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SEMISIMPLICITY OF FREE CENTRED EXTENSIONS

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ABSTRACT. We prove that a free centred extension $R[E]$ is a semisimple ring if R is a semisimple ring and $C[E]$ is semisimple for every field C which is the extended centroid of a primitive factor of R.

1. **Introduction.** Let K be a field and G a group. Much work has been done to study the (Jacobson) semisimplicity of the group ring KG (see, for example [5], Chapter 7, 268–303). Assume that for some group G, the group ring KG is semisimple for every field (resp. every field of a given characteristic) K. In this note we will show that under this assumption the semisimplicity of the group ring RG follows for every semisimple ring (resp. semisimple algebra of the same characteristic) R. Some results in this direction for tensor products were obtained in [4].

More generally, let $S = R[E]$ be a free centred extension of the ring R [1]. We denote by $\mathcal{P}$ the set of all the primitive ideals of R. Note that for every $P \in \mathcal{P}$, the maximal right quotient ring Q_P of the ring R/P is defined ([6], Chapter IX) and the extended centroid C_P of R/P, *i.e.* the center of Q_P, is a field. Then the rings $Q_P[E]$ and $C_P[E]$ are well-defined ([1], Section 2). Put $\mathcal{F} = \{C_P : P \in \mathcal{P}\}$.

The purpose of this note is to prove the following

THEOREM. *Let R be a semisimple ring and $S = R[E]$ a free centred extension of R. Assume that $C[E]$ is semisimple for every field $C \in \mathcal{F}$. Then S is also semisimple.*

Now we recall some results from [2] (see also [1]) that we will need in the proof. Let R be a prime ring, Q the maximal right quotient ring of R and C the extended centroid of R. For a free centred extension $S = R[E]$ of R with the basis $E = (e_i)_{i\in\Omega}$, we denote by $T = Q[E]$ the extension of S to a free centred extension of Q and we put $V = C[E]$, an algebra over C.

Throughout this note submodule means sub-bimodule. If N is an R-submodule of S, the closure of N is defined as the submodule

$$\begin{aligned}[N] &= \{x \in S : \text{ there exists } 0 \neq A \lhd R \text{ with } xA \subseteq N\} \\ &= \{y \in S : \text{ there exists } 0 \neq B \lhd R \text{ with } By \subseteq N\}.\end{aligned}$$

The submodule N is said to be *closed* if $[N] = N$.

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A one-to-one correspondence between the following sets was proved in [2] (Theorems 2.5, 2.15):

i) The set of all the closed R-submodules of S.

ii) The set of all the closed Q-submodules of T.

iii) The set of all the C-subspaces of V.

This correspondence associates the closed R-submodule N of S with the closed Q-submodule N^* of T and the C-subspace N_0 of V if $N^* \cap S = N$ and $N^* = N_0 Q$. Moreover, the correspondence preserves left closed ideals, closed (two-sided) ideals, R-disjoint prime ideals, intersections as well as inclusions (see also [1], Theorem 2.7; [2], Theorem 5.3 and Remark 5.4, iii).

We use the same notation and terminology used in [1]. In particular, ideal means two sided ideal. If $x \in S$, the support of x is denoted by $\operatorname{supp}(x)$ and the e-coefficient of x by $x(e)$. A submodule N of S is said to be *R-disjoint* if $N \cap R = 0$. The minimality of N is defined by $\operatorname{Min}(N) = \{\operatorname{supp}(a) : a \in M(N)\}$, where $M(N)$ is the set of all the elements of minimal support in N.

Throughout this note primitive means left primitive. Then an ideal P of R is said to be *primitive* if there exists a maximal left ideal H of R such that $(H : R) = \{a \in R : aR \subseteq H\} = P$. Clearly, semisimplicity means Jacobson semisimplicity.

A submodule L of $S = R[E]$ which is also a left ideal will be denoted by ${}_S L_R$. An R-disjoint ideal I of S will be called here a *quasi-primitive ideal* of S if there exists ${}_S L_R$ which is maximal with respect to $L \cap R = 0$ such that $(L : S) = I$.

2. **Results.** Let R be a prime ring, $S = R[E]$ a free centred extension of R and put, as above, $T = Q[E]$ and $V = C[E]$. For an R-disjoint closed ideal P of S we denote by P^* the extension of P to T and by P_0 the contraction $P^* \cap V$. Under this notation we have

LEMMA 1. *If P_0 is a primitive ideal of V, then P is a quasi-primitive ideal of S.*

PROOF. Let L_0 be a maximal left ideal of V such that $(L_0 : V) = P_0$. Then there exists a closed submodule L^* of T such that $L^* \cap V = L_0$. Since L_0 is a left ideal, so is L^*. Then $L = L^* \cap S$ is a closed R-submodule of S which is also a left ideal. Take ${}_S N_R \supseteq L$ maximal with respect to $N \cap R = 0$. Thus $[N]$ is also an S-left R-right submodule of S with $[N] \supseteq N$ and $[N] \cap R = 0$. Therefore $[N] = N$, *i.e.*, N is closed. Consequently there exists a left ideal N_0 of V corresponding to N. Since $N_0 \supseteq L_0$, by the maximality of L_0 we obtain $N_0 = L_0$ and so $N = L$. Thus L is closed and maximal with respect to $L \cap R = 0$.

