^n \mid N \in \mathbb{N}\}\). Also, the following characterisation of Pisot numbers among real numbers greater than 1 is shown: \( \theta \) is a Pisot number if and only if there exists \( \lambda \neq 0 \) such that \( \|\lambda \alpha\| < \frac{1}{2} \) for all \( \alpha \in \{\sum_{n=0}^{N} a_n \theta^n \mid a_n \in \{0, 1\}, N \in \mathbb{N}\} \), where \( \|\lambda \alpha\| = \min\{((\lambda \alpha)), 1 - ((\lambda \alpha))\} \).

2000 Mathematics Subject Classification. 11R80, 11J71, 11R06.

1. Introduction. For a point \( t \) of the real line \( \mathbb{R} \) we denote by \([t]\) the largest element of the ring \( \mathbb{Z} \) of rational integers, not exceeding \( t \). We also denote by \(((t))\) and \( \|t\| \) the difference \( t - [t] \) and the minimum of the set \( \{((\lambda \alpha)), 1 - ((\lambda \alpha))\} \), respectively. Namely, \([t]\) is the integer part of \( t \), \(((\ ))\) is the fractional part function and \( \|t\| \) is the usual distance from \( t \) to \( \mathbb{Z} \).

Let throughout \( \theta \in (1, \infty), \lambda \in \mathbb{R} \setminus \{0\} \) and \( n \in \mathbb{N} := \mathbb{Z} \cap [0, \infty) \). Dubickas in \([3]\) defined a subset \( G \) of \( (1, \infty) \), with the property that for each \( \theta \in \mathbb{Z} \), there is \( \lambda = \lambda(\theta) \) such that \((((\lambda \theta^n))) < \frac{1}{2}\) for all \( n \). An element of \( G \) is called a \( G \)-number. A result due to Tijdeman, and cited in \([3]\) gives immediately that \( (3, \infty) \subset Z \). Set \( \mathcal{Y} := (1, \infty) \setminus Z \). Some classes of algebraic integers, which belong to \( \mathbb{Z} \cap (1, 3) \), or to \( \mathcal{Y} \cap (1, 2) \), are exhibited in \([3]\), and from this one can easily deduce that \( 2 \) is a left-hand limit point of \( Z \), and \( 1 \) is a limit point of \( \mathcal{Y} \). Dubickas proved in particular that strong Pisot numbers are \( Z \)-numbers. Recall that a Pisot number is a real algebraic integer greater than 1 whose other conjugates are of modulus less than 1. The set of Pisot numbers is usually noted \( S \). A Pisot number \( \theta \) of degree \( d \) is called a strong Pisot number if \( d = 1 \), or if \( d \geq 2 \) and \( \theta \) has a conjugate belonging to the interval \((0, 1)\), which is greater than the absolute values of \( d - 2 \) remaining conjugates of \( \theta \) \([2]\). We denote by \( S_{\theta} \) the set of strong Pisot numbers.

Let

\[
A_0 = A_0(\theta) := \{\theta^n \mid n \in \mathbb{N}\} \cup \left\{ \sum_{n=0}^{N} \theta^n \mid N \in \mathbb{N}\right\},
\]

\[
A_m = A_m(\theta) := \left\{ \sum_{n=0}^{N} a_n \theta^n \mid a_n \in \{0, \ldots, m\}, N \in \mathbb{N}\right\}.
\]
The first aim of this note is to show that strong Pisot numbers are a kind of ‘strong Z-numbers’:

**Theorem 1.** The following are equivalent.

(i) $\theta \in S_\sigma$.

(ii) For any $\epsilon > 0$ and any $m$, there is $\lambda$ such that $((\lambda \alpha)) < \epsilon$ for all $\alpha \in A_m(\theta)$.

(iii) There exist $m$ and $\lambda$ such that $((\lambda \alpha)) < \frac{1}{2}$ for all $\alpha \in A_m(\theta)$.

(iv) There is $\lambda$ such that $((\lambda \alpha)) < \frac{1}{2}$ for all $\alpha \in A_0(\theta)$.

In terms of fractional part function, Theorem 1 may be viewed as a characterisation of strong Pisot numbers among real numbers greater than 1. This contrasts with the famous characterisation of Pisot numbers among real numbers, due to Pisot [4], which says: If there is $\lambda$ such that $\sum_{n \in \mathbb{N}} ||\lambda \theta^n||^2 < \infty$, then $\theta \in S$. The important question whether there is a transcendental number $\theta$ satisfying $\lim_{n \to \infty} ||\lambda \theta^n|| = 0$ for some $\lambda$, is still unsolved [1]. We shall mainly use this last mentioned result of Pisot to prove Theorem 1 and the result below:

**Theorem 2.** The following are equivalent.

(i) $\theta \in S$.

