Canad. Math. Bull. Vol. **47** (3), 2004 pp. 417–430

A Hopf Type Lemma and a CR Type Inversion for the Generalized Greiner Operator

Niu Pengcheng, Han Yanwu and Han Junqiang

Abstract. In this paper we establish a Hopf type lemma and a CR type inversion for the generalized Greiner operator. Some nonlinear Liouville type results are given.

1 Introduction

Our aim in this paper is to consider some properties associated with generalized Greiner operators

$$\triangle_L = \sum_{j=1}^{n}(X_j^2 + Y_j^2), \tag{1.1}$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j\,|z|^{2k-2}\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2kx_j\,|z|^{2k-2}\frac{\partial}{\partial t},$$

$j = 1, \dots, n$, $x, y \in R^n$, $t \in R$, $z = x + \sqrt{-1}\,y$, $|z| = \left[\sum_{j=1}^n (x_j^2 + y_j^2)\right]^{1/2}$, $k \geq 1$. When $k = 1$, (1.1) becomes the Heisenberg Laplacian (see Folland [7]); when $k = 2, 3, \dots$, (1.1) is the Greiner operator (see [10]). As is well known, if $k > 1$, then the vector fields $X_j, Y_j (j = 1, \dots, n)$ do not satisfy the left translation invariance and, if $k \neq 1, 2, 3, \dots$, they do not meet the Hörmander condition (see [11]).

Beals, Gaveau and Greiner [1] constructed an explicit fundamental solution for a class of subelliptic operators containing the operators (1.1) as a particular case. Recently, Zhang, Niu and Luo in [14] obtained the Hardy type inequality and the Pohozaev type identity of $\triangle_L$.

For any second order partial differential operator

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

Received by the editors November 28, 2002.

This work supported by National Natural Science Foundation of China and Natural Science Foundation of Shaanxi Province

AMS subject classification: 35H20.

Keywords: Hopf type lemma, CR inversion, Liouville type theorem, generalized Greiner operator.

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

with $(a_{ij}(x))$ a positive semi-definite matrix, the weak Maximum Principle is true (see [13]). If the operator is in divergence form and is generated by vector fields satisfying the Hörmander condition, then the Strong Maximum Principle holds (see [6]). In [4], a Hopf type lemma for the Heisenberg Laplacian was proved. We will establish a similar result for the operator (1.1).

The CR inversion associated with the Heisenberg Laplacian was introduced by Jerison and Lee; see [12, 5]. We will develop the analogue for the operator (1.1).

Let us now describe the contents of the paper. In Section 2 we collect various facts that are used subsequently. In Section 3 we establish the Hopf type lemma for the operator $\triangle_L$. The key ingredient in the proof of the result is the quasi distance defined in Section 2. It allows us to take an effective auxiliary function. Section 4 contains the CR type transform for the operator (1.1). Clearly it plays the role of the "Kelvin transform". In Section 5 we consider some Liouville type results for nonnegative solutions of semilinear equations of the form

$$\triangle_L u + f(\xi, u) \leq 0.$$

These results generalize those of [2, 3] in the Heisenberg Laplacian setting.

2 Preliminary Facts

This section is devoted to giving some known facts (see [14]) about the operator $\triangle_L$ and the family of vector fields $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ which will be useful later on.

Denote the generalized gradient by $\nabla_L = \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$. Let us denote by δ_λ the natural dilations in R^{2n+1}, *i.e.*,

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^{2k} t), \lambda > 0. \tag{2.1}$$

Let $A = (a_{ij})$ be a symmetrical matrix, where

$$a_{ij} = \delta_{ij}, \quad i, j = 1, \dots, n;$$

$$a_{2n+1,j} = 2ky_j\,|z|^{2k-2}, \quad j = 1, \dots, n;$$

$$a_{2n+1,n+j} = -2kx_j\,|z|^{2k-2}, \quad j = 1, \dots, n;$$

$$a_{2n+1,2n+1} = 4k^2|z|^{4k-2}.$$

Then it is easy to observe that

$$\triangle_L = \operatorname{div}(A\nabla), \tag{2.2}$$

where div and ∇ are the usual divergence and gradient in R^{2n+1} respectively.

Let

$$\sigma = \begin{pmatrix} I_n & 0 & 2ky|z|^{2k-2} \\ 0 & I_n & -2kx|z|^{2k-2} \end{pmatrix},$$

where I_n is the identity matrix in R^n. Obviously one has $A = \sigma^T\sigma$ and then

$$\triangle_L = \operatorname{div}(\sigma^T\sigma\nabla). \tag{2.3}$$

The homogeneous dimension with respect to the dilations (2.1) is

$$Q = 2n + 2k.$$

Define a quasi distance between two points ξ, η in R^{2n+1} by setting

$$d(\xi, \eta) = \left[|z|^{4k} + |z'|^{4k} + (t - t')^2\right]^{\frac{1}{4k}}, \tag{2.4}$$

where $\xi = (z, t)$, $\eta = (z', t') \in R^{2n+1}$. Clearly in this definition the quasi ball centered at ξ with radius R is denoted by

$$B_L(\xi, R) = \left\{\eta \in R^{2n+1} : d(\xi, \eta) < R\right\}. \tag{2.5}$$

Note that for $R > 0$ sufficiently large, if $B(0, R)$ is the Euclidean ball of radius R centered at the origin, then

$$B(0, R) \subset B_L(0, R) \subset B(0, R^2). \tag{2.6}$$

We can now state some useful properties concerning the operator $\triangle_L$. One verifies directly that

$$\begin{aligned} \triangle_L = \sum_{j=1}^{n} \Big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4k y_j |z|^{2k-2} \frac{\partial^2}{\partial x_j \partial t} - 4k x_j |z|^{2k-2} \frac{\partial^2}{\partial y_j \partial t} \Big) \\ + 4k^2 |z|^{4k-2} \frac{\partial^2}{\partial t^2}. \end{aligned} \tag{2.7}$$

