A MEAN VALUE RELATED TO D. H. LEHMER’S PROBLEM AND THE RAMANUJAN’S SUM∗

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(Received 20 August 2010; accepted 24 September 2011)

Abstract. Let \( q > 1 \) be an odd integer and \( c \) be a fixed integer with \( (c, q) = 1 \). For each integer \( a \) with \( 1 \leq a \leq q - 1 \), it is clear that there exists one and only one \( b \) with \( 0 \leq b \leq q - 1 \) such that \( ab \equiv c \pmod{q} \). Let \( N(c, q) \) denotes the number of all solutions of the congruence equation \( ab \equiv c \pmod{q} \) for \( 1 \leq a, b \leq q - 1 \) in which \( a \) and \( b \) are of opposite parity, where \( b \) is defined by the congruence equation \( b^2 \equiv 1 \pmod{q} \). The main purpose of this paper is using the mean value theorem of Dirichlet \( L \)-functions to study the mean value properties of a summation involving \( (N(c, q) - \frac{1}{2}\phi(q)) \) and Ramanujan’s sum, and give two exact computational formulae.

2000 Mathematics Subject Classification. Primary 11L40, 11F20.

1. Introduction. Let \( p \) be an odd prime and \( c \) be a fixed integer with \( (c, p) = 1 \). For each integer \( a \) with \( 1 \leq a \leq p - 1 \), it is clear that there exists one and only one \( b \) with \( 0 \leq b \leq p - 1 \) such that \( ab \equiv c \pmod{p} \). Let \( M(c, p) \) denotes the number of cases in which \( a \) and \( b \) are of opposite parity. In [6], Professor D. H. Lehmer asked to study \( M(1, p) \) or at least to say something non-trivial about it. It is known that \( M(1, p) \equiv 2 \) or \( 0 \pmod{4} \) when \( p \equiv \pm 1 \pmod{4} \). For general odd number, \( q \geq 3 \), Wenpeng [7] and [8] studied the asymptotic properties of \( M(1, q) \), and obtained a sharp asymptotic formula:

\[
M(1, q) = \frac{1}{2} \phi(q) + O\left(q^{\frac{1}{2}}d(q)\ln^2 q\right),
\]

where \( \phi(q) \) denotes the Euler function, and \( d(q) \) is the number of divisors of \( q \).

Wenpeng [11] also studied the asymptotic properties of the mean square value of the error term \( M(a, p) - \frac{q-1}{2} \) and gave the asymptotic formula

\[
\sum_{a=1}^{p-1} \left(M(a, p) - \frac{q-1}{2}\right)^2 = \frac{3}{4} p^2 + O\left(p \exp\left(\frac{3 \ln p}{\ln \ln p}\right)\right).
\]

Now, we let \( q \) be an odd integer, \( c \) be any integer with \( (c, q) = 1 \), \( N(c, q) \) denotes the number of pairs of integers \( a, b \) with \( ab \equiv c \pmod{q} \) for \( 1 \leq a, b \leq q - 1 \) in which \( a \) and \( b \) are of opposite parity, and

\[
E(c, q) = N(c, q) - \frac{1}{2} \phi(q).
\]

∗This work is supported by the N.S.F. (11071194) of P.R.China.
The main purpose of this paper is to use the analytic method and the properties of the Dirichlet $L$-functions to study the mean value computational problem of the function $R_q(c + 1)E(c, q)$, where $R_q(c)$ is the Ramanujan's sum, defined as follows (see Theorem 8.6 in [1]):

$$R_q(c) = \sum_{k=1}^{q} e^{\frac{2\pi i kc}{q}} = \sum_{d|\gcd(c,q)} d \mu(q/d),$$

where $\mu(n)$ is the famous M"obius function.

About the mean value of $R_q(c + 1)E(c, q)$, it seems that none has studied it yet, at least we have not seen any related result till now. In this paper we shall prove the following two conclusions.