(ii) For any $\epsilon > 0$ and any $m \in \mathbb{N}$, there is $\lambda$ such that $||\lambda \beta|| < \epsilon$ for all $\beta \in B_m(\theta)$.

(iii) For any $\epsilon > 0$ and any $m \in \mathbb{N}$, there is $\lambda$ such that $||\lambda \alpha|| < \epsilon$ for all $\alpha \in A_m(\theta)$.

(iv) There is $\lambda$ such that $||\lambda \alpha|| < \frac{1}{3}$ for all $\alpha \in A_1(\theta)$.

In these pages when we speak about conjugates, minimal polynomial and degree of an algebraic number we mean over the field of the rationals $\mathbb{Q}$. For a Pisot number $\theta$ of degree $d$, we denote by $\theta := \theta_1, \ldots, \theta_d$, the conjugates of $\theta$, and by $\sigma_1, \ldots, \sigma_d$, the embeddings of $\mathbb{Q}(\theta)$ into the complex field $\mathbb{C}$, where $\sigma_1$ is the identity of $\mathbb{Q}(\theta)$. As usual, for an element $\alpha$ of the field $\mathbb{Q}(\theta)$, we denote by $Trace(\alpha)$ the sum $\sigma_1(\alpha) + \cdots + \sigma_d(\alpha)$, namely the trace of $\alpha$ for the extension $\mathbb{Q}(\theta)/\mathbb{Q}$. The proofs of Theorems 1 and 2 appear in the following sections, consecutively. It is interesting to determine whether the constant 1/3 in Theorem 2(iv) is optimal, or whether we may replace $A_1(\theta)$ by one of its proper subsets without affecting the conclusion. Analogue questions may be posed for Theorem 1. Distribution in $\mathbb{R}$ of the elements of the set $S_{st}$ is another problem related to Theorem 1. Some computations suggest the following conjecture:

min $S_{st} = 2$, min($S_{st\setminus\{2\}}$) = $(3 + \sqrt{5})/2$ and min $S_{st}' = 3$, where $S_{st}'$ is the derived set of $S_{st}$. From the proof of the result below, one can easily deduce that 3 is a left-hand limit point of $S_{st}$.

**Proposition.** The set $S_{st}'$ contains $\mathbb{N} \cap [3, \infty)$.

**Proof.** Let $b$ be a rational integer greater than 2 and let $P_n(x) := x^n(x - b) + 1$, where $n \geq b$. Since $|bz^n| = b > 2 \geq |z^{n+1} + 1|$ when the complex number $z$ runs through the unit circle, Rouché’s theorem gives that $P_n$ has $n$ roots with modulus less than 1, and so the polynomial $P_n$ has a unique root, say $\theta_{(n)}$, of modulus greater than 1, as $P_n(0) = 1$. Hence, $P_n$ is irreducible over $\mathbb{Q}$ and is the minimal polynomial of $\theta_{(n)}$. Notice also that the real function $P_n(t)$ is decreasing on the interval $(0, nb/(n + 1))$ and is increasing on $(nb/(n + 1), \infty)$ because its formal derivative is $(n + 1)t^{n-1}(t - nb/(n + 1))$. It follows by the relations $P_n(0) = 1$, $P_n(1) = 2 - b$, $b - 1 < nb/(n + 1)$ and $P_n(b) = 1$ that $P_n$
has a unique root, say $\rho$, in the interval $(0, 1)$ and $\theta_{(n)} \in (b - 1, b)$. Consequently, $\theta_{(n)}$ is a Pisot number, and
\[
\lim_{n \to \infty} \theta_{(n)} = b,
\]
since $0 < b - \theta_{(n)} = \frac{1}{\theta_{(n)}^{d-1}} < \frac{1}{(b - 1)^{d-1}} \leq \frac{1}{2}$. If $\alpha$ is a conjugate of $\theta_{(n)}$ such that $\alpha \neq \theta_{(n)}$, then $b|\alpha|^n = |\alpha^{n+1}| + 1 \leq |\alpha^{n+1}| + 1$; hence $P(\alpha) = |\alpha|^n + b|\alpha|^n + 1 \geq 0$ and so $|\alpha| \leq \rho$, as $P_n(0) = 0$ and $P_n(t)$ is decreasing on $(0, 1)$. Moreover, the equality $|\alpha| = \rho$ holds only if $|\alpha^{n+1}| + 1 = |\alpha|^n + 1$ that is when $\alpha^{n+1}$ is a positive real number. It follows in this case by the equality $\alpha^{n+1} = b\alpha^n - 1$ that $\alpha^n > 1/b > 0$, and so $\alpha = \alpha^{n+1}/\alpha^n$ is also a positive real number; thus, $\alpha = \rho$ and so $\theta_{(n)} \in S_{st}$.