A routine calculation shows that the operator $\triangle_L$ is homogeneous of degree 2 with respect to the dilations δ_λ defined in (2.1), namely $\triangle_L(\delta_\lambda) = \lambda^2 \delta_\lambda(\triangle_L)$. For u, a smooth function depending only on $\rho = |\xi|_L = d(\xi, 0)$, one obtains

$$\triangle_L u(\rho) = \psi \Big(\frac{\partial^2 u}{\partial \rho^2} + \frac{Q-1}{\rho} \frac{\partial u}{\partial \rho} \Big),$$

where $\psi = |z|^{4k-2}/\rho^{4k-2}$. Clearly $0 \le \psi \le 1$. If u is a smooth function of $d = d(\xi, \eta)$, then we have

$$\triangle_L u(d) = u'(d) \triangle_L d + u''(d) |\nabla_L d|^2. \tag{2.8}$$

The following is an important Gauss-Green formula:

$$\begin{aligned} \int_\Omega \triangle_L u \cdot v \, d\xi &= -\int_\Omega \nabla_L u \cdot \nabla_L v \, d\xi + \int_{\partial\Omega} v A \nabla u \cdot \vec{\nu} \, d\sigma \\ &= -\int_\Omega \nabla_L u \cdot \nabla_L v \, d\xi + \int_{\partial\Omega} v \nabla_L u \cdot \nu_L \, d\xi, \end{aligned} \tag{2.9}$$

where $\vec{\nu}$ is the exterior normal to $\partial\Omega$ and $\nu_L(\xi) = \sigma(\xi)\nu(\xi)$.

3 A Hopf Type Lemma

In this section we want to examine a version of Hopf lemma for the operator ΔL. Let us start with the following definition which is a natural generalization of interior sphere condition concepts in the Euclidean space and in the Heisenberg group, respectively.

Definition 3.1 *Let $\Omega \subset R^{2n+1}$ be a connected open set. Then Ω satisfies the interior Greiner's sphere condition at $\xi_0 \in \partial\Omega$ if there exist a constant $R > 0$ and $\eta \in \Omega$ such that the quasi-ball $B_L(\eta, R) \subset \Omega$ and $\xi_0 \in \partial B_L(\xi, R)$.*

Lemma 3.1 *Let Ω be a bounded smooth domain of R^{2n+1} possessing the interior Greiner's sphere condition at $\xi_0 \in \partial\Omega$. If*

(1) *$u \in C^2(\Omega) \bigcap C(\bar{\Omega})$ and is continuous in ξ_0,*
(2) *$-\triangle_L u + cu \geq 0$ in Ω, where c is bounded in Ω,*
(3) *$u(\xi) > u(\xi_0) = 0$ for $\xi \in B_L(\xi_0, R) \bigcap \Omega$ for some $R > 0$,*

then, for any $\vec{n}$ exterior direction to $\partial\Omega$ at ξ_0, we have

$$\limsup_{h \to 0^+} \frac{u(\xi_0) - u(\xi_0 - h\vec{n})}{h} < 0$$

and if it exists, it holds

$$\frac{\partial u(\xi_0)}{\partial n} < 0.$$

Moreover $A\nabla u(\xi_0) \cdot \vec{\nu}(\xi_0) < 0$ where $\vec{\nu}$, the exterior normal to $\partial\Omega$ at ξ_0, is not in the direction of the t-axis.

Proof Let $\xi = (z, t) = (x, y, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t)$, $\eta = (z', t') = (x', y', t') = (x'_1, \dots, x'_n, y'_1, \dots, y'_n, t')$ and $R > 0$ as in the Definition 3.1. Let $d = d(\xi, \eta) = \left[|z|^{4k} + |z'|^{4k} + (t - t')^2\right]^{\frac{1}{4k}}$. It is clear that

$$|\nabla_L d|^2 = \sum_{j=1}^{n} (|X_j d|^2 + |Y_j d|^2) = d^{2-8k} |z|^{4k-2} \left[|z|^{4k} + (t - t')^2\right], \tag{3.1}$$

$$\triangle_L d = (1 - 4k) d^{1-8k} |z|^{4k-2} \left[|z|^{4k} + (t - t')^2\right] + 2(n + 3k - 1) d^{1-4k} |z|^{4k-2}. \tag{3.2}$$

We set

$$v(d) = e^{-aR^2} - e^{-ad^2}, \quad \text{for } 0 < \rho < d < R, \tag{3.3}$$

and claim the existence of $m > 0$ such that

$$-\triangle_L v - 4m(x_1 - x'_1) X_1 v \geq 0 \tag{3.4}$$

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

for a sufficiently large. Indeed, in view of

$$v'(d) = 2ade^{-ad^2},$$
$$v''(d) = (2a - 4ad^2)e^{-ad^2},$$

we have from (3.1) and (3.2),

$$\begin{aligned}(3.5)\quad -\triangle_L v - 4m(x_1 - x_1')X_1 v &= -v''(d)|\nabla_L d|^2 - v'(d)\triangle_L d - 4m(x_1 - x_1')X_1 v\\ &= 2ae^{-ad^2}\left[2ad^2|\nabla_L d|^2 - |\nabla_L d|^2 - d\triangle_L d - 4m(x_1 - x_1')X_1 d\right]\\ &= 2ae^{-ad^2}\Big\{2ad^2|\nabla_L d|^2 + (4k-2)d^{2-8k}|z|^{4k-2}[|z|^{4k} + (t-t')^2]\\ &\quad - d^{2-4k}|z|^{4k-2}[2n + 6k - 2 + 4m(x_1 - x_1')x_1]\\ &\quad - 4md^{2-4k}|z|^{2k-2}(x_1 - x_1')(t-t')y_1\Big\}.\end{aligned}$$

First case: $|\nabla_L d|^2 > 0$. Clearly (3.4) follows from (3.5) for a sufficiently large.