**Theorem 1.** Let $q \geq 3$ is an odd square-full number (that is, for any prime $p$, $p|q$ if and only if $p^2|q$), we have the identity

$$\sum_{c=1}^{q} R_q(c + 1)E(c, q) = \frac{1}{2} \phi^2(q) \prod_{p|q} \left(1 + \frac{1}{p}\right),$$

where $\phi(q)$ is Euler function, $\prod_{p|q}$ denotes the product over all prime divisor of $q$.

**Theorem 2.** For any prime $p \geq 3$, we have the identity

$$\sum_{c=1}^{p-1} R_p(c + 1)E(c, p) = \frac{1}{2} p(p-1).$$

For general odd number $q \geq 3$, whether there exists a computational formula for $\sum_{c=1}^{q} R_q(c + 1)E(c, q)$ is an open problem.

We believe that it is true. But now, we can only give an asymptotic formula.

2. **Several lemmas.** In this section we shall give several Lemmas, which are necessary for the proof of our theorems. First we have the following.

**Lemma 1.** Let $\chi$ be a primitive character modulo $m$ with $\chi(-1) = -1$. Then we have

$$\frac{1}{m} \sum_{b=1}^{m} b \chi(b) = \frac{i}{\pi} \tau(\chi)L(1, \overline{\chi}),$$

where $\tau(\chi)$ is the Gaussian sum associated with $\chi$, $e(y) = e^{2\pi i y}$, and $L(1, \chi)$ denotes the Dirichlet $L$-function corresponding to $\chi$.

**Proof.** This can be easily deduced by Theorems 12.11 and 12.20 in [1].

**Lemma 2.** Suppose $\chi$ is an odd character mod $q$, then we have the identity

$$(1 - 2\chi(2)) \sum_{a=1}^{q} a \chi(a) = \chi(2)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$
Proof. See [5].

**Lemma 3.** Let \( q > 1 \) be an odd number, then we have the identity

\[
\sum_{c=1}^{q} R_q(c + 1)E(c, q) = \frac{2}{\phi(q)} \sum_{\chi \, \text{mod} \, q, \chi(-1) = -1} |\tau(\chi)|^2 \cdot |1 - 2\chi(2)|^2 \cdot \left| \frac{1}{q} \sum_{a=1}^{q} a\chi(a) \right|^2,
\]

where \( \sum_{\chi \, \text{mod} \, q, \chi(-1) = -1} \) denotes the summation over all Dirichlet characters \( \chi \mod q \) such that \( \chi(-1) = -1 \).

**Proof.** From the orthogonality relation for character sums mod \( q \) and the definition of \( N(c, q) \), we have

\[
N(c, q) = \frac{1}{2} \sum_{a=1}^{q} \sum_{b=1}^{q} (1 - (-1)^{a+b}) = \frac{1}{2} \phi(q) - \frac{1}{2} \sum_{a=1}^{q} \sum_{b=1}^{q} (-1)^{a+b}
\]

\[
= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \, \text{mod} \, q} \chi(c) \left( \sum_{a=1}^{q} (-1)^{a} \chi(a) \right) \left( \sum_{b=1}^{q} (-1)^{b} \chi(b) \right)
\]

\[
= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \, \text{mod} \, q} \chi(c) \left( \sum_{a=1}^{q} (-1)^{a} \chi(a) \right) \left( \sum_{b=1}^{q} (-1)^{b} \chi(b) \right)
\]

\[
= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \, \text{mod} \, q} \chi(c) \left| \sum_{a=1}^{q} (-1)^{a} \chi(a) \right|^2.
\]

If \( \chi(-1) = 1 \), then

\[
\sum_{a=1}^{q} (-1)^{a} \chi(a) = 0.
\]

If \( \chi(-1) = -1 \), then

\[
\sum_{a=1}^{q} (-1)^{a} \chi(a) = 2\chi(2) \sum_{a=1}^{\frac{q-1}{2}} \chi(a).
\]