**Remark.** A simple computation shows that any polynomial of the form $x^2 - bx + k$, where $b \in \mathbb{N} \cap [3, \infty]$ and $k \in \{1, \ldots, b - 2\}$, is the minimal polynomial of a quadratic strong Pisot number, say $\theta_k$, satisfying $b - 1 < \theta_k < b$. Similarly, as in the above proof, by considering the sequence of polynomials $x^n(x^2 - bx + k) + 1$, we easily obtain that $\theta_k$ is a left-hand limit point of the set $S_{st}$, when $b \geq 4$ and $k \leq b - 3$. Consequently, each interval of the form $[n, n + 1]$, where $n \geq 3$, contains at least $n$ elements of the set $S_{st}$.

**2. Proof of Theorem 1.** To make clear the proof of Theorem 1, let us recall some results on Pisot numbers. The first two results are due to Pisot [4] and Smyth [6].

**Lemma 1.** ([4]) If $\sum_{n=0}^{\infty} \|\lambda \theta^n\|^2 < \infty$ for some $\lambda$, then $\theta \in S$ and $\lambda \in \mathbb{Q}(\theta)$.

**Lemma 2.** ([6]) Two distinct conjugates of a Pisot number having the same modulus are complex conjugates.

Theorem (ii) and Lemma 2 of [7] yield the following:

**Lemma 3.** If $\lambda$ satisfies $\lim_{n \to \infty} \|\lambda \theta^n\| = 0$ for some $\theta \in S$, then $\lambda \in \mathbb{Q}(\theta)$ and there is $N \in \mathbb{N}$ such that $\text{Trace}(\lambda \theta^n) \in \mathbb{Z}$ for all $n \geq N$.

Finally, let us show a simple argument on the conjugates of a Pisot number.

**Lemma 4.** Let $\theta$ be a Pisot number of degree $d$. Then for any positive rational integer $p$, $\theta^p$ is a Pisot number of degree $d$. If $\rho e^{i\alpha}$ is a conjugate of $\theta$, where $\rho^2 = -1$ and $(\rho, a) \in (0, 1) \times (0, 1)$, then for any $b \in \mathbb{R}$, the sequence $((na + b))_{n}$ is dense in $[0, 1]$.

**Proof.** Let $p$ be a positive rational integer. Then, $\theta^p \in \mathbb{Q}(\theta)$, and the conjugates of $\theta^p$ are among the numbers $\theta^p, \theta^p_2, \ldots, \theta^p_k$. Since $|\theta^p_k| < 1$ for all $k \in \{2, \ldots, p\}$, $\theta^p$ is not repeated by the action of embeddings $\sigma_1, \ldots, \sigma_d$; thus $\mathbb{Q}(\theta) = \mathbb{Q}(\theta^p)$ and $\theta^p$ is a Pisot number of degree $d$. Let $\rho e^{i\alpha}$ be a non-real conjugate of $\theta$, then $\rho e^{-i\alpha}$ is also another conjugate of $\theta$, and so (by the first part of Lemma 4) $\rho^p \rho^{p} e^{i\alpha}$ and $\rho^p \rho^{-p} e^{-i\alpha}$ are two distinct conjugates of $\theta^p$. Hence, $\alpha \notin \mathbb{Q}$, and the result follows immediately by Kronecker’s theorem (see for instance Appendix 8 in [5]).