Second case: $|\nabla_L d|^2 = 0$. We get that $|z| = 0$ from (3.1) and the right hand side of (3.5) becomes zero. Consequently the claim (3.4) is concluded.

Let $\xi_0 \in \partial B_L(\eta, R)$ and $\vec{n}$ be an exterior direction to $\partial\Omega$ at ξ_0. Define an auxiliary function

$$(3.6)\qquad w = e^{-m(x_1 - x_1')^2}u, \text{ for } m > 0.$$

Then the following inequality holds:

$$(3.7)\qquad -\triangle_L w - 4m(x_1 - x_1')X_1 w \geq 0.$$

In fact, it is easy to check that

$$\frac{\partial^2 w}{\partial x_1^2} = e^{-m(x_1 - x_1')^2}\left[4m^2(x_1 - x_1')^2 u - 2mu - 4m(x_1 - x_1')\frac{\partial u}{\partial x_1} + \frac{\partial^2 u}{\partial x_1^2}\right],$$
$$\frac{\partial^2 w}{\partial x_j^2} = e^{-m(x_1 - x_1')^2}\frac{\partial^2 u}{\partial x_j^2},\ j = 2, \dots, n,$$
$$\frac{\partial^2 w}{\partial y_j^2} = e^{-m(x_1 - x_1')^2}\frac{\partial^2 u}{\partial y_j^2},\ j = 1, 2, \dots, n,$$
$$\frac{\partial^2 w}{\partial x_1 \partial t} = e^{-m(x_1 - x_1')^2}\left[-2m(x_1 - x_1')\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x_1 \partial t}\right],$$
$$\frac{\partial^2 w}{\partial x_j \partial t} = e^{-m(x_1 - x_1')^2}\frac{\partial^2 u}{\partial x_j \partial t},\ j = 2, \dots, n,$$
$$\frac{\partial^2 w}{\partial y_j \partial t} = e^{-m(x_1 - x_1')^2}\frac{\partial^2 u}{\partial y_j \partial t},\ j = 1, 2, \dots, n.$$

Then one infers that

$$\text{(3.8)}\qquad \triangle_L w + 4m(x_1 - x_1')X_1 w = e^{-m(x_1-x_1')^2}\left[-4m^2(x_1 - x_1')^2 u + \triangle_L u - 2mu\right] \le e^{-m(x_1-x_1')^2}(\triangle_L u - 2mu).$$

The claim (3.7) is proved from the condition (2).

If we show that

$$\text{(3.9)}\qquad \frac{\partial w(\xi_0)}{\partial n} < 0,$$

then $\frac{\partial w(\xi_0)}{\partial n} = e^{-m\left((x_0)_1 - x_1'\right)^2}\frac{\partial u(\xi_0)}{\partial n}$ gives the first statement of the Lemma.

In light of (3.4) and (3.8), we have

$$\text{(3.10)}\qquad -\triangle_L(w+\epsilon v) - 4m(x_1 - x_1')X_1(w+\epsilon v) \ge 0, \text{ in } B_L(\eta, R)\setminus B_L(\eta,\rho)$$

and $w + \epsilon v \ge 0$, on $\partial B_L(\eta, R)$. Furthermore, for ϵ sufficiently small, $w + \epsilon v \ge 0$, on $\partial B_L(\eta,\rho)$. Thus, from the weak maximum principle (see[10]), we deduce that

$$\text{(3.11)}\qquad w + \epsilon v \ge 0, \text{ in } B_L(\eta, R)\setminus B_L(\eta,\rho).$$

Now note that $w(\xi_0) = -\epsilon v(\xi_0) = 0$. Furthermore, for any $\vec{n}\cdot\vec{\nu} > 0$ and for small $h > 0$, $w(\xi_0 - h\vec{n}) \ge -\epsilon v(\xi_0 - h\vec{n})$. Using the fact that v_d' is strictly positive, (3.9) is concluded. If $\vec{\nu}$ is not in the t-axis direction, then

$$A(\xi_0)\vec{\nu}\cdot\vec{\nu} = \sigma(\xi_0)^T\sigma(\xi_0)\vec{\nu}\cdot\vec{\nu} = \sigma(\xi_0)\vec{\nu}\cdot\sigma(\xi_0)\vec{\nu} > 0.$$

This implies that $A\vec{\nu}$ is an exterior direction at ξ_0 and then

$$A(\xi_0)\nabla u(\xi_0)\cdot\vec{\nu}(\xi_0) < 0.$$

The proof of the lemma is completed. ■

Based on Lemma 3.1, we give the following Strong Maximum Principle, whose proof is similar to one in elliptic context (see Gilbang-Trudinger [9] or Garofalo-Vassilev [8]).