Note that the identity

\[
\sum_{c=1}^{q} \chi(c)R_q(c + 1) = \sum_{a=1}^{q} e\left(\frac{a}{q}\right) \sum_{c=1}^{q} \chi(c)e\left(\frac{ac}{q}\right) = \chi(-1) |\tau(\chi)|^2,
\]
combining (1), (2), (3) and Lemma 2, we may immediately deduce

\[
\sum_{c=1}^{q} R_q(c + 1) E(c, q) = \frac{2}{\phi(q)} \sum_{\chi \mod q} |\tau(\chi)|^2 \cdot |1 - 2\chi(2)|^2 \cdot \left| \frac{1}{q} \sum_{a=1}^{q} a\chi(a) \right|^2.
\]

This proves Lemma 3.

To introduce Lemma 4, we need to give the definition of the Dedekind sums. For a positive integer \( q \) and an arbitrary integer \( h \), the classical Dedekind sums \( S(h, q) \) is defined by

\[
S(h, q) = \sum_{a=1}^{q} \left( \left( \frac{a}{q} \right) \right) \left( \left( \frac{ah}{q} \right) \right).
\]

where

\[
\left( (x) \right) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer} \end{cases}
\]

The various properties of \( S(h, k) \) have been studied by many authors (see [2–4, 9, 10]). For this sum, there is also another kind of expression, which is as follows.

**Lemma 4.** Let \( q > 2 \) be an integer, then for any integer \( a \) with \( (a, q) = 1 \), we have the identity

\[
S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} d^2 \sum_{\chi \mod d} \chi(a)|L(1, \chi)|^2.
\]

**Proof.** See Lemma 2 in [9].

**Lemma 5.** Let \( q > 2 \) be an odd square-full number (that is, for any prime \( p \), \( p|q \) if and only if \( p^2|q \)), then we have the identities

(I) \[
\sum_{\chi \mod q}^* |L(1, \chi)|^2 = \frac{\pi^2 \phi^3(q)}{12} \prod_{p|q} \left( 1 + \frac{1}{p} \right);
\]

(II) \[
\sum_{\chi \mod q}^* \chi(2)|L(1, \chi)|^2 = \frac{\pi^2 \phi^3(q)}{24} \prod_{p|q} \left( 1 + \frac{1}{p} \right),
\]

where \( \sum_{\chi \mod q, \chi(-1)=-1}^* \) denotes the summation over all odd primitive characters \( \mod q \).
Proof. From the definition of the Dedekind sums, Lemma 4 and the Möbius inversion formula (see Theorem 2.9 in [1]) we have

\[
\sum_{\chi \mod q} \chi(a)|L(1, \chi)|^2 = \frac{\phi(q)}{q^2} \pi^2 \sum_{d \mid q} \mu(d) \frac{q}{d} S\left(a, \frac{q}{d}\right)
\]

\[
= \pi^2 \frac{\phi(q)}{q} \sum_{d \mid q} \frac{\mu(d)}{d} S\left(a, \frac{q}{d}\right).
\]

(4)

If \(a = 1\), then it is easy to compute

\[
S(1, q) = \sum_{k=1}^{q-1} \left(\frac{k}{q} - \frac{1}{2}\right)^2 = \frac{1}{12} \left(q - 3 + \frac{2}{q}\right).
\]

So from this formula and (4) we have

\[
\sum_{\chi \mod q} \chi(-1) = -1 |L(1, \chi)|^2 = \frac{\pi^2}{12} \frac{\phi(q)}{q} \sum_{d \mid q} \frac{\mu(d)}{d} \left(\frac{q}{d} - 3 + \frac{2d}{q}\right)
\]

\[
= \frac{\pi^2}{12} \frac{\phi(q)}{q} \sum_{d \mid q} \frac{\mu(d)}{d^2} - \frac{\pi^2}{4} \frac{\phi(q)}{q} \sum_{d \mid q} \frac{\mu(d)}{d} + \frac{\pi^2}{6} \frac{\phi(q)}{q^2} \sum_{d \mid q} \mu(d)
\]

\[
= \frac{\pi^2}{12} \frac{\phi^2(q)}{q} \left[ \prod_{p \mid q} \left(1 + \frac{1}{p}\right) - \frac{3}{q}\right].
\]