**Proof of Theorem 1.** Let $\theta$ be a strong Pisot number with degree $d$, and let $\epsilon > 0$. If $d = 1$, then $A_m \subset \mathbb{N}$ and so $(\langle \alpha \rangle) = 0 < \epsilon$ for all $\alpha \in A_m$. Now, suppose $d \geq 2$, and $\theta_2 > |\theta_1| \geq \ldots \geq |\theta_d|$. Then, $t_n := \text{Trace}(\theta^n) = \theta^n + \theta^n_2 + \ldots + \theta^n_d \in \mathbb{Z}$, $\theta^n_2 + \ldots + \theta^n_d = t_n - \theta^n \in \mathbb{R}$, $\theta^n_2 + \ldots + \theta^n_d \leq \theta^n_2$ and $\text{lim}_{n \to \infty} \sum_{k=2}^{d}(\frac{\theta^n}{\theta^n_2})^k = 1$. Let $n_1$ be a positive
rational integer such that for all \( n \geq n_1 \) we have
\[
0 < \theta_2^n + \cdots + \theta_d^n \tag{1}
\]
and
\[
md\theta_2^n \frac{1}{1 - \theta_2} < \min\{1, \varepsilon\}. \tag{2}
\]

Setting \( \lambda := -\theta_1^n \), we have \( \lambda \theta_2^n = -\theta_{2}^{n+1} = -t_{n+1} + \theta_2^{n+1} + \cdots + \theta_{d}^{n+1} \), and the relations (1) and (2) give \( 0 < \theta_{2}^{n+1} + \cdots + \theta_{d}^{n+1} < d\theta_2^{n+1} < (1 - \theta_2)/m < 1 \) for all \( n \).

Hence, \( -t_{n+1} = [\lambda \theta_2^n] \), \( ((\lambda \theta_2^n)) = \theta_2^{n+1} + \cdots + \theta_{d}^{n+1} \), and so \( ((\lambda \theta_2^n)) < (1 - \theta_2)/\varepsilon < \varepsilon \) for all \( n \). Similarly, if \( \alpha = \sum_{n=0}^{N} a_n \theta_2^n \), where \( a_n \in \{0, 1, \ldots, m\} \) and \( N \in \mathbb{N} \), then \( \lambda \alpha = \sum_{n=0}^{N} a_n \lambda \theta_2^n = -\sum_{n=0}^{N} a_n t_{n+1} + \sum_{n=0}^{N} a_n (\theta_2^{n+1} + \cdots + \theta_{d}^{n+1}) \), and the inequalities (1) and (2) again yield \( 0 < \sum_{n=0}^{N} a_n (\theta_2^{n+1} + \cdots + \theta_{d}^{n+1}) < \sum_{n=0}^{N} a_n (d\theta_2^{n+1}) \leq md\theta_2^n \sum_{n=0}^{N} \theta_2^n \frac{m-1}{m} < \min\{1, \varepsilon\} \); thus, \( ((\lambda \alpha)) = \sum_{n=0}^{N} a_n (\theta_2^{n+1} + \cdots + \theta_{d}^{n+1}) < \varepsilon \), as \( -\sum_{n=0}^{N} a_n t_{n+1} \in \mathbb{Z} \), and so Theorem 1(ii) holds. The implications (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) in Theorem 1 are trivially true, since \( A_0 \subset A_1 \subset A_m \). To show that the proposition (iv) \( \Rightarrow \) (i), is true, let us first verify the equalities
\[
\sum_{n=0}^{N} ((\lambda \theta_2^n)) = \left( \left( \lambda \sum_{n=0}^{N} \theta_2^n \right) \right), \tag{3}
\]
where \( \lambda \) satisfies \( ((\lambda \alpha)) < \frac{1}{\varepsilon} \) for all \( \alpha \in A_0 \), and \( N \in \mathbb{N} \). It is clear that (3) holds for \( N = 0 \). By the relations \( \lambda \sum_{n=0}^{N+1} \theta_2^n = [(\lambda \sum_{n=0}^{N} \theta_2^n) + [((\lambda \theta_2^{N+1}) + ((\lambda \theta_2^{N+1})) + (\lambda \theta_2^{N+1}) + (\lambda \theta_2^{N+1})) \leq \lambda \sum_{n=0}^{N} \theta_2^n + \lambda \theta_2^{N+1} + \cdots + \lambda \theta_2^{N+1} \), and a simple induction gives (3). Letting \( N \) tends to infinity in (3), we obtain
\[
\sum_{n=0}^{\infty} ((\lambda \theta_2^n)) \leq \frac{1}{2}
\]
and so \( \sum_{n=0}^{\infty} \| \lambda \theta_2^n \| \leq \frac{1}{2} \). It follows by Lemma 1 that \( \theta \in S \) and \( \lambda \in \mathbb{Q}(\theta) \). The last inequality also gives \( \lim_{n \to \infty} \| \lambda \theta_2^n \| = 0 \), and so by Lemma 3, there is \( n_2 \in \mathbb{N} \) such that \( t_n := \text{Trace}(\lambda \theta_2^n) \in \mathbb{Z} \) for all \( n \geq n_2 \). Let \( d \) be the degree of \( \theta \). If \( d = 1 \), then \( \theta \in S_{1} \).