Theorem 3.1 *Let $u \in C^2(\Omega)\cap C(\bar{\Omega})$ satisfying $\triangle_L u \le 0$ ($\triangle_L u \ge 0$). If u is not a constant identically, then u can not have a nonpositive minimum (nonnegative maximum) at a point in Ω.*

Corollary 3.1 *If $u \in C^2(\Omega)\cap C(\bar{\Omega})$ satisfying $\triangle_L u = 0$ and u is not a constant identically , then throughout Ω,*

$$\min_{\partial\Omega} u < u(\xi) < \max_{\partial\Omega} u, \qquad \text{for any } \xi \in \Omega.$$

4 A CR Type Inversion

A function u is said to be cylindrical in R^{2n+1} with respect to the operator ΔL, if for any $(x, y, t) \in R^n \times R^n \times R$, it has $u(x, y, t) = u(r, t)$ with $r = \sqrt{x^2 + y^2}$.

We define the CR type inversion of a regular function $u(x, y, t)$ in R^{2n+1} to be

$$v(x, y, t) = \frac{1}{\rho^{Q-2}} u(\tilde{x}, \tilde{y}, \tilde{t}) \tag{4.1}$$

with $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ and $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$, where

$$\tilde{x}_i = \frac{x_i t + |z|^{2k} y_i}{\rho^{2k+2}}, \tilde{y}_i = \frac{y_i t - |z|^{2k} x_i}{\rho^{2k+2}}, \tilde{t} = -\frac{t}{\rho^{4k}}.$$

Note that v is a regular function in $R^{2n+1} \backslash \{0\}$. Denote $\tilde{z} = (\tilde{x}, \tilde{y})$ in the sequel.

Theorem 4.1 *Let $u(x, y, t)$ be a solution of*

$$\triangle_L u(x, y, t) = f(x, y, t). \tag{4.2}$$

Then v defined by (4.1) *satisfies*

$$\triangle_L v(x, y, t) = \frac{1}{\rho^{Q+2}} f(\tilde{x}, \tilde{y}, \tilde{t}). \tag{4.3}$$

Proof Since

$$\tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2} = \frac{r}{\rho^2}, \ \tilde{\rho} = (|\tilde{z}|^{4k} + \tilde{t}^2)^{\frac{1}{4k}} = \frac{1}{\rho},$$

we know that if u is cylindrical, then so is v. For the sake of simplicity we will prove (4.3) only for cylindrical functions.

A short computation gives the following equalities

$$\frac{\partial \rho}{\partial r} = \frac{r^{4k-1}}{\rho^{4k-1}}; \qquad \frac{\partial \rho}{\partial t} = \frac{t}{2k\rho^{4k-1}};$$

$$\frac{\partial \tilde{r}}{\partial r} = \frac{t^2 - r^{4k}}{\rho^{4k+2}}; \qquad \frac{\partial \tilde{r}}{\partial t} = \frac{-tr}{k\rho^{4k+2}};$$

$$\frac{\partial \tilde{t}}{\partial r} = \frac{4ktr^{4k-1}}{\rho^{8k}}; \qquad \frac{\partial \tilde{t}}{\partial t} = \frac{t^2 - r^{4k}}{\rho^{8k}};$$

$$\frac{\partial}{\partial r}\left(\frac{1}{\rho^{Q-2}}\right) = \frac{(2-Q)r^{4k-1}}{\rho^{Q+4k-2}}; \qquad \frac{\partial}{\partial t}\left(\frac{1}{\rho^{Q-2}}\right) = \frac{(2-Q)t}{2k\rho^{Q+4k-2}}.$$

Therefore $v(r, t) = \frac{1}{\rho^{Q-2}} u(\tilde{r}, \tilde{t})$ satisfies

$$\frac{\partial v}{\partial r} = \frac{(2-Q)r^{4k-1}}{\rho^{Q+4k-2}} u + \frac{1}{\rho^{Q-2}}\left[\frac{\partial u}{\partial \tilde{r}}\left(\frac{t^2 - r^{4k}}{\rho^{4k+2}}\right) + \frac{\partial u}{\partial \tilde{t}}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right)\right] \tag{4.4}$$

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

and hence

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{(2-Q)r^{4k-1}}{\rho^{Q+4k-2}}\right)u \\ + \frac{2(2-Q)r^{4k-1}}{\rho^{Q+4k-2}}\left[\frac{\partial u}{\partial \tilde{r}}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right) + \frac{\partial u}{\partial \tilde{t}}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right)\right] \\ + \frac{t^2-r^{4k}}{\rho^{Q+4k}}\left[\frac{\partial^2 u}{\partial \tilde{r}^2}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right) + \frac{\partial^2 u}{\partial \tilde{r}\partial \tilde{t}}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right)\right] \\ + \frac{4ktr^{4k-1}}{\rho^{Q+8k-2}}\left[\frac{\partial^2 u}{\partial \tilde{t}^2}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right) + \frac{\partial^2 u}{\partial \tilde{t}\partial \tilde{r}}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right)\right] \\ + \frac{1}{\rho^{Q-2}}\left[\frac{\partial u}{\partial \tilde{r}}\frac{\partial}{\partial r}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right) + \frac{\partial u}{\partial \tilde{t}}\frac{\partial}{\partial r}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right)\right]. \tag{4.5}$$