(5)

Note that \(q\) is a square-full number, \(\mu(q)\) and \(\phi(q)\) are two multiplicative functions,

\[
\sum_{d \mid q} \mu(d) \frac{\phi^2(q/d)}{q^2/d^2} = 0 \quad \text{and} \quad \sum_{\chi \mod q} |L(1, \chi)|^2 = \sum_{d \mid q} \sum_{\chi \mod \frac{q}{d}, \chi(-1) = -1} |L(1, \chi \chi_0)|^2,
\]

from the Möbius inversion formula and (5) we may immediately get

\[
\sum_{\chi \mod \frac{q}{2}} |L(1, \chi)|^2 = \sum_{d \mid q} \mu(d) \sum_{\chi \mod \frac{q}{2}, \chi(-1) = -1} |L(1, \chi \chi_0)|^2
\]

\[
= \sum_{d \mid q} \mu(d) \sum_{\chi \mod \frac{q}{2}, \chi(-1) = -1} |L(1, \chi)|^2
\]

\[
= \sum_{d \mid q} \mu(d) \left\{ \frac{\pi^2}{12} \frac{\phi^2(q/d)}{q/d} \left[ \prod_{p \mid \frac{q}{d}} \left(1 + \frac{1}{p}\right) - \frac{3}{q/d}\right] \right\}
\]

\[
= \frac{\pi^2}{12} \frac{\phi^3(q)}{q^2} \prod_{p \mid q} \left(1 + \frac{1}{p}\right),
\]
where we have used the fact that
\[ \sum_{d \mid q} \mu(d) \sum_{\chi \mod q} \frac{1}{2} \chi(-1) = -1 \mid |L(1, \chi \chi_0)|^2 = \sum_{d \mid q} \mu(d) \sum_{\chi \mod q} \frac{1}{2} \chi(-1) = -1 \mid |L(1, \chi)|^2 \] if \( q \) be a square-full number, and \( \chi_0 \) denotes the principal character mod \( q \). This proves the Formula (I) of Lemma 5.

If \( a = 2 \), then note that \( q \) is an odd number, so from the definition of the Dedekind sums we have
\[
S(2, q) = \sum_{k=1}^{q-1} \left( \frac{k}{q} - \frac{1}{2} \right) \left( \frac{2k}{q} - \frac{1}{2} \right) + \sum_{k=\frac{q+1}{2}}^{q-1} \left( \frac{k}{q} - \frac{1}{2} \right) \left( \frac{2k}{q} - \frac{3}{2} \right)
= \frac{1}{24} \left( q - 6 + \frac{5}{q} \right).
\]

Then from this identity and (4) we have
\[
\sum_{\chi \mod q} \chi(2) \mid |L(1, \chi)|^2 = \frac{\pi^2 \phi(q)}{24} \sum_{d \mid q} \mu(d) \left( \frac{q}{d} - 6 + \frac{5d}{q} \right)
= \frac{\pi^2 \phi(q)}{24} \sum_{d \mid q} \mu(d) \frac{d}{d} - \frac{\pi^2 \phi(q)}{4} \sum_{d \mid q} \mu(d) + \frac{5\pi^2 \phi(q)}{24} \sum_{d \mid q} \mu(d)
= \frac{\pi^2 \phi^2(q)}{24} \left( \prod_{p \mid q} \left( 1 + \frac{1}{p} \right) - \frac{6}{q} \right).
\]