Suppose \( d \geq 2 \), and \( |\theta_2| \geq \ldots \geq |\theta_d| \). Lemma 2 says that we have to prove that \( \theta_2 \) is a positive real number. Assume that \( n \geq n_2 \). Then, \( t_n = \lambda \theta_2^n + \lambda_2 \theta_2^n + \cdots + \lambda_d \theta_d^n \), where \( \lambda_k = \sigma_k(\lambda) \) for \( k \in \{1, \ldots, d\} \), \( \lambda_2 \theta_2^n + \cdots + \lambda_d \theta_d^n = t_n - \lambda \theta_2^n \in \mathbb{R} \), and
\[
t_n - [\lambda \theta_2^n] = ((\lambda \theta_2^n)) + \lambda_2 \theta_2^n + \cdots + \lambda_d \theta_d^n. \tag{4}
\]
Let \( n_3 \) be the smallest element of \( \mathbb{N} \) satisfying
\[
d \cdot |\theta_2|^{n_3} \max_{2 \leq j \leq d} |\lambda_j| < 1/2.
\]
Then, (4) gives, for \( n \geq \max\{n_2, n_3\} \),
\[
-1/2 < t_n - [\lambda \theta_2^n] < 1/2 + 1/2.
\]
since $|\lambda_2 \theta_2^n + \cdots + \lambda_d \theta_d^n| < d |\theta_2|^n \max_{1 \leq j \leq d} |\lambda_j|$ (recall that $0 \leq (\lambda \theta^n) < 1/2$ for all $n$); thus $t_n - [\lambda \theta^n] = 0$ and so

$$\lambda_2 \theta_2^n + \cdots + \lambda_d \theta_d^n = -(\lambda \theta^n) \leq 0.$$  \hspace{1cm} (5)

Now we claim that the result follows directly from (5) and Lemmas 2 and 4. Indeed, if $\theta_2 \notin \mathbb{R}$, then $d \geq 3$, $\theta_3 = \theta_2$ and $\lambda_3 = \lambda_2$. Set $\theta_2 := \rho e^{i \pi}$ and $\lambda_2 = \eta e^{i \pi}$ where $\rho^2 = -1$, $(\rho, \eta) \in (0, 1) \times (0, 1)$, $b \in (-1, 1]$ and $\eta > 0$. Then, Lemma 4 states that there are infinitely many $n$ such that the corresponding quantities $2n \cos((na + b)\pi) + \sum_{k=4}^{d} \lambda_k \rho^k \theta^n$ are all positive because $\lim_{n \to \infty} \rho^k \theta^n = 0$ for all $k \in \{4, \ldots, d\}$; this leads to a contradiction since by (5) we have $\rho^n(2n \cos((na + b)\pi) + \sum_{k=4}^{d} \lambda_k \rho^k \theta^n) = \lambda_2 \theta_2^n + \lambda_3 \theta_3^n + \cdots + \lambda_d \theta_d^n \leq 0$. Finally, if $\theta_2 \in \mathbb{R}$, then the relation (5), together with Lemma 2, again gives $\lim_{n \to \infty} -(\lambda \theta^n) = \lambda_2$, and so $\theta_2 > 0$ and $\lambda_2 < 0$, as $\lambda = \sigma^{-1}(\lambda_2) = 0$ when $\lambda_2 = 0$; thus $\theta \in \mathcal{S}_n$. \hfill \Box

3. Proof of Theorem 2. Let $\epsilon > 0$, and let $\beta = \sum_{n=0}^{N} b^n \theta^n$, where $\theta$ is a Pisot number of degree $d$, $N \in \mathbb{N}$ and $b_n \in \{-m, \ldots, -m + 1, \ldots, m\}$. If $d = 1$, then $B_m \subset \mathbb{N}$ and so $|\beta| < \epsilon$. Suppose $d \geq 2$. It is clear that $\beta$ is an integer of the field $\mathbb{Q}(\theta)$, the conjugates of $\beta$ are among the numbers $\beta_k := \sigma_k(\beta) = \sum_{n=0}^{N} b_n \theta_k^n$, where $k \in \{1, \ldots, d\}$, and

$$|\beta_k| \leq m \sum_{n=0}^{N} |\theta_k|^n \frac{m}{1 - |\theta_k|} \text{ for } k \in \{2, \ldots, d\}. \hspace{1cm} (6)$$