One easily infers

$$\frac{\partial v}{\partial t} = \frac{(2-Q)t}{2k\rho^{Q+4k-2}}u + \frac{1}{\rho^{Q-2}}\left[\frac{\partial u}{\partial \tilde{r}}\left(\frac{-tr}{k\rho^{4k+2}}\right) + \frac{\partial u}{\partial \tilde{t}}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)\right]$$

and then

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{(2-Q)t}{2k\rho^{Q+4k-2}}\right)u \\ + \frac{(2-Q)t}{k\rho^{Q+4k-2}}\left[\frac{\partial u}{\partial \tilde{r}}\left(\frac{-tr}{k\rho^{4k+2}}\right) + \frac{\partial u}{\partial \tilde{t}}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)\right] \\ + \frac{-tr}{k\rho^{Q+4k}}\left[\frac{\partial^2 u}{\partial \tilde{r}^2}\left(\frac{-tr}{k\rho^{4k+2}}\right) + \frac{\partial^2 u}{\partial \tilde{r}\partial \tilde{t}}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)\right] \\ + \frac{t^2-r^{4k}}{\rho^{Q+8k-2}}\left[\frac{\partial^2 u}{\partial \tilde{t}^2}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right) + \frac{\partial^2 u}{\partial \tilde{t}\partial \tilde{r}}\left(\frac{-tr}{k\rho^{4k+2}}\right)\right] \\ + \frac{1}{\rho^{Q-2}}\left[\frac{\partial u}{\partial \tilde{r}}\frac{\partial}{\partial t}\left(\frac{-tr}{k\rho^{4k+2}}\right) + \frac{\partial u}{\partial \tilde{t}}\frac{\partial}{\partial t}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)\right]. \tag{4.6}$$

It is evident to see that

$$\frac{\partial}{\partial r}\left(\frac{(2-Q)r^{4k-1}}{\rho^{Q+4k-2}}\right) = \frac{(2-Q)r^{4k-2}}{\rho^{Q+8k-2}}\left[(4k-1)\rho^{4k} - (Q+4k-2)r^{4k}\right],$$

$$\frac{\partial}{\partial t}\left(\frac{(2-Q)t}{2k\rho^{Q+4k-2}}\right) = \frac{2-Q}{2k}\left(\frac{1}{\rho^{Q+4k-2}} - \frac{(Q+4k-2)t^2}{2k\rho^{Q+8k-2}}\right),$$

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

$$\frac{\partial}{\partial r}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right)=-\frac{4kr^{4k-1}}{\rho^{4k+2}}-\frac{(4k+2)r^{4k-1}(t^2-r^{4k})}{\rho^{8k+2}},$$
$$\frac{\partial}{\partial r}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right)=4ktr^{4k-2}\left(\frac{4k-1}{\rho^{8k}}-\frac{8kr^{4k}}{\rho^{12k}}\right),$$
$$\frac{\partial}{\partial t}\left(\frac{-tr}{k\rho^{4k+2}}\right)=\frac{r}{k}\left(\frac{(2k+1)t^2}{k\rho^{8k+2}}-\frac{1}{\rho^{4k+2}}\right),$$
$$\frac{\partial}{\partial t}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)=\frac{2t}{\rho^{8k}}-\frac{4t(t^2-r^{4k})}{\rho^{12k}}.$$

By (2.7), we obtain readily

$$\triangle_L v(r,t)=\frac{\partial^2 v}{\partial r^2}+\frac{2n-1}{r}\cdot\frac{\partial v}{\partial r}+4k^2r^{4k-2}\frac{\partial^2 v}{\partial t^2}.$$

Consequently, from (4.4), (4.5) and (4.6), we have

$$\triangle_L v(r,t)=a_0u+a_1\frac{\partial^2 u}{\partial\tilde r^2}+a_2\frac{\partial^2 u}{\partial\tilde r\partial\tilde t}+a_3\frac{\partial^2 u}{\partial\tilde t^2}+b_1\frac{\partial u}{\partial\tilde r}+b_2\frac{\partial u}{\partial\tilde t},\tag{4.7}$$

where a_0, a_1, a_2, a_3, b_1 and b_2 are the coefficients to be determined. Using the previous computations, we deduce that the coefficients satisfy the following:

$$\begin{aligned}
a_0&=\frac{\partial}{\partial r}\left(\frac{(2-Q)r^{4k-1}}{\rho^{Q+4k-2}}\right)+\frac{2n-1}{r}\cdot\frac{(2-Q)r^{4k-2}}{\rho^{Q+4k-2}}+4k^2r^{4k-2}\frac{\partial}{\partial t}\left(\frac{(2-Q)t}{2k\rho^{Q+4k-2}}\right)\\
&=0,\\
a_1&=\frac{t^2-r^{4k}}{\rho^{Q+4k}}\cdot\frac{t^2-r^{4k}}{\rho^{4k+2}}+4k^2r^{4k-2}\left(\frac{-tr}{k\rho^{Q+4k}}\right)\cdot\left(\frac{-tr}{k\rho^{4k+2}}\right)=\frac{1}{\rho^{Q+2}},\\
a_2&=\left(\frac{t^2-r^{4k}}{\rho^{Q+4k}}\right)\frac{4ktr^{4k-1}}{\rho^{8k}}+\frac{4ktr^{4k-1}}{\rho^{Q+8k-2}}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right)\\
&\quad+4k^2r^{4k-2}\left(\frac{-tr}{k\rho^{Q+4k}}\right)\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)+4k^2r^{4k-2}\left(\frac{t^2-r^{4k}}{\rho^{Q+8k-2}}\right)\left(\frac{-tr}{k\rho^{4k+2}}\right)\\
&=0,\\
a_3&=\frac{4ktr^{4k-1}}{\rho^{Q+8k-2}}\cdot\frac{4ktr^{4k-1}}{\rho^{8k}}+4k^2r^{4k-2}\left(\frac{t^2-r^{4k}}{\rho^{Q+8k-2}}\right)\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)=\frac{1}{\rho^{Q+2}}\cdot\frac{4k^2r^{4k-2}}{\rho^{8k-4}},\\
b_1&=\frac{2(2-Q)r^{4k-1}}{\rho^{Q+4k-2}}\cdot\frac{t^2-r^2}{\rho^{4k+2}}+\frac{1}{\rho^{Q-2}}\cdot\frac{\partial}{\partial r}\left(\frac{t^2-r^{4k}}{\rho^{4k+2}}\right)+\frac{2n-1}{r}\cdot\frac{1}{\rho^{Q-2}}\cdot\frac{t^2-r^{4k}}{\rho^{4k+2}}\\
&\quad+4k^2r^{4k-2}\left[\frac{(2-Q)t}{k\rho^{Q+4k-2}}\left(\frac{-tr}{k\rho^{4k+2}}\right)+\frac{1}{\rho^{Q-2}}\frac{\partial}{\partial t}\left(\frac{-tr}{k\rho^{4k+2}}\right)\right]=\frac{1}{\rho^{Q+2}}\frac{(2n-1)\rho^2}{r},
\end{aligned}$$