Applying the Möbius inversion formula and (6) we can also deduce that
\[
\sum_{\chi \mod q} * \chi(2) \mid |L(1, \chi)|^2 = \sum_{d \mid q} \mu(d) \sum_{\chi \mod q} \chi(2) \chi_0(2) \chi(1, \chi \chi_0)|^2
= \sum_{d \mid q} \mu(d) \sum_{\chi \mod q} \chi(2) \chi(1, \chi)|^2
= \sum_{d \mid q} \mu(d) \left\{ \frac{\pi^2 \phi^2(q/d)}{24} \left[ \prod_{p \mid q} \left( 1 + \frac{1}{p} \right) - \frac{6}{q/d} \right] \right\}
= \frac{\pi^2 \phi^3(q)}{24} \sum_{p \mid q} \left( 1 + \frac{1}{p} \right).
\]

This completes the proof of Lemma 5.

\[ \square \]

3. Proof of the theorems. In this section, we will use lemmas from Section 2 to prove our theorems. First we prove Theorem 1. For any odd square-full number \( q \geq 3 \) and \( \chi \mod q \), note that the Gauss sum \( \tau(\chi) = 0 \), if \( \chi \) is not a primitive character mod \( q \);

If \( \chi \) is a primitive character mod \( q \), then we have \( |\tau(\chi)|^2 = q \). So from Lemmas 1, 3
and 5 we have

\[ \sum_{c=1}^{q} R_q(c + 1)E(c, q) = \frac{2}{\phi(q)} \sum_{\chi \mod q, \chi(-1)=1} \tau(\chi)^2 \cdot |1 - 2\chi(2)|^2 \cdot \left| \frac{1}{q} \sum_{a=1}^{q} a\chi(a) \right|^2 \]

\[ = \frac{2}{\pi^2} \frac{q^2}{\phi(q)} \sum_{\chi \mod q, \chi(-1)=1} (5 - 2\chi(2) - 2\overline{\chi}(2)) |L(1, \chi)|^2 \]

\[ = \frac{2}{\pi^2} \frac{q^2}{\phi(q)} \sum_{\chi \mod q, \chi(-1)=1} (5 - 4\chi(2)) |L(1, \chi)|^2 \]

\[ = \frac{2}{\pi^2} \frac{q^2}{\phi(q)} \left[ \frac{5\pi^2 \phi^3(q)}{12} \prod_{p \mid q} \left(1 + \frac{1}{p}\right) - \frac{4\pi^2 \phi^3(q)}{24} \prod_{p \mid q} \left(1 + \frac{1}{p}\right) \right] \]

\[ = \frac{1}{2} \phi^2(q) \prod_{p \mid q} \left(1 + \frac{1}{p}\right). \]

This proves Theorem 1.

Applying Lemma 4 we may get identities

\[ \sum_{\chi \mod p, \chi(-1)=1} |L(1, \chi)|^2 = \frac{\pi^2 (p - 1)^2(p - 2)}{12} \frac{p^2}{p^2} \]

and

\[ \sum_{\chi \mod p} \chi(2)|L(1, \chi)|^2 = \frac{\pi^2 (p - 1)^2(p - 5)}{24} \frac{p^2}{p^2}. \]

From these two formulae, Lemmas 1 and 3 we can deduce that

\[ \sum_{c=1}^{p-1} R_p(c + 1)E(c, p) \]

\[ = \frac{2}{\pi^2} \frac{p^2}{p - 1} \left( \frac{5}{12} \sum_{\chi \mod p, \chi(-1)=1} |L(1, \chi)|^2 - 4 \sum_{\chi \mod p, \chi(-1)=1} \chi(2)|L(1, \chi)|^2 \right) \]

\[ = \frac{2}{\pi^2} \frac{p^2}{p - 1} \left[ \frac{5\pi^2 (p - 1)^2(p - 2)}{12} \frac{p^2}{p^2} - \frac{4\pi^2 (p - 1)^2(p - 5)}{24} \frac{p^2}{p^2} \right] \]

\[ = \frac{1}{2} p(p - 1). \]

This completes the proof of Theorem 2.
REFERENCES