Set $\lambda := \theta^p$, where $p \in \mathbb{N}$ and satisfies $|\theta_k|^p < \frac{m(1 - |\theta_k|)}{m(d - 1)}$ for all $k \in \{2, \ldots, d\}$. Then, $t := \text{Trace}(\lambda \beta) = \theta^p \beta + \theta_2^p \beta_2 + \cdots + \theta_d^p \beta_d \in \mathbb{Z}$, and by the relation (6) we obtain

$$||\lambda \beta|| \leq ||\lambda \beta - t|| = |\theta^p \beta_2 \cdots + \theta_d^p \beta_d| < \epsilon;$$

thus, Theorem 2(ii) holds. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) in Theorem 2 are trivially true because $A_1 \subset A_m \subset B_m$. Now assume that there is $\lambda$ such that $||\lambda \alpha|| < \frac{1}{3}$ for all $\alpha \in A_1$. We shall use Lemma 1 to prove that $\theta \in \mathcal{S}$. Set $\lambda \theta^n := x_n + y_n$, where $x_n \in \mathbb{Z}$ and $|y_n| = ||\lambda \theta^n||$. If $s_N = \sum_{n=0}^{N} a_n \theta^n$, where $a_n \in \{0, 1\}$ and $N \in \mathbb{N}$, then $\lambda s_N = \sum_{n=0}^{N} a_n x_n + \sum_{n=0}^{N} a_n y_n$ and $\sum_{n=0}^{N} a_n x_n \in \mathbb{Z}$. Similarly, as in the proof of Theorem 1, let us show the relation

$$\left| \sum_{n=0}^{N} a_n x_n \right| < \frac{1}{3} \text{ for all } N. \hspace{1cm} (7)$$

If $N = 0$, then $a_0 y_0 \in \{0, y_0\}$, and so $-1 < a_0 y_0 < 1/3$, as $|y_0| = ||\lambda||$. Suppose that (7) holds for some $N \in \mathbb{N}$, and let $s_{N+1} = \sum_{n=0}^{N+1} a_n \theta^n$, where $(a_n)_{0 \leq n \leq N+1}$ is a sequence of elements of the set $\{0, 1\}$. By the hypothesis and the induction hypothesis we have $|\sum_{n=0}^{N+1} a_n y_n| \leq |\sum_{n=0}^{N} a_n y_n| + |a_{N+1} y_{N+1}| < \frac{1}{3} + \frac{1}{3}$. Since $\lambda s_{N+1} = x + y$, where $x \in \mathbb{Z}$ and $|y| = ||\lambda s_{N+1}|| < 1/3$, and $\lambda s_{N+1} = s_{N+1} + a_n x_n + \sum_{n=0}^{N+1} a_n y_n$, we see that $\sum_{n=0}^{N+1} a_n x_n - y \in \mathbb{Z}$. It follows by the inequalities $|\sum_{n=0}^{N+1} a_n y_n - y| \leq |\sum_{n=0}^{N+1} a_n y_n| + |y| < \frac{2}{3} + \frac{1}{3}$ that $\sum_{n=0}^{N+1} a_n x_n - y = 0$, $|\sum_{n=0}^{N+1} a_n y_n| = |y| < \frac{1}{3}$, and so (7) is true. Now fix (for a moment) a positive rational integer $N$, and consider the subsets, say $U$ and
of \{0, 1, \ldots, N\} defined as follows: \( n \in U \iff y_n > 0 \) and \( n \in V \iff y_n < 0 \). Then,

\[
\sum_{n=0}^{N} \|\lambda^n\theta\| = \sum_{n=0}^{N} |y_n| = \sum_{n \in U} y_n + \sum_{n \in V} (-y_n),
\]

and so

\[
\sum_{n=0}^{N} \|\lambda^n\theta\| < \frac{2}{3}, \tag{8}
\]

since by (7) we have \( |\sum_{n \in U} y_n| < \frac{1}{3} \) and \( |\sum_{n \in V} (-y_n)| < \frac{1}{3} \). Letting \( N \) tend to infinity in (8), we obtain the result by Lemma 1.

\[\square\]

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