$$b_2 = \frac{2(2-Q)r^{4k-1}}{\rho^{Q+4k-2}} \cdot \frac{4ktr^{4k-1}}{\rho^{8k}} + \frac{1}{\rho^{Q-2}} \cdot \frac{\partial}{\partial r}\left(\frac{4ktr^{4k-1}}{\rho^{8k}}\right)$$
$$+ \frac{2n-1}{r} \cdot \frac{1}{\rho^{Q-2}} \cdot \frac{4ktr^{4k-1}}{\rho^{8k}}$$
$$+ 4k^2 r^{4k-2}\left[\frac{(2-Q)t}{k\rho^{Q+4k-2}} \cdot \frac{t^2 - r^{4k}}{\rho^{8k}} + \frac{1}{\rho^{Q-2}} \frac{\partial}{\partial t}\left(\frac{t^2-r^{4k}}{\rho^{8k}}\right)\right]$$
$$= 0.$$

Applying these to (4.7) gives

$$\begin{aligned}\triangle_L v(r,t) &= \frac{1}{\rho^{Q+2}} \cdot \frac{\partial^2 u}{\partial \tilde{r}^2} + \frac{1}{\rho^{Q+2}} \cdot \frac{(2n-1)\rho^2}{r} \cdot \frac{\partial u}{\partial \tilde{r}} + \frac{1}{\rho^{Q+2}} \cdot \frac{4k^2 r^{4k-2}}{\rho^{8k-4}} \cdot \frac{\partial^2 u}{\partial \tilde{t}^2} \\ &= \frac{1}{\rho^{Q+2}}\left(\frac{\partial^2 u}{\partial \tilde{r}^2} + \frac{2n-1}{\tilde{r}} \cdot \frac{\partial u}{\partial \tilde{r}} + 4k^2 \tilde{r}^{4k-2} \cdot \frac{\partial^2 u}{\partial \tilde{t}^2}\right) \\ &= \frac{1}{\rho^{Q+2}} \triangle_L u(\tilde{r}, \tilde{t})\end{aligned}$$

so the result is proved. ■

Remark 4.1 If a cylindrical function u satisfies the equation

$$\triangle_L u + u^p = 0 \text{ in } R^{2n+1},$$

then v, the CR inversion of u, which is given by

$$v(r,t) = \frac{1}{\rho^{Q-2}} u\left(\frac{r}{\rho^2}, -\frac{t}{\rho^4}\right)$$

satisfies the equation

$$\triangle_L v + \frac{1}{\rho^{Q+2-p(Q-2)}} v^p = 0 \text{ in } R^{2n+1} \backslash \{0\}.$$

5 Liouville Type Theorems

In this section we study the Liouville type behaviors of the operator (1.1) which were considered for positive solutions of superlinear equations associated to the sub-Laplacian on the Heisenberg group in [2, 3]. The following Theorem 5.1 is an application of Theorem 3.1.

Theorem 5.1 *Let $u \in C^2(R^{2n+1})$ be a nonnegative solution of*

$$\triangle_L u + h(\xi) u^p \leq 0 \text{ in } R^{2n+1}, \tag{5.1}$$

where $h(\xi) \in C(R^{2n+1})$ and $h(\xi) > K\psi|\xi|_L^\nu$ with $K > 0$, $|\xi|_L = d(\xi, 0)$ and $\nu > -2$. If $1 < p < \frac{Q+\nu}{Q-2}$, then $u \equiv 0$ in R^{2n+1}.

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

Proof We take a cut-off function $\phi_R(\rho) = \phi\left(\frac{\rho}{R}\right)$, where $\rho = |\xi|_L$, $R > 0$ and ϕ satisfies:

- $\phi \in C^\infty[0, +\infty)$, $\qquad 0 \le \phi \le 1$;
- $\phi \equiv 1$ on $[0, \frac{1}{2}]$ and $\phi \equiv 0$ on $[1, +\infty)$;
- $-\frac{C}{R} \le \frac{\partial \phi_R}{\partial \rho} \le 0$ and $\left|\frac{\partial^2 \phi_R}{\partial \rho^2}\right| \le \frac{C}{R^2}$ for some constant $C > 0$.