If $x \in [(L : S)]$ we have $Ax \subseteq (L : S)$ for some $0 \neq A \lhd R$. Then $AxS \subseteq L$ and so since L is closed, $xS \subseteq L$. Consequently $x \in (L : S)$ and thus $(L : S)$ is closed. Also $P \subseteq (L : S) \subseteq L$. Using again the one-to-one correspondence we have $P_0 \subseteq (L : S)_0 \subseteq L_0$ and therefore $P_0 = (L : S)_0$. It follows that $P = (L : S)$ and the proof is complete. ■

If A is an ideal (resp. left ideal) of R we denote by AE the set of all the elements of the type $\sum_{i=1}^{n} a_i e_i$, where $a_i \in A$. Then AE is also an ideal (resp. left ideal) of S.

The idea of the proof of the next lemma is contained in a previous paper ([3], Lemma 3).

LEMMA 2. *Let H be a maximal left ideal of R with $(H : R) = 0$ and let I be an R-submodule of S with $I \cap R = 0$. Then $(HE + I) \cap R = H$.*

PROOF. If $I = 0$ the result is clear. So we may assume $I \neq 0$ and, by contradiction, that $(HE + I) \cap R = R$. Then there exist $a_i \in H$, $i = 0, 1, \ldots, n$, and $x \in I$ with $1 = \sum_{i=0}^{n} a_i e_i + x$, where we put $e_0 = 1$. So $x = (1 - a_0) - \sum_{i=1}^{n} a_i e_i \in I$. Thus we may choose an element $y = b_0 + \sum_{i=1}^{t} b_i e_i \in I$ such that $b_0 \notin H$, $b_i \in H$ for $i = 1, \ldots, t$, and for which t is minimal. Take $\Gamma \in \mathrm{Min}(I)$ with $\Gamma \subseteq \{e_0, e_1, \ldots, e_t\}$. Since I is R-disjoint there exists $i \neq 0$ such that $e_i \in \Gamma$, say $e_1 \in \Gamma$.

The set of all the elements $r \in R$ such that there exists $z \in I$ with $\mathrm{supp}(z) = \Gamma$ and $z(e_1) = r$, together with 0, is a non-zero ideal A of R. So $A \not\subseteq H$ and therefore there exists an element $z \in I$ with $\mathrm{supp}(z) = \Gamma$ and $c_1 = z(e_1) \notin H$. Also, since $b_0 \notin H$ we have $Rb_0 + H = R$. Let $r \in R$, $h \in H$ be such that $rb_0 + h = 1$. Then $c_1 rb_0 + c_1 h = c_1$ and it follows that $c_1 rb_0 \notin H$. Then we put $v = c_1 ry - zrb_1 \in I$. We easily see that such a v contradicts the choice of y. ∎

LEMMA 3. *Assume that R is a primitive ring and P is a quasi-primitive ideal of S. Then P is a primitive ideal of S.*

PROOF. Let H be a maximal left ideal of R with $(H : R) = 0$ and let ${}_S L_R \subseteq S$ which is maximal with respect to $L \cap R = 0$ such that $(L : S) = P$. By Lemma 2, $(HE + L) \cap R = H$. Then there exists a left ideal N of S such that $N \supseteq HE + L$ which is maximal with respect to $N \cap R = H$. Hence N is a maximal left ideal of S. We show that $(N : S) = P$.

Put $(N : R) = \{x \in S : xR \subseteq N\}$. Then $(N : R)$ is a left ideal and a right R-submodule of S with $L \subseteq (N : R) \subseteq N$. Also we can easily see that $(N : R) \cap R = 0$ because $N \cap R = H$. Therefore $L = (N : R)$ follows by the maximality of L. Consequently $(N : S) \subseteq L$ and so $(N : S) = (L : S) = P$. The proof is complete. ∎

The following is clear.

COROLLARY 4. *Assume that R is a primitive ring with extended centroid C. If P_0 is a primitive ideal of $C[E]$, then P is a primitive ideal of S. In particular, if $C[E]$ is primitive (resp. semisimple), then so is $R[E]$.*

Now we are able to prove the Theorem.

PROOF OF THEOREM. Since R is semisimple we have $\bigcap\{P : P \in \mathcal{P}\} = 0$. We factor out from R and S the ideals P and PE, respectively. By the assumption $C_P[E]$ is semisimple. Then $(R/P)[E] \simeq R[E]/PE$ is also semisimple, by Corollary 4. Therefore PE is a semiprimitive ideal of S and consequently S is semisimple because $\bigcap\{PE : P \in \mathcal{P}\} = 0$. ∎

By factoring out the Jacobson radical $J(R)$ of R we easily obtain.

COROLLARY 5. *Let R be any ring and $S = R[E]$ a free centred extension of R. Assume that $C[E]$ is semisimple for any field $C \in \mathcal{F}$. Then the Jacobson radical $J(S)$ of S is contained in $J(R)E$.*

Note that if R is an algebra over the field of p elements, p a prime integer, then the characteristic of C_P is p for every $P \in \mathcal{P}$. Similarly, if R is an algebra over the field of rational integers, the characteristic of C_P is zero. So, denoting by Ch(F) the characteristic of the field F we have

COROLLARY 6. *Let R be a semisimple algebra over a prime field F and $S = R[E]$ a free centred extension of R. Assume that for every field K such that the ring $K[E]$ is well-defined and* $\mathrm{Ch}(K) = \mathrm{Ch}(F)$*, the ring $K[E]$ is semisimple. Then S is also a semisimple ring.*

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