Denote

$$I_R = \int_{R^{2n+1}} h(\xi) u^p \phi_R^q \, d\xi, \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \tag{5.2}$$

Observe that $I_R \ge 0$. Moreover, by using (5.1) and the assumptions on ϕ_R,

$$I_R \le -\int_{B_L(0,R)} \triangle_L u \phi_R^q \, d\xi.$$

Hence an integration by parts yields

(5.3)
$$\begin{aligned}
I_R &\le -\int_{\partial B_L(0,R)} \phi_R^q \nabla_L u \cdot \nu_L \, d\Sigma + \int_{B_L(0,R)} \nabla_L u \cdot \nabla_L \phi_R^q \, d\xi \\
&= -\int_{\partial B_L(0,R)} \phi_R^q \nabla_L u \cdot \nu_L \, d\Sigma + \int_{\partial B_L(0,R)} u \cdot \nabla_L \phi_R^q \cdot \nu_L \, d\Sigma - \int_{B_L(0,R)} u \triangle_L \phi_R^q \, d\xi \\
&= \int_{\partial B_L(0,R)} q u \phi_R^{q-1} \phi_R' \nabla_L \rho \cdot \nu_L \, d\Sigma - \int_{B_L(0,R)} u \triangle_L \phi_R^q \, d\xi \\
&= -\int_{B_L(0,R)} u \triangle_L \phi_R^q \, d\xi.
\end{aligned}$$

where $\nu_L(\xi) = \sigma(\xi)\nu(\xi)$ and $\nu(\xi)$ is the normal to $\partial\Omega$, $d\Sigma$ denotes the $2n$-dimensional Hausdorff measure. On the other hand, in view of (2.7),

$$\begin{aligned}
\triangle_L \phi_R^q &= \psi \left(\frac{\partial^2 \phi_R^q}{\partial \rho^2} + \frac{Q-1}{\rho} \cdot \frac{\partial \phi_R^q}{\partial \rho} \right) \\
&= \psi \left[q(q-1)\phi_R^{q-2} (\phi_R')^2 + q\phi_R^{q-1}\phi_R'' + \frac{Q-1}{\rho} \cdot q\phi_R^{q-1}\phi_R' \right].
\end{aligned} \tag{5.4}$$

Thus, we get, using the assumptions on ϕ_R and denoting by $\Omega_R = B_L(0,R) \backslash B_L(0, \frac{R}{2})$,

$$\begin{aligned}
I_R &\le -\int_{\Omega_R} u\psi \left(q\phi_R^{q-1}\phi_R'' + \frac{Q-1}{\rho} \cdot q\phi_R^{q-1}\phi_R' \right) d\xi \\
&\le \frac{C}{R^2} \int_{\Omega_R} u\psi \phi_R^{q-1} \, d\xi \\
&\le \frac{C}{R^2} \left[\int_{\Omega_R} \psi u^p \rho^\nu \phi_R^{(q-1)p} \, d\xi \right]^{\frac{1}{p}} \left[\int_{\Omega_R} \psi \rho^{-\frac{q}{p}\nu} \, d\xi \right]^{\frac{1}{q}}.
\end{aligned} \tag{5.5}$$

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

where we have used the Hölder inequality.

Picking $R > 0$ sufficiently large in Ω_R, and noting that h satisfy $h > K\psi|\xi|_L^{\nu}$, we have

$$I_R \le C\left[\int_{\Omega_R} hu^p \phi_R^q \, d\xi\right]^{\frac{1}{p}} R^{\frac{Q}{p}-\frac{\nu}{p}-2}, \tag{5.6}$$

and this implies

$$I_R^{1-\frac{1}{p}} \le CR^{\frac{Q}{p}-\frac{\nu}{p}-2}. \tag{5.7}$$

Hence, if $1 < p < \frac{Q+\nu}{Q-2}$, letting $R \to \infty$, we arrive at

$$I = \int_{R^{2n+1}} hu^p \, d\xi = 0. \tag{5.8}$$

This yields $u \equiv 0$ for ρ large, since h is strictly positive outside of a set of measure zero and u is a prior nonnegative.

The claim follows now by the Strong Maximum Principle (see Theorem 3.1). In fact, choose $\bar{R} > 0$ in such a way that, for $\rho \ge \bar{R}$, $h > 0$. Then $u \equiv 0$ on the complementary of $B_L(0, \bar{R})$, as we proved. Hence, u satisfies

$$u \ge 0, \triangle_L u \le 0, \text{ in } B_L(0, \bar{R} + \delta),$$
$$u \equiv 0, \text{ for } \bar{R} \le \rho \le \bar{R} + \delta,$$

for some $\delta > 0$. Therefore, by Theorem 3.1, since u is not strictly positive, u has to be identically zero.

If $p = \frac{Q+\nu}{Q-2}$, then (5.7) implies that I is finite and that the right hand side of (5.5) tends to zero when R goes to infinity. This shows $I = 0$ and we conclude as above. ∎

Our next Liouville-type results concern the case where D are half spaces. Let us start with the following:

Lemma 5.1 *Let $D \subset R^{2n+1}$ be a domain with smooth boundary ∂D. Assume that $\eta \in C^2(D) \cap C(\bar{D})$ satisfies*

$$\begin{cases} \eta \ge 0, \ \triangle_L \eta \ge 0 & \text{in } D \\ \eta = 0, & \text{on } \partial D \end{cases} \tag{5.9}$$

and let $u \in C^2(D) \bigcap C(\bar{D})$ be a solution of

$$u \ge 0, \ \triangle_L u + g(\xi)u^{\alpha} \le 0 \text{ in } D, \ \alpha > 1 \tag{5.10}$$

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

with $g > 0$ *in* D *and* $g \in C(\bar{D})$. *If* $\Omega_R = (B_L(0,R)\backslash B_L(0,\frac{R}{2})) \cap D \neq \phi$ *for some* $R > 0$, *then*

$$I_R = \int_D g u^\alpha \phi_R^\beta \eta^p \, d\xi, \ p \geq 1,$$

with β *such that* $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$, *satisfies the following estimate:*

$$(5.11) \quad I_R \leq I_R^{\frac{1}{\alpha}} \left\{ \frac{C}{R^2} \left[\int_{\Omega_R} \eta^p g^{-\frac{\beta}{\alpha}} \, d\xi \right]^{\frac{1}{\beta}} + \frac{C}{R} \left[\int_{\Omega_R} \eta^{p-\beta} |\nabla_L \eta \cdot \nabla_L \rho|^\beta g^{-\frac{\beta}{\alpha}} \, d\xi \right]^{\frac{1}{\beta}} \right\}$$

Theorem 5.2 *Let* $D = \{\xi \in R^{2n+1} : x_1 > 0\}$. *For* $\tau \geq 0$, $\varphi(\xi) \geq C|\xi|_L^\nu$ *with* $\nu > -1$ *and* $\varphi \in C(\bar{D})$. *Assume one of the following conditions holds :*

$$(1) \qquad \frac{Q+\nu-1}{Q-2} \leq \tau + 2 \quad \text{and} \quad 1 < \alpha < \frac{Q+\nu-1}{Q-2};$$

$$(2) \qquad \frac{Q+\nu-1}{Q-2} > \tau + 2 \quad \text{and} \quad 1 < \alpha \leq \frac{Q+\nu+\tau+1}{Q-1}.$$

Then the only nonnegative solution $u \in C^2(D) \cap C(\bar{D})$ *of*

$$(5.12) \qquad \triangle_L u + x_1^\tau \varphi(\xi) u^\alpha \leq 0 \ \text{in} \ D$$

is $u \equiv 0$.

Theorem 5.3 *Let* $D = \{\xi \in R^{2n+1} : t > 0\}$. *Assume* $1 < \alpha < \frac{Q+2k+\nu}{Q+2k-2}$ *and* $\varphi(\xi) \geq C|\xi|_L^\nu$ *with* $\nu > -2$ *and* $\varphi \in C(\bar{D})$. *Then the only nonnegative solution* $u \in C^2(D) \cap C(\bar{D})$ *of*

$$(5.13) \qquad \triangle_L u + \varphi(\xi) u^\alpha \leq 0 \ \text{in} \ D$$

is $u \equiv 0$.

Theorem 5.4 *Let* $D = \{\xi \in R^{2n+1} : t > 0\}$. *Assume one of the following hypotheses holds:*

$$(1) \qquad n \geq 3 \quad \text{and} \quad 1 < \alpha < \frac{Q-2k}{Q-2k-2};$$

$$(2) \qquad n < 3 \quad \text{and} \quad 1 < \alpha < \frac{Q+4k}{Q+2k-2}.$$

Then the only nonnegative solution $u \in C^2(D) \bigcap C(\bar{D})$ *of*

$$(5.14) \qquad \triangle_L u + t u^\alpha \leq 0 \ \text{in} \ D$$

is $u \equiv 0$.

The proofs of these results follow by arguments parallel to those used in [3].

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press

References

[1] R. Beals, B. Gaveau and P. Greiner, *Uniforms hypoelliptic Green's functions.* J. Math. Pures Appl. **77**(1998), 209–248.

[2] I. Birindelli, I. Capuzzo Dolcetta and A. Cutri, *Liouville theorems for semilinear equations on the Heisenberg group.* Ann. Inst. H. Poincaré Anal. Non Linéaire **14**(1997), 295–308.

[3] ———, *Indefinite semi-linear equations on the Heisenberg group:a priori bounds and existence.* Comm. Partial Differential Equations **23**(1998), 1123–1157.

[4] I. Birindelli and A. Cutri, *A semilinear problem for the Heisenberg Laplacian.* Rend. Sem. Mat. Univ. Padova, **94**(1995), 137–153.

[5] I. Birindelli and J. Prajapat, *Nonlinear Liouville theorems in the Heisenberg group via the moving plane method.* Comm. Partial Differential Equations, **24**(1999), 1875–1890.

[6] J. M. Bony, *Principe du Maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés.* Ann. Inst. Fourier Grenobles **19**(1969), 277–304.

[7] G. B. Folland, *A fundamental solution for a subelliptic operator.* Bull. Amer. Math. Soc. **79**(1973), 373–376.

[8] N. Garofalo and D. Vassilev, *Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type.* Duke Math. J. **106**(2001), 411–448.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. Grundlehren Math. Wiss 224, Springer-Verlag, New York, 1983.

[10] P. C. Greiner, *A fundamental solution for a nonelliptic partial differential operator.* Can. J. Math. **31**(1979), 1107–1120.

[11] L. Hörmander, *Hypoelliptic second order differential equations.* Acta Math. **119**(1967), 147–171.

[12] D. S. Jerison and J. M. Lee, *The Yamabe problem on CR manifolds.* J. Differential Geom. **25**(1987), 167–197.

[13] O. A. Oleĭnik and E. V. Radkevič, *Second order equations with nonnegative characteristic form.* Plenum Press, New York, 1973.

[14] H. Zhang, P. Niu and X, Luo, *Mean value theorem and Pohozaev type identities of generalized Greiner operator and their applications.* J. Partial Differential Equations **14**(2001), 220–234.

Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi, 710072
P.R. China

Department of Mathematics
and Physics Science
Nanhua University
Hengyang, Hunan, 421200
P.R. China

Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi, 710072
P.R. China

https://doi.org/10.4153/CMB-2004-041-4 Published online by Cambridge